Theory of the anomalous skin effect in metals with complicated Fermi surfaces

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We give examples of metallic Fermi surfaces with geometric structures for which the impedance ζ under anomalous-skin-effect conditions does not become independent of the mean l free path as the latter increases, and estimate the impedance of these metals. We also elucidate how the overall Fermi surface topology affects the transition to the anomalous-skin-effect regime in normal metals. © *1994 American Institute of Physics.*

1. INTRODUCTION

The theory of the anomalous skin effect developed by Reuter and Sondheimer,[1] which was based on the free-electron approximation and a very simple treatment of the interaction between electrons and the metal surface, has been generalized by many authors in many directions (clarifying the role of surface structure, going beyond the τ-approximation, treating Fermi surfaces of arbitrary shape, etc.).

Anomalous behavior of the skin effect is most evident in the limit $l \gg \delta$, where l is the mean-free path of electrons and δ is the skin depth, when the expression for the surface impedance allows one to pass to the limit $l \to \infty$. The dissipation of electromagnetic energy that leads to attenuation of the electric field intensity into the bulk of the metal is determined by the loss of electrons from the skin-depth layer, and not by real collisions with irregularities of the crystal lattice in the bulk metal. Because the electrons that play the primary role in producing the anomalous skin effect are those that move almost parallel to the metal surface, the character of their interaction with the boundary is not very important: according to Reuter and Sondheimer,[1] the surface impedances corresponding to specular and diffuse scattering differ only by a factor 8/9.

The geometric approach to calculating various characteristics of metals has played an important role in understanding the high-frequency properties of the latter.[2–6]

Thus, according to this interpretation, for $l \gg \delta$ the electrons that participate in generating the surface impedance of a metal lie in a "belt" on the Fermi surface:

$$\mathbf{nv}(\mathbf{p})=0; \quad \varepsilon(\mathbf{p})=\varepsilon_{\mathrm{F}}, \tag{1}$$

where $\varepsilon(\mathbf{p})$ is the energy of an electron with quasimomentum $\mathbf{p}$, and $\mathbf{v}=\partial\varepsilon/\partial\mathbf{p}$ is its velocity; ε_{F} is the Fermi energy, and $\mathbf{n}$ is the normal to the metal surface.[1)]

In what follows we will assume that the metal occupies the half-space $z>0$, and that $\mathbf{n}\equiv(0,0,1)$ so that $\mathbf{nv}(\mathbf{p})\equiv v_z(\mathbf{p})$.

Note: for a nonquadratic energy spectrum (i.e., $\varepsilon \neq \alpha_{ik}p_ip_k$, where α_{ik} are constants) the first of Eqs. (1) is not the equation of a plane in $\mathbf{p}$ space. Thus, the "belt" is not a planar curve in this case. The structure of the belt is a strong function not only of the geometry of the Fermi surface but also the direction of the vector $\mathbf{n}$. As the direction of $\mathbf{n}$ changes, the structure of the belt can be significantly altered: indeed, even its connectivity can change. In general (for the case of a surface of general topology, to use the mathematical terminology), the connectivity of the belt can change in two ways: either (a) closed loops can appear (disappear) in the belt, or (b) a bridge between two loops can rupture (or rejoin). As shown by Avanesyan *et al.*,[7] the kinetic characteristics of a metal have singularities in this case, which they refer to as O-type singularities (for case "a") and X-type singularities (for case "b"). In both cases the change in topology of the belt gives "local information" about the Fermi surface: when a belt is initiated its radius is initially zero, while when a bridge in the belt is ruptured (or formed) there is a point of self-intersection. We will refer to a point in momentum space that is responsible for a change in connectivity of the belt as a critical point ($\mathbf{p}_c$). Avanesyan *et al.*[7] showed that critical points $\mathbf{p}=\mathbf{p}_c$ are located along curves of parabolic points. Therefore, singularities of O- and X-type can occur only for those metals whose Fermi surfaces have parabolic points.[2)]

When a metal is subjected to an external probe, its Fermi surface can change its dimensionality and shape. If the connectivity of the Fermi surface changes in this case, then a phase transition of order 2 1/2 occurs in the metal at $T=0$, as was shown by I. M. Lifshitz,[9] (i.e., a topological or Lifshitz transition; see the review by Blauter *et al.*[6] and the bibliography contained therein). However, even in the absence of changes in the connectivity of the metal Fermi surface, we may observe qualitative changes in its properties under the action of an external probe. Kaganov *et al.*[10] showed that these changes can be due to a change in the connectivity of the line of parabolic points at the Fermi surface. Changes in connectivity of curves of parabolic points, and the set of phenomena that accompany these changes, were referred to there as generalized topological transitions. In a generalized topological transition the bulk thermodynamic potential of the electrons (and the other thermodynamic characteristics of the metal) do not have anomalies in the absence of an external magnetic field. However, the kinetic properties of the metal are sensitive to generalized topological transformations, especially those that involve electrons of the belt.

In this paper, we will discuss two examples where the surface impedance of the metal should exhibit a sensitivity to the local geometry of the Fermi surface; for simplicity and clarity we will limit our discussion to hypothetical metals

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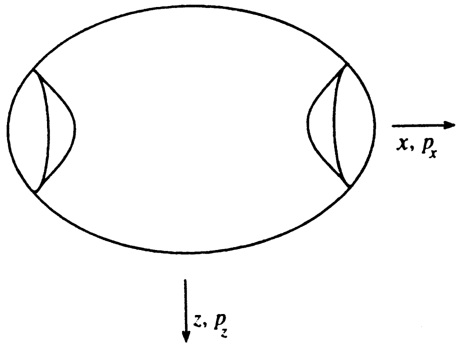

FIG. 1. A Fermi surface with a "crater" (the belt is shown).

whose Fermi surfaces possess axial symmetry (i.e., consisting of surfaces of revolution). The Fermi surfaces of interest to us are shown schematically in Figs. 1–4.

A. (Figs. 1, 2). The Fermi surface has a "crater." We choose the surface of the crystal so that the belt (1) has points of self-intersection. External perturbations can convert this surface into the usual ovaloid shape (Fig. 2); in this process the surface passes through a stage at which points of flattening appear on the Fermi surface, where two loops of the belt contract into points.

It is not difficult to show that the belt changes its topology even for rather small changes of the direction n (we may speak of a "broken crosspiece," although it is rather complicated to represent this pictorially for those spatial positions of the loops of the belt where this occurs). We will not discuss this type of change in the belt topology, because it requires rather small controlled changes in the position of the sample boundary relative to the crystallographic axis, which are in practice not feasible.

B. (Figs. 3, 4). The Fermi surface has a crosspiece and is situated so that the belt (1) consists of three loops. Under an external perturbation the Fermi surface may be converted into an ovaloid, passing through a stage in which the surface is quasi-cylindrical, i.e., three belts collapse into one.

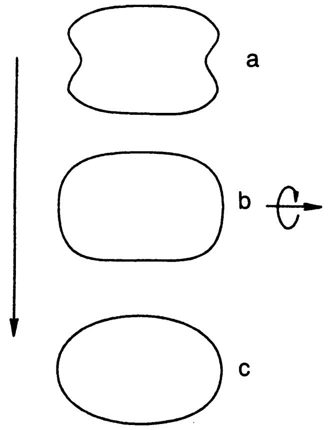

FIG. 2. Transformation of a Fermi surface from a crater (a) through a point of flattening (b) to an elliptic point (c). A side view is shown.

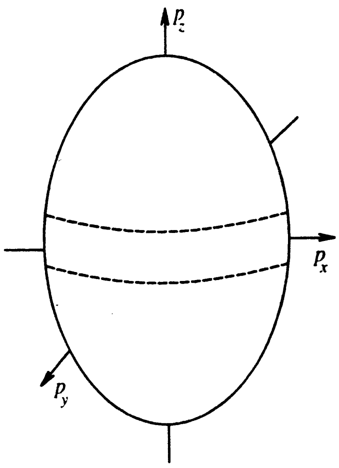

FIG. 3. A Fermi surface with a quasicylindrical portion (indicated by the dashed curves).

Both cases (A and B) exhibit generalized topological transformations: when the Fermi surface is converted into an ovaloid, loops present in the curves of parabolic points on it disappear (shown in Figs. 1 and 3). However, because of our assumption of axial symmetry, they are a special exception to the general rule. This can be seen with particular clarity for the second example, because generalized topological transformations in case B do not involve a "local event" in p-space: all the loops that appear in the parabolic-point curves have finite radius.

2. KINETIC INVESTIGATION

For the anomalous skin effect, it is necessary to solve the Maxwell equations to calculate the impedance:

$$\text{curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{j}, \quad \text{curl } \mathbf{E} = \frac{i\omega}{c} \mathbf{H}, \tag{2}$$

where ω is the frequency of the electromagnetic wave incident on the metal surface $z=0$, and $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{j}$ are the Fourier components of the electric and magnetic field intensities and the current density (respectively); we omit the factor $e^{-i\omega t}$. Equation (2) must be supplemented by a current equation:

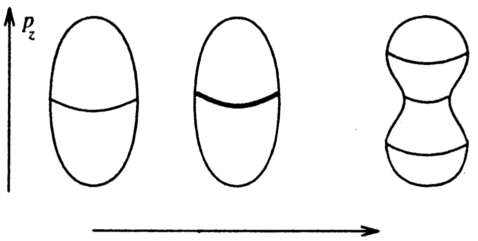

FIG. 4. Transformation of the Fermi surface: conversion of a single belt into three.

$$\mathbf{j}=\frac{2e}{(2\pi\hbar)^3}\int \mathbf{v}f_1 d^3p, \tag{3}$$

where f_1 is the nonequilibrium portion of the electron distribution function, which satisfies the Boltzmann equation (f_F is the equilibrium Fermi function):

$$v_z\frac{\partial f_1}{\partial z}+\frac{f_1}{\tau}=-\frac{\partial f_F}{\partial\varepsilon}ev_xE_x(z). \tag{4}$$

We used the τ approximation[3] and the assumption that the electric field in the wave is polarized along the symmetry axis of the metal, here along the x axis. Throughout this paper we will assume that, due to the postulated symmetry of the metal, no component of the electric field intensity appears that is normal to the metal surface ($E_z\equiv 0$), so that only the following field and current components are nonzero: $E_x(z)$, $H_y(z)$, $j_x(z)$. In the kinetic equation (4) we have omitted a term $-i\omega f_1$, because it is assumed that $\omega\tau\ll 1$. This is not a fundamental simplification; rather, it emphasizes that spatial dispersion of the metallic conductivity takes precedence over its temporal dispersion (for the range of applicability of these results, refer to the Conclusion).

In order to solve the kinetic equation (4) and compute the current density (3) we must formulate boundary conditions, in particular those that describe the interaction of electrons with the metal surface. However, noting that the quantities of interest to us depend weakly on the boundary conditions (see above), we will make the following (not entirely logical!) assumption: we will assert that the connection between **E** and **j** is "forgotten" at the boundary, thereby assuming that Eq. (4) is true throughout space, i.e., $E_x(z)\to 0$ as $|z|\to\infty$, while in computing $E_x(z)$ we "remember" the metal-vacuum boundary by assuming that the electric field has a discontinuous derivative (a kink) at $z=0$.

Thus, from Eq. (4), passing to Fourier components, we have

$$f_1(k)=-e\tau\frac{\partial f_F}{\partial\varepsilon}\frac{v_x}{1+ikv_z\tau}E_x(k). \tag{5}$$

The function and its spatial Fourier transforms we denote by the same letters but with different arguments. From Eq. (3) we have

$$j_x(k)=\sigma_{xx}(k)E_x(k),$$
$$\sigma_{xx}(k)=-\frac{2e^2}{(2\pi\hbar)^3}\int\tau\frac{\partial f_F}{\partial\varepsilon}\frac{v_x^2}{1+(kv_z\tau)^2}d^3p. \tag{6}$$

In obtaining expression (6) we have used the fact that the equal-energy surfaces have an inversion center: $\varepsilon(-\mathbf{p})=\varepsilon(\mathbf{p})$ and $\mathbf{v}(-\mathbf{p})=-\mathbf{v}(\mathbf{p})$. As a result, $\sigma_{xx}(k)$ is an even function of wave vector, i.e., $\sigma_{xx}(-k)=\sigma_{xx}(k)$.

Eliminating the magnetic field, for $E_x=E_x(z)$ we obtain from Eq. (2)

$$\frac{d^2E_x(z)}{dz^2}+\frac{4\pi i\omega}{c^2}j_x(z)=0. \tag{7}$$

Let us extend the function $E_x=E_x(z)$ in an even fashion onto the left half-axis: $E_x(-z)=E_x(z)$. Because $\sigma_{xx}(k)$ is an even function of its argument, according to (6), we also have $j_x(-z)=j_x(z)$. This allows us to assume that Eq. (7) is correct on the entire axis ($-\infty<z<\infty$). For the incident electromagnetic wave on the metal surface we specify the magnetic field at $z=0$ $H_y(z=+0)\equiv H_y(0)$ $[H_y(z=-0)\equiv -H_y(0)]$; according to (2) $H_y(z)$ is an odd function, i.e., $dE_x/dz|_{z=-0}=-dE_x/dz|_{z=0}$. Consequently, $E_x=E_x(z)$ has a kink at $z=0$. Passing to Fourier components, and taking into account the existence of the kink in the function $E_x=E_x(z)$, we find from (6) and (7):

$$\frac{E_x(0)}{H_y(0)}=\zeta_{xx},\quad \zeta_{xx}=\frac{2i\omega}{\pi c}\int_0^\infty\frac{dk}{[4\pi i\omega\sigma_{xx}(k)]/c^2-k^2}. \tag{8}$$

From Eq. (8) it is clear that in order to compute the impedance we must know the Fourier components of the electrical conductivity; all the singularities of the impedance (if there are any) arising from the geometry of the Fermi surface can be analyzed using Eq. (6).

We note that $\sigma_{xx}=\sigma_{xx}(k)$ can be used not only to calculate the impedance ξ_{xx}, but also directly, e.g., to calculate the coefficient of ultrasonic attenuation. In this case we have $k=\omega/u$, where u is the velocity of sound (see Ref. 12).

3. A FERMI SURFACE WITH A CRATER (A)

The simplest way to describe a Fermi surface that contains a crater is through the expression

$$\varepsilon_F=|p_x|v-\frac{p_\perp^2}{2m}+\frac{p_\perp^4}{4mp_0^2},\quad p_\perp\equiv(0,p_y,p_z), \tag{9}$$

where $p_\perp=\sqrt{p_y^2+p_z^2}$; m, v, and p_0 are positive constants. In Fig. 5 we show all the characteristic dimensions; the velocity components are as follows:

$$v_x=v\,\mathrm{sign}\,p_x,\quad v_\perp=-\frac{p_\perp}{m}\left(1-\frac{p_\perp^2}{p_0^2}\right). \tag{10}$$

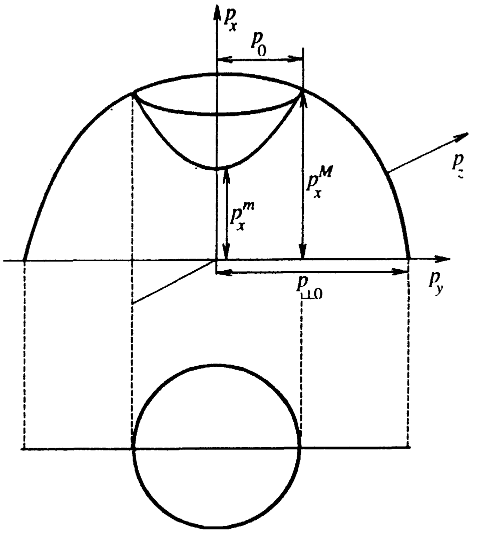

FIG. 5. Half of a Fermi surface with a crater; belt structure.

It is convenient to go to cylindrical coordinates:

$$p_\perp=\sqrt{p_y^2+p_z^2},\quad \varphi,\quad p_x,\quad \operatorname{tg}\varphi=\frac{p_y}{p_z}. \tag{11}$$

In cylindrical coordinates the equation of the belt reduces to the equation of a circle:

$$p_\perp=p_0,\quad p_x=p_x^M \tag{12}$$

(see Fig. 5) and the equation of a straight line

$$\varphi=\frac{\pi}{2}, \tag{12'}$$

to which we must add the equation of the Fermi surface (9).

Equation (12) describes a circle along a "path" around the "crater," while (12′) is the intersection of the Fermi surface (9) with the plane $p_z=0$ [for $p_z=0$ and $v_z=0$ see (10)]. We emphasize that the belt is located on planes that are perpendicular to one another.

We will set $-\partial f_F/\partial\varepsilon=\delta(\varepsilon-\varepsilon_F)$, which implies that $T\ll\varepsilon_F$, where T is the temperature. It is easy to carry out the integration over $p_\perp$ due to the δ function, and

$$\sigma_{xx}(k)=\frac{e^2\tau m v^2}{\pi^2\hbar^3}\left\{\int_0^{p_x^M}\frac{dp_x}{\sqrt{\left[1+\frac{4mv}{p_0^2}\left(\frac{\varepsilon_F}{v}-p_x\right)\right]\left[1+\left(\frac{k\tau p_+}{m}\right)^2\left(1+\frac{4mv}{p_0^2}\left(\frac{\varepsilon_F}{v}-p_x\right)\right)\right]}}\right.$$
$$\left.+\int_{p_x^m}^{p_x^M}\frac{dp_x}{\sqrt{\left[1+\frac{4mv}{p_0^2}\left(\frac{\varepsilon_F}{v}-p_x\right)\right]\left[1+\left(\frac{k\tau p_-}{m}\right)^2\left(1+\frac{4mv}{p_0^2}\left(\frac{\varepsilon_F}{v}-p_x\right)\right)\right]}}\right\}, \tag{13}$$

where

$$p_\pm^2=p_0^2\left(1\pm\sqrt{1+\frac{4mv}{p_0^2}\left(\frac{\varepsilon_F}{v}-p_x\right)}\right),$$
$$p_x^m=\frac{\varepsilon_F}{v};\quad p_x^M=\frac{\varepsilon_F}{v}+\frac{p_0^2}{4mv}. \tag{14}$$

Expressions (13) and (14) determine the dependence of the xx component of the conductivity on the wave vector k. For $k=0$

$$\sigma_{xx}=\frac{e^2\tau v p_0^2}{4\pi^2\hbar^3}\left\{\left(1+\frac{4m\varepsilon_F}{p_0^2}\right)^{1/2}+1\right\}. \tag{15}$$

To complete our picture let us write down the value of the transverse conductivity:

$$\sigma_{yy}=\sigma_{zz}=\frac{e^2\tau p_0^4}{4\pi^2\hbar^3 m^2 v}\left\{\frac{1}{2}\left(1+\frac{4m\varepsilon_F}{p_0^2}\right)^2+\frac{2}{3}\left(1+\frac{4m\varepsilon_F}{p_0^2}\right)^{3/2}+\frac{1}{3}\right\}, \tag{15'}$$

and also the volume of the Fermi surface Ω_F that contains the conduction electrons:

$$n_e=\frac{2\Omega_F}{(2\pi\hbar)^3},$$
$$\Omega_F=\frac{\pi p_0^4}{2mv}\left\{\frac{4m\varepsilon_F}{p_0^2}+\frac{2}{3}\left(1+\frac{4m\varepsilon_F}{p_0^2}\right)^{3/2}+\frac{2}{3}\right\}. \tag{16}$$

If the Fermi surface of the metal has other sheets besides (9), then naturally n_e is the density of electrons belonging to the sheet under study. The macroscopic (static) conductivities (15), (15′) have their usual orders of magnitude [provided, of course, that we have not made any special assumptions about the parameters entering into Eq. (9)], i.e., the existence of the crater does not affect the value of the static electrical conductivity.

Kaganov *et al.*[13] discussed how the existence of a belt with self-intersection affects the electrical conductivity of a thin film ($d\ll l$, where d is the film thickness). It is found that the conductivity grows by a factor $\ln(l/d)$ compared to the usual case.[14,15] This growth effect also occurs in $\sigma_{xx}(k)$ when $kl\gg 1$. In fact, it is clear from Eq. (13) that once we remove the factor $|k|\tau p_0/m$ from under the square root and take it outside the integral, we cannot pass to the limit in the remaining integral (i.e., we cannot take k to infinity), because the integral diverges logarithmically at its upper limit (for $p_x=p_x^M$).

Calculating the integral (13) for $|k|l\gg 1$, $l=\tau p_0/m$ leads to the following result:

$$\sigma_{xx}|_{|k|l\gg 1}\approx\frac{e^2\tau v p_0^2}{2\pi^2\hbar^3}\frac{\ln[2(1+4m\varepsilon_F/p_0^2)^{1/4}|k|l]}{|k|l}. \tag{17}$$

It is easy to show that for $|k|l\gg 1$, σ_{yy} increases logarithmically as well.

Note: if the belt (1) is simply connected, then for $|k|l\gg 1$ the conductivity σ_{xx} decays like const/$|k|l$.

The result (17) is found to agree with the general discussion of Avanesyan *et al.*:[7] when the belt has a point of self-intersection (a singularity of X-type), all the kinetic coefficients of conductivity type increase logarithmically as the mean-free path l increases.

4. A FERMI SURFACE WITH FLATTENING (A)

As we said in the Introduction, an external perturbation can make the "crater" at the Fermi surface disappear; at the instant it does so the equation for the Fermi surface takes the form (Fig. 2)

$$\varepsilon_{\mathrm{F}}=|p_x|v+\frac{p_1^4}{4mp_0^2}, \quad p_1=\sqrt{p_y^2+p_z^2}. \tag{18}$$

The parameters v, m, and p_0 in Eq. (18) can differ somewhat from the corresponding parameters in Eq. (9).

Making use of Eq. (6), it is not difficult to obtain

$$\sigma_{xx}(k)=\frac{e^2\tau v p_0(mv)^{1/2}}{8\pi^2\hbar^3}\int_0^{\varepsilon_{\mathrm{F}}/v}\frac{dp_x}{\sqrt{\left(\frac{\varepsilon_{\mathrm{F}}}{v}-p_x\right)\left\{1+(kl)^2\left[\frac{4mv}{p_0^2}\left(\frac{\varepsilon_{\mathrm{F}}}{v}-p_x\right)\right]^{3/2}\right\}}}, \tag{19}$$

where, as before, $l=\tau p_0/m$.

For $kl=0$,

$$\sigma_{xx}=\frac{e^2\tau v p_0(m\varepsilon_{\mathrm{F}})^{1/2}}{4\pi^2\hbar^3}, \tag{20}$$

while for $|k|l\to\infty$

$$\sigma_{xx}(k)\propto\beta\,\frac{e^2\tau v p_0^2}{\pi^2\hbar^3}\,\frac{1}{(|k|l)^{2/3}}, \quad \beta\sim1.4. \tag{21}$$

The existence of a point of flattening at the Fermi surface changes the asymptotic behavior of the conductivity quite dramatically: in place of the usual asymptotic behavior ($\sigma\propto\sigma_0/|k|l$) we have Eq. (21).

It is important to emphasize that it is the electrons located near the isolated points on the Fermi surface that are, in fact, "responsible" for the special asymptotic behavior of the conductivity [see expressions (17) and (21)], i.e., near a point of self-intersection of the belt [formula (17)] or a point of flattening [formula (21)]. Therefore, the primary result (the dependence of σ_{xx} on $|k|l$ as $|k|l\to\infty$) is practically independent of the model chosen (if, of course, we do not "absolutize" the value of the coefficients in these formulas).

However, it is necessary to keep in mind that the structure of the Fermi surface near these isolated points that determine the asymptotic behavior of $\sigma_{xx}(k)$ is very important. This is easy to demonstrate for the example discussed here of a Fermi surface with flattening. The higher the order of tangency of the Fermi surface to the tangent plane perpendicular to the axis p_x, the larger the exponent of $p_\perp$ is in Eq. (18). Suppose it equals 2β (for the usual case of an elliptic point, $\beta=1$; here we have discussed $\beta=2$). If we do the calculation for an arbitrary value of $\beta\geqslant1$, it is not difficult to show that for $|k|l\gg1$ the conductivity $\sigma_{xx}(k)$ is proportional to the expression

$$I_\beta(|k|l)=\frac{1}{(|k|l)^{2/(2\beta-1)}}\int_0^{|k|l}\frac{dy}{y^{(2\beta-3)(2\beta-1)}(1+y^2)^{1/2}}.$$

From this it is clear that $\sigma_{xx}(k)=\mathrm{const}/|k|l$ for $\beta=1$, as we should expect, while for $\beta>1$

$$\sigma_{xx}(k)=\frac{\mathrm{const}}{(|k|l)^{2/(2\beta-1)}}. \tag{22}$$

The exponent reduces to zero as β increases.

If there is a "truly" flat portion of the Fermi surface with an area equal to ΔS, then the electrons that belong to it are not subject to the anomalous skin effect, and these are the ones that determine the high-frequency properties of the metal in the geometry under discussion (provided, of course, that $\Delta S/S\gg1/|k|l$, where S is the area of the Fermi surface).

5. FERMI SURFACE WITH A QUASICYLINDRICAL PORTION (B)

In the process of changing from an ovaloid into a surface with a "waist," the Fermi surface should go through a stage where three belts merge into one, and the cylinder that encircles such a surface is tangent to it to a much higher degree than usual (Fig. 4).

The simplest equation that models such a surface is the following:

$$\varepsilon_{\mathrm{F}}=p_\perp v+\frac{p_z^4}{4mp_0^2}, \quad p_\perp=\sqrt{p_x^2+p_y^2}, \tag{23}$$

here, m, v, p_0 are constants [of course, they do not equal the parameters in Eq. (9)]. As always, the z axis is directed perpendicular to the metal-vacuum boundary.

In quasi-two-dimensional metals like graphite, the Fermi surfaces have quasicylindrical portions which, for most problems, are treated as true cylinders. If such a cylinder is located so that normals to its surface (electron velocities) are parallel to the sample surface (a metallic half-space), then we will not observe the anomalous skin effect in such a sample. More precisely: only electrons of the cylindrical portion of the Fermi surface will determine the high-frequency conductivity σ_{hf}, while the contribution of the remaining electrons will be of the same order as the quantity $1/|k|l$, for $l\gg\delta$ the equations of electrodynamics (for the impedance or skin depth) will be the same as in the theory of the normal skin effect if we replace the static conductivity by

$\sigma_{hf}=e^2 l/4\pi^2(p_y\Delta p_z/\hbar^3)$, where p_y is the radius of the cylinder and Δp_z is its height (the number of electrons within the cylinder equals $p_y^2\Delta p_z/4\pi^2\hbar^3$).

From Eqs. (6) and (23) we can proceed as we did previously to obtain

$$\sigma_{xx}(k)=\frac{e^2\tau v^2 m}{\pi^2\hbar^3 p_0}\left(\frac{p_0^2}{4mv}\right)^{3/4}\int_0^{\varepsilon_F/v}\frac{p_\perp dp_\perp}{\left(\frac{\varepsilon_F}{v}-p_\perp\right)^{3/4}\left\{1+(|k|l)^2\left[\frac{4mv}{p_0^2}\left(\frac{\varepsilon_F}{v}-p_\perp\right)\right]^{3/2}\right\}}. \tag{24}$$

Here, as before, $l=\tau p_0/m$.

For $|k|l=0$,

$$\sigma_{xx}=\frac{4\sqrt{2}}{5}\frac{e^2\tau p_0\varepsilon_F}{\pi^2\hbar^3}\left(\frac{m\varepsilon_F}{p_0^2}\right)^{1/4}, \tag{25}$$

while for $|k|l\to\infty$

$$\sigma_{xx}(k)\simeq\frac{e^2\tau p_0\varepsilon_F}{12\pi\hbar^3}\frac{1}{(|k|l)^{1/3}}. \tag{26}$$

The exponent ν of $|k|l$ in the denominator of expression (26) (in this case $\nu=1/3$) depends on the order of tangency of the cylinder encircling the Fermi surface. As the exponent 2α of p_z increases in Eq. (23) (in our case $\alpha=2$) the exponent ν decreases. If we set $\sigma_{xx}=\text{const}/|k|l$, then $\gamma=1/2\alpha-1$. As $\alpha\to\infty$ the exponent satisfies $\gamma\to 0$.

6. IMPEDANCE OF A METALLIC HALF-SPACE (SPECIAL CASES)

The Fourier components of the conductivity are interesting in their own right (see above). However, it is clear that the most important reason we have for obtaining these expressions in this paper [keeping in mind Eqs. (17), (21), and (24)] is so that we can calculate the surface impedance of a metal whose Fermi surface possesses singularities of the type described above. Because the static conductivity possesses no singular properties, we will naturally discuss the anomalous skin effect. In fact, the expressions we will introduce are in the anomalous skin-effect limit ($l\to\delta$).

Naturally, in order to compute the impedance we use Eq. (8), writing the Fourier component of the conductivity in the following way:

$$\sigma_{xx}(k)=\tilde{\sigma}f(|k|l). \tag{27}$$

By comparing with Eqs. (17), (21), and (24), in each case it is clear how we should interpret $\tilde{\sigma}$ and what the form of $f(|k|l)$ must be. Note that if we do not specify parameters that lead to model expressions for exotic Fermi surfaces, then σ will be close in order of magnitude to an ordinary real metallic conductivity.

If we discuss only the anomalous-skin-effect limit, then it is sufficient to give the asymptotic values of the function $f(|k|l)$:

$$f(|k|l)\approx\begin{cases}\ln(|k|l)/(|k|l) & \text{if the Fermi surface has a crater,}\\ 1/(|k|l)^{2/3} & \text{if the Fermi surface has a flattening point,}\\ 1/(|k|l)^{1/3} & \text{for a quasicylindrical Fermi surface.}\end{cases} \tag{28}$$

Substituting the value of $\sigma_{xx}(k)$ into Eq. (8), we write it as follows:

$$\zeta_{xx}=\frac{2i\omega}{\pi c}\int_0^\infty\frac{dk}{\frac{4\pi i\omega\tilde{\sigma}}{c^2}\frac{\ln^\gamma|kl|}{(|k|l)^q}-k^2}. \tag{29}$$

Comparing (28) with the asymptotic values of $f(|k|l)$, we see: $\gamma=0$ or 1, while $q=1$, 2/3, and 1/3.

Let us begin with $\gamma=0$. Changing to dimensionless integration variables, we obtain

$$\zeta_{xx}=\frac{2\omega}{\pi c}\tilde{\delta}\left(\frac{l}{\tilde{\delta}}\right)^{q/(q+2)}\int_0^\infty\frac{x^q dx}{1+ix^{q+2}};\qquad \tilde{\delta}=\frac{c}{\sqrt{4\tau\tilde{\sigma}\omega}}. \tag{30}$$

For $q=1$ we have the standard expression for the surface impedance ζ_∞ in the anomalous-skin-effect limit, in which the dependence on the mean-free path becomes $\tilde{\delta}(l/\tilde{\delta})^{1/3}=\tilde{\delta}^{2/3}l^{1/3}\sim(l/\sigma)^{1/3}$. Therefore, to within a complex factor β, where $|\beta|\sim 1$,

$$\zeta_{xx}=\beta|\zeta_\infty|\left(\frac{\tilde{\delta}}{l}\right)^{(2/3)\,(1-q)/(2+q)} \tag{31}$$

Thus,

$$\zeta_{xx}=\beta|\zeta_\infty|\begin{cases}(\delta/l)^{1/12} & \text{if the Fermi surface has a flattening point,}\\ (\delta/l)^{4/21} & \text{for a quasicylindrical Fermi surface.}\end{cases} \tag{32}$$

The case $\gamma=1$ and $q=1$ (the presence of a crater on the Fermi surface) must be discussed in slightly more detail. Here

$$\zeta_{xx}=\frac{2\omega}{\pi c}\int_0^\infty\frac{kdk}{\frac{4\pi\tilde{\sigma}\omega}{c^2 l}\ln(k|l)+ik^3}.$$

Let us change, as before, to dimensionless variables of integration:

$$\zeta_{xx}=\frac{2\omega}{\pi c}\left(\frac{c^2 l}{4\pi\tilde{\sigma}\omega}\right)^{1/3}\int_0^\infty \frac{x\,dx}{\ln\left[\left(\frac{l}{\tilde{\delta}}\right)^{2/3}x\right]+ix^3}.$$

To logarithmic accuracy[4)] the impedance of a metal whose Fermi surface has a crater is

$$\zeta_{xx}\approx\beta|\zeta_\infty|\left[\ln\left(\frac{l}{\tilde{\delta}}\right)\right]^{-1/3},\quad |\beta|\sim 1. \tag{33}$$

Expressions (32), (33) are the first terms of an expansion in powers of the ratio $\tilde{\delta}/l$. We recall that $\tilde{\delta}\sim l^{-1/2}$, and ζ_∞ does not depend on l. Thus, if we separate out the dependence on the mean-free path l, from Eq. (31) we obtain

$$\zeta_{xx}=\beta|\zeta_\infty|\left(\frac{l_0}{l}\right)^{(1-q)/(2+q)}, \tag{34}$$

where

$$l_0=(\tilde{\delta}^2 l)^{1/3}. \tag{35}$$

The parameter l_0 does not depend on the mean-free path (see above) and is the penetration depth of the electromagnetic field into the metal in the "ordinary" anomalous skin-effect limit, for which $l_0\ll l$ (see Ref. 16, para. 86). From (34) and (33) we have

$$\zeta_{xx}=\beta|\zeta_\infty|\begin{cases}[(3/2\ \ln(l/l_0)]^{-1/3} & \text{if the Fermi surface has a crater,}\\ (l_0/l)^{1/8} & \text{if the Fermi surface has a flattening point,}\\ (l_0/l)^{2/7} & \text{for a quasicylindrical Fermi surface.}\end{cases} \tag{36}$$

Let us say a few words about the accuracy of these expressions (36). Of course, the first of them is by far the most accurate [see example 4)].

In deriving all these expressions we have, first of all, retained only leading asymptotic terms for $\sigma_{xx}(k)$ [see Eq. (28)]; secondly, we have neglected contributions from "ordinary" sheets of the Fermi surface (if the metal Fermi surface has any such ordinary sheets, i.e., besides the anomalous one).

To complete this section, we note that the absolute value of the impedance is not so important as the fact that, in the cases we have discussed, its dependence on mean-free path does not disappear for $l\gg\delta$. Consequently, this quantity continues to depend on temperature, the presence of impurities, and everything that determines the value of the mean-free path (or the static conductivity and static resistivity; see Fig. 6).

7. GENERALIZED TOPOLOGICAL TRANSITION

Among the Fermi surfaces[5)] of various metals, we find those that possess singularities of the sort described above. However, it is quite possible that such a singular surface can also be generated by an external perturbation, i.e., as a result of a generalized topological transition.[10]

Using the expressions obtained and Figs. 2 and 4, we can trace how the surface impedance should change as we pass through a generalized topological transition.

Here we consider two different topological transitions:

A. Formation of a crater by passage through a point of flattening (Fig. 2);

B. Formation of a "waist" (Fig. 4).

The following features are common to these two cases:

1) on both sides of the transition the structure of the Fermi surface and the belts on it are different and the impedances of the metal are different;

2) at the transition point the conductivity is higher (impedance is smaller) than on either side of the transition.

A. When a crater forms before the transition, the impedance has its usual value, while it decreases as we approach the transition (as analysis shows). At the transition point there is a point of flattening on the Fermi surface, and the impedance is smaller than normal by a factor of $(l/\tilde{\delta})^{1/12}$ [see (32)]. After the transition, the belt has a point of self-intersection and the impedance is $[\ln(l/\tilde{\delta})]^{1/3}$ times smaller than normal [see (33)].

B. Formation of a waist is accompanied by "splitting" of one of the belts into three. At the transition point, the Fermi surface has a quasicylindrical portion and the impedance is

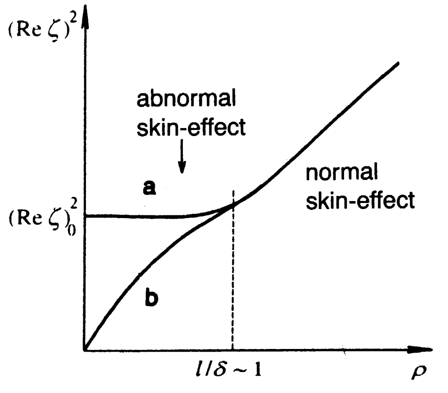

FIG. 6. Schematic dependence of the surface impedance on the mean-free path l: (a) standard, (b) singular.

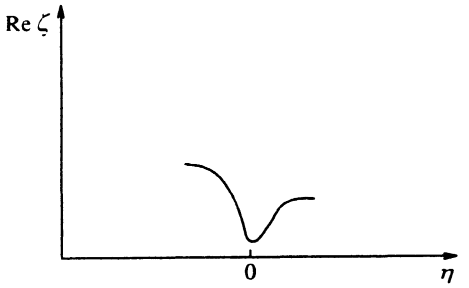

FIG. 7. Dependence of the surface impedance ζ on the parameter η used to specify closeness to a generalized topological transition (sketch).

$(l/\tilde{\delta})^{4/21}$ times smaller than normal. On that side of the transition where there is a waist in the Fermi surface, the impedance is somewhat smaller than on the side where there is no waist (if the parameters of the belt before and after the transition were to coincide, then the impedance would be smaller by a factor of $\sqrt[3]{3}$).

In Fig. 7 we show schematically the dependence of the impedance on the parameter $\eta=(\varepsilon_F-\varepsilon_c)/\varepsilon_c$ that defines the closeness to the generalized topological transition; here ε_c is the value of the Fermi energy at which the generalized topological transition takes place. The width of the transition region is not large ($|\eta|\sim 1/|kl|^q$, where the power q depends on the geometry of the transition; see above). It is necessary to understand that this generalized topological transition affects the high-frequency properties of the metal (i.e., the impedance will have an anomaly) only when an electromagnetic wave is reflected from that face of a single crystal for which the belt (1) undergoes the metamorphosis described above.

8. CONCLUSION

In this paper we have discussed the surface impedance of metals in the normal state, a problem that is not exactly the center of attention for experimental physicists. Among other things, it is possible that this is due to some feeling that this area of physics is "closed." In our view, the origin of this feeling is the suspicion that agreement between the theory of the anomalous skin effect and experiment is not entirely adequate.

Among the many problem areas in which there is still a prospect of improving the agreement between theory and experiment, the question of the role of local geometry of the Fermi surface in electrodynamics of metals is by no means the least interesting. We have undertaken to demonstrate the connection between the geometric structure of the Fermi surface and the value of the impedance in a number of the simplest cases. Unfortunately, we know of no experimental work that might require use of the expressions derived here. On the other hand, special experiments could be formulated based on contemporary values of the electron energy spectrum of metals that could undoubtedly reveal the existence of belts with complicated topologies, flattening points, and quasicylindrical segments, i.e., all the singular geometries that ought to appear in the high-frequency properties of a metal.

We have intentionally limited our discussion to the RF wavelength range,[6)] more precisely the condition $\omega\tau\ll 1$. The behavior of the kinetic coefficients [e.g., the components of the conductivity tensor $\sigma_{ik}(\omega,k)$] depends sensitively on the ratio of the spatial and temporal dispersion parameters. In particular, it is well known (see Ref. 2, Sec. 23) that we may "forget" about Fermi-liquid interactions between electrons for $\omega\tau\ll 1$, with the understanding, however, that the values used for $\varepsilon(\mathbf{p})$ and $\mathbf{v}(\mathbf{p})$ refer to quasiparticle electrons. In other words, all the expressions obtained above are correct if we take into account Fermi-liquid interactions between the electrons. On the other hand, for $\omega\tau\gg 1$ the singularities that occur are very sensitive to Fermi-liquid interaction.[18] For pure metals at low temperatures the combination of conditions

$$\omega\tau\ll 1, \quad l\gg\delta$$

does not limit the choice of frequencies very much: there exists a "window"

$$\left(\frac{\delta_L}{l}\right)^2\frac{1}{\tau}\ll\omega\ll\frac{1}{\tau}, \quad \delta_L=\frac{c}{\omega_L}\sim 10^{-5}\ \text{cm},$$

that is "wide open" for $l\gg\delta_L$.

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[1)]It is convenient to represent this "belt" as a boundary between light and shadow when the Fermi surface is illuminated by parallel rays of light in the direction **n** (in the case of a Fermi surface with dents and crosspieces it is necessary to assume that the surface is semitransparent). The corresponding figures can be seen in Refs. 4, 5.

[2)]It is only rarely that the Fermi surface of a metal is so simple that there are no parabolic points on it [e.g., the Fermi surfaces of Na, K, Rb, Cs are spheres, and the Fermi surface of Bi is a system of ellipses (see Ref. 2, Appendix III, and Ref. 8)].

[3)]For an analysis of the feasibility of using the τ approximation, see Ref. 11.

[4)]Strictly speaking, for our estimate we have neglected terms of order $\ln\ln(l/\tilde{\delta})$ compared to $\ln(l/\tilde{\delta})$. Obviously, this is a very crude approximation.

[5)]In the majority of metals the Fermi surface is broken up into several sheets; it is possible that only one of the sheets will have one of the properties that we have discussed.

[6)]An approach that is not limited by the condition $\omega\tau\ll 1$ makes up the contents of N. A. Zembovskiĭ's dissertation.[17]

[1]G. E. Reuter and E. H. Sondheimer, Proc. Roy. Soc. A **195**, 336 (1948).
[2]I. M. Lifshitz, N. Ya. Azbel', and M. I. Kaganov, *Electronic Theory of Metals*, Consultants Bureau, New York (1973).
[3]M. I. Kaganov and I. M. Lifshitz, Usp. Fiz. Nauk **129**, 487 (1979) [Sov. Phys. Usp. **22**, 904 (1979)].
[4]M. I. Kaganov, Usp. Fiz. Nauk **145**, 507 (1985) [Sov. Phys. Usp. **28**, 257 (1985)].
[5]M. I. Kaganov and Yu. V. Gribkova, Fiz. Nizk. Temp. **17**, 907 (1991) [Sov. J. Low Temp. Phys. **17**, 473 (1991)].
[6]Ya. M. Blauter, M. I. Kaganov, A. V. Pantsulaya, and A. A. Varlamov, Phys. Reports, 1994.
[7]G. T. Avanesyan, M. I. Kaganov, and T. Yu. Lisovskaya, JETP Lett. **25**, 355 (1977); Zh. Eksp. Teor. Fiz. **75**, 1786 (1978) [Sov. Phys. JETP **48**, 900 (1978)].
[8]A. P. Cracknell, *The Fermi Surface of Metals*, Taylor and Francis, Ltd., London, 1971.
[9]I. M. Lifshitz, Zh. Eksp. Teor. Fiz. **38**, 1569 (1960) [Sov. Phys. JETP **12**, 1130 (1960)].

[10] M. I. Kaganov and A. A. Nurmagambetov, Zh. Eksp. Teor. Fiz. **83**, 2296 (1982) [Sov. Phys. JETP **56**, 1331 (1982)].
[11] M. I. Kaganov, G. Ya. Lubarskiĭ, and E. Chervonko, Zh. Eksp. Teor. Fiz. **101**, 1351 (1992) [Sov. Phys. JETP **74**, 725 (1992)].
[12] V. M. Kontorovich, *Dynamic Equations of the Theory of Elasticity. Conduction Electrons* [in Russian], vol. **P**, p. 44. Coll. ed. by M. I. Kaganov and V. S. Edel'man. Nauka, Moscow, 1985.
[13] M. I. Kaganov, D. V. Kamshilin, and A. A. Nurmagambetov, Fiz. Nizk. Temp. **15**, 289 (1989) [Sov. J. Low Temp. Phys. **15**, 162 (1989)].
[14] K. Fuchs, Proc. Cambridge Phil. Soc. **34**, 100 (1938).
[15] Ya. Azbel' and M. I. Kaganov, Zh. Eksp. Teor. Fiz. **27**, 762 (1954).
[16] E. M. Lifshitz and L. P. Pitaevskiĭ, *Physical Kinetics*, Pergamon, Oxford (1981).
[17] N. A. Zimbovskaya, *High-Frequency Effects in the Electron Liquids of Metals with Singularities in the Shape of their Fermi Surfaces* [in Russian] (Abst. diss., partial req. for the degree of Doc. of Phys. and Math. Sci., N. I. Lobachevskiĭ State Univ., Nizhniĭ Novgorod, 1994).
[18] M. I. Kaganov and A. G. Plyavenek, Zh. Eksp. Teor. Fiz. **88**, 249 (1985) [Sov. Phys. JETP **61**, 145 (1985)].

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