# First, second, and fourth sound in relativistic superfluidity theory with dissipation

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Equations are derived describing the propagation of first, second, and fourth sound in relativistic superfluidity theory with dissipation, and velocity expressions that allow for damping are obtained for each. © *1994 American Institute of Physics.*

The main purpose of this paper is to examine the propagation of waves of first, second, and fourth sound in relativistic superfluid systems with dissipation. Lebedev and Khalatnikov[1] have given a relativistic generalization of the phenomenological approach describing superfluid systems. They first obtained the equations of two-velocity relativistic hydrodynamics and thermodynamics in the nondissipative approximation, and then introduced dissipative terms into the equations. Fomin and Shadura,[2] in a different approach and in terms of different thermodynamic variables, obtained equations of relativistic superfluidity theory in the nondissipative approximation that generalize Landau's phenomenological equations.[3] Their results are in full agreement with those of Lebedev and Khalatnikov.[1] The present paper proceeds from the results of Refs. 1 and 2 to derive equations describing the propagation of possible sound vibrations in the system and find the velocities of propagation, allowing for damping. The method used was developed for a similar problem in nonrelativistic superfluidity theory.[4]

Consider a quantum system in a local equilibrium state below the critical point, in which the system contains both superfluid ("condensate") and normal ("excitation gas") microscopic components, each with its own density $\rho_s$ and $\rho_n$ and velocity field $v^\nu$ and $u^\nu$, respectively. The basic equations of relativistic superfluid two-velocity thermodynamics and hydrodynamics describing such a system are[1,2] a) the energy–momentum conservation law

$$\partial_\nu T^{\mu\nu}=0,$$

$$T^{\mu\nu}=(TS_n+\mu\rho_n)u^\mu u^\nu+\mu\gamma^{-1}\rho_s v^\nu v^\mu-Pg^{\mu\nu}+\tau^{\mu\nu},$$

$$\tau^{\mu\nu}=-\xi^{\mu\lambda\nu}\partial_\lambda(\mu T^{-1})+\xi_s^{\mu\lambda\nu}\partial_\lambda(u^s T^{-1}), \tag{1}$$

where $T^{\mu\nu}$ is the energy–momentum tensor, $\mu$ is the invariant chemical potential, $T$ is the invariant temperature, $P$ is the pressure, $S_n$ is the density of the normal-component entropy, $g^{\mu\nu}$ is the metric tensor, $\tau^{\mu\nu}$ is the part of the energy–momentum tensor describing dissipation, and

$$\begin{bmatrix} \xi^{\mu\lambda} & \xi_s^{\mu\lambda} \\ \xi^{\mu\lambda\nu} & \xi_s^{\mu\lambda\nu} \end{bmatrix}$$

is the kinetic coefficient matrix, which is symmetric in $\mu$ and $\nu$ and in $\lambda$ and $\nu$; b) the current conservation law

$$\partial_\nu j^\nu=0, j^\nu=\rho_n u^\nu+\rho_s v^\nu+\lambda^\nu, \tag{2}$$

where $\lambda^\nu=-\xi^{\nu\lambda}\partial_\lambda\lambda^{\nu[}=-\xi^{\nu[\lambda[}\partial_\lambda(\mu T^{-1})+\xi_s^{\nu\mu}\partial_\mu(u^s T^{-1})$ is the dissipation correction to the current vector; and c) the law of increasing entropy in dissipative systems

$$\partial_\nu S^\nu=R, \tag{3}$$

where $S^\nu=S_n u^\nu-\mu T^{-1}\lambda^\nu+u_\mu\tau^{\mu\nu}$ is the entropy flux vector, and

$$R=\xi_s^{\mu\lambda\nu}\partial_\mu(u_\nu T^{-1})\partial_\lambda(u^s T^{-1})-\xi^{\mu\lambda\nu}\partial_\mu(u_\nu T^{-1})\partial_\lambda(\mu T^{-1})$$
$$-\xi^{\mu\lambda\nu}\partial_\lambda(u_\nu T^{-1})\partial_\mu(\mu T^{-1})+\xi^{\mu\lambda}\partial_\lambda(\mu T^{-1})\partial_\mu(\mu T^{-1})$$

is the term responsible for the increase in entropy due to dissipative processes. The equations describing the motion of the superfluid component are

$$\mu\gamma^{-1}v^\nu\partial_\nu v^\lambda=\Delta_s^{\lambda\nu}\partial_\nu(\mu\gamma^{-1}), \tag{4}$$

$$\Delta_s^{\mu\nu}\partial_\nu v^s\Delta_s^{\mu\nu[}\partial_{\nu[}v^{s[}-\Delta_s^{s\nu}\partial_\nu v^\mu=0, \Delta_s^{\mu\nu}=g^{\mu\nu}-v^\mu v^\nu. \tag{5}$$

The relations

$$dP=S_n dT+\rho_n d\mu+\rho_s d(\mu\gamma^{-1}), \tag{6}$$

$$d\varepsilon=TdS_n+\mu\gamma^{-1}d(\rho_s+\gamma\rho_n)-\mu\gamma^{-1}\rho_n d\gamma \tag{7}$$

determine two-velocity thermodynamics in Lorentz invariant form, with $\varepsilon$ the invariant energy density ($\varepsilon=u_\nu v_\lambda T^{\lambda\nu}$),

$$\gamma=u^\nu v_\nu, \qquad u^\nu u_\nu=1, \qquad v^\nu v_\nu=1,$$
$$\Delta_s^{\mu\nu}v_\nu=0, \quad \Delta_n^{\mu\nu}=g^{\mu\nu}-u^\mu u^\nu, \quad \Delta_n^{\mu\nu}u_\nu=0. \tag{8}$$

We will describe the sounds in manifestly covariant form. The subscript 0 denotes equilibrium values of quantities, assumed to be independent of coordinates and time, and the subscript 1 denotes small deviations from equilibrium values. The equations are linearized with the simplifying assumption that in an equilibrium state the superfluid and normal components move with the same velocity ($u_0^\nu=v_0^\nu$). Since

$$u^\nu u_\nu=1, u_{0\nu}u_0^\nu\approx 1, v^\nu v_\nu=1, v_{0\nu}v_0^\nu\approx 1,$$

both $u_1^\nu$ and $v_1^\nu$ are orthogonal to $u_0^\nu$ in the linear approximation:

$$u_0^\nu u_{1\nu}=u_0^\nu v_{1\nu}=0. \tag{9}$$

Using these orthogonality relations, we have to second order

$$\gamma=u^\nu v_\nu=(u_0^\nu+u_1^\nu)(v_{0\nu}+v_{1\nu})\approx 1+u_0^\nu v_{1\nu}=1. \tag{10}$$

We introduce the notation


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$$\omega_s=\mu\rho_s, \qquad \omega_n=TS_n+\mu\rho_n,$$

$$\varepsilon=\omega_s+\omega_n-P, \quad S_n=\sigma\rho=\sigma(\rho_n+\rho_s).$$

After linearizing Eqs. (1)–(8), eliminating the derivatives of $u_1^\nu$ and $v_1^\nu$, and choosing $\sigma$ and $\rho$ as the independent variables, we arrive at a system of two equations describing the propagation of first and second sound in the system:

$$\left(\frac{\partial\varepsilon}{\partial\sigma}\right)_\rho \partial_u^2\sigma_1+\left(\frac{\partial\varepsilon}{\partial\rho}\right)_\sigma \partial_u^2\rho_1+\left(\frac{\partial P}{\partial\sigma}\right)_\rho \Delta_0^{\lambda\nu}\partial_\lambda\partial_\nu\sigma_1$$
$$+\left(\frac{\partial P}{\partial\sigma}\right)_\sigma \Delta_0^{\lambda\nu}\partial_\lambda\partial_\nu\rho_1=\beta_{11}^{\lambda\nu}\left[\left(\frac{\partial T}{\partial\sigma}\right)_\rho \partial_\lambda\partial_\nu\partial_u\sigma_1\right.$$
$$\left.+\left(\frac{\partial T}{\partial\rho}\right)_\sigma \partial_\lambda\partial_\nu\partial_u\rho_1\right]+\beta_{12}^{\lambda\nu}\left[\left(\frac{\partial P}{\partial\sigma}\right)_\rho \partial_\lambda\partial_\nu\partial_u\sigma_1\right.$$
$$\left.+\left(\frac{\partial P}{\partial\rho}\right)_\sigma \partial_\lambda\partial_\nu\partial_u\rho\right]-\beta_3^{\nu\lambda\mu}\left[\left(\frac{\partial P}{\partial\sigma}\right)_\rho \partial_\lambda\partial_\mu\partial_\nu\sigma_1\right.$$
$$\left.+\left(\frac{\partial P}{\partial\rho}\right)_\sigma \partial_\lambda\partial_\mu\partial_\nu\rho_1\right]-\beta_4^{\nu\lambda\mu}\left[\left(\frac{\partial T}{\partial\sigma}\right)_\rho \partial_\lambda\partial_\mu\partial_\nu\sigma_1\right.$$
$$\left.+\left(\frac{\partial T}{\partial\rho}\right)_\sigma \partial_\lambda\partial_\mu\partial_\nu\rho_1\right], \tag{11}$$

$$\partial_u^2\sigma_1-\alpha_1\left(\frac{\partial P}{\partial\sigma}\right)_\rho \Delta_0^{\lambda\nu}\partial_\lambda\partial_\nu\sigma_1-\alpha_1\left(\frac{\partial P}{\partial\sigma}\right)_\sigma \Delta_0^{\lambda\nu}\partial_\lambda\partial_\nu\rho_1$$
$$+\alpha_2\left(\frac{\partial T}{\partial\sigma}\right)_\rho \Delta_0^{\lambda\nu}\partial_\lambda\partial_\nu\rho_1+\alpha_2\left(\frac{\partial T}{\partial\rho}\right)_\sigma \Delta_0^{\lambda\nu}\partial_\lambda\partial_\nu\rho_1$$
$$=\beta_1^{\lambda\nu}\left[\left(\frac{\partial T}{\partial\sigma}\right)_\rho \partial_\lambda\partial_\nu\partial_u\sigma_1+\left(\frac{\partial T}{\partial\rho}\right)_\sigma \partial_\lambda\partial_\nu\partial_u\rho_1\right]$$
$$+\beta_2^{\lambda\nu}\left[\left(\frac{\partial P}{\partial\sigma}\right)_\rho \partial_\lambda\partial_\nu\partial_u\sigma_1+\left(\frac{\partial P}{\partial\rho}\right)_\sigma \partial_\lambda\partial_\nu\partial_u\rho_1\right]$$
$$-\beta_1^{\lambda\nu\mu}\left[\left(\frac{\partial T}{\partial\sigma}\right)_\rho \partial_\lambda\partial_\mu\partial_\nu T_1+\left(\frac{\partial T}{\partial\rho}\right)_\sigma \partial_\lambda\partial_\mu\partial_\nu\sigma_1\right]$$
$$-\beta_2^{\lambda\mu\nu}\left[\left(\frac{\partial P}{\partial\sigma}\right)_\rho \partial_\lambda\partial_\mu\partial_\nu\sigma_1+\left(\frac{\partial P}{\partial\rho}\right)_\sigma \partial_\lambda\partial_\mu\partial_\nu\rho_1\right]. \tag{12}$$

Here

$$\Delta_0^{\lambda\nu}=g^{\lambda\nu}-u_0^\lambda u_0^\nu, \quad \partial_u=u_\nu\partial^\nu,$$

$$\alpha_1=T\sigma^2\rho_s(\mu_0\rho_0\omega_{n0})^{-1}, \quad \alpha_2=\sigma^2\rho_s(\mu_0\rho_0\omega_{n0})^{-1},$$

$$\beta_1^{\lambda\nu}=\rho_0^{-1}(\beta_{31}^{\lambda\nu}-\sigma_0\beta_{32}^{\lambda\nu}), \quad \beta_2^{\lambda\nu}=\rho_0^{-2}(\beta_{32}^{\lambda\nu}-\beta_{12}^{\lambda\nu}),$$

$$\beta_1^{\lambda\mu\nu}=\sigma_0\rho_{s0}\rho_0^{-1}\beta_{21}^{\lambda\nu\mu}, \quad \beta_2^{\lambda\nu\mu}=\omega_{n0}\rho_0^{-1}\beta_{22}^{\lambda\nu\mu},$$

$$\beta_3^{\lambda\nu\mu}=\sigma_0\rho_{s0}\rho_0^{-2}\beta_{22}^{\lambda\nu\mu}, \quad \beta_4^{\lambda\nu\mu}=\omega_{n0}(\beta_{21}^{\lambda\nu\mu}-\sigma_0\beta_{22}^{\lambda\nu\mu}),$$

$$\beta_{12}^{\nu\lambda}=T_0^{-2}u_{0\mu}(\xi_\sigma^{\lambda\mu\nu}u_0^\sigma-\xi^{\lambda\mu\nu}),$$

$$\beta_{32}^{\lambda\nu}=T_0^{-1}u_{0\mu}\xi^{\lambda\mu\nu}-T_0^{-2}\mu_0\xi^{\lambda\nu},$$

$$\beta_{31}^{\lambda\nu}=T_0^{-3}\mu_0(\xi^{\lambda\nu}\mu_0-\xi_\sigma^{\lambda\nu}u_0^\sigma)\lambda\nu+T_0^{-2}u_{0\mu}(\xi^{\lambda\mu\nu}\mu_0$$
$$-\xi_\sigma^{\lambda\mu\nu}u_0^\sigma),$$

$$\beta_{21}^{\nu\lambda\mu}=T_0^{-2}\Delta_0^{\sigma\nu}(\xi_{\gamma\sigma}^{\lambda\mu}u_0^\gamma-\mu_0\xi_\sigma^{\lambda\mu}), \quad \beta_{22}^{\nu\lambda\mu}=T_0^{-1}\xi_\sigma^{\lambda\mu}\Delta_0^{\sigma\nu}.$$

Assuming a plane-wave solution

$$\sigma_1=\sigma_{10}\exp(ik_\nu x^\nu), \quad \rho_1=\rho_{10}\exp(ik_\nu x^\nu)$$

($k^\nu$ is the four-dimensional wave vector), and introducing the notation

$$\kappa^2=\frac{k_u^2}{k_l^2}, \quad k_u^2=(k_\nu u_0^\nu)^2, \quad k_l^2=k_u^2-k_\nu^2,$$

$$\tau_{11}=\left[\left(\frac{\partial T}{\partial\sigma}\right)_\rho \beta_{11}^{\lambda\nu}k_\lambda k_\nu+\left(\frac{\partial P}{\partial\sigma}\right)_\rho \beta_{12}^{\lambda\nu}k_\lambda k_\nu\right.$$
$$\left.-\left(\frac{\partial P}{\partial\sigma}\right)_\rho \beta_3^{\lambda\nu\mu}k_\lambda k_\nu k_\mu-\left(\frac{\partial T}{\partial\sigma}\right)_\rho \beta_4^{\lambda\nu\mu}k_\lambda k_\nu k_\mu\right]k_l^{-2}, \tag{13}$$

$$\tau_{12}=\left[\left(\frac{\partial T}{\partial\rho}\right)_\sigma \beta_{11}^{\lambda\nu}k_\lambda k_\nu+\left(\frac{\partial P}{\partial\rho}\right)_\sigma \beta_{12}^{\lambda\nu}k_\lambda k_\nu\right.$$
$$\left.-\left(\frac{\partial P}{\partial\rho}\right)_\sigma \beta_3^{\lambda\nu\mu}k_\lambda k_\nu k_\mu-\left(\frac{\partial T}{\partial\rho}\right)_\sigma \beta_4^{\lambda\nu\mu}k_\lambda k_\nu k_\mu\right]k_l^{-2}, \tag{14}$$

$$\tau_{21}=\left[\left(\frac{\partial T}{\partial\sigma}\right)_\rho \beta_1^{\lambda\nu}k_\lambda k_\nu+\left(\frac{\partial P}{\partial\sigma}\right)_\rho \beta_2^{\lambda\nu}k_\lambda k_\nu\right.$$
$$\left.-\left(\frac{\partial T}{\partial\sigma}\right)_\rho \beta_1^{\lambda\nu\mu}k_\lambda k_\nu k_\mu-\left(\frac{\partial P}{\partial\sigma}\right)_\rho \beta_2^{\lambda\nu\mu}k_\lambda k_\nu k_\mu\right]k_l^{-2}, \tag{15}$$

$$\tau_{22}=\left[\left(\frac{\partial T}{\partial\rho}\right)_\sigma \beta_1^{\lambda\nu}k_\lambda k_\nu+\left(\frac{\partial P}{\partial\rho}\right)_\sigma \beta_2^{\lambda\nu}k_\lambda k_\nu\right.$$
$$\left.-\left(\frac{\partial T}{\partial\rho}\right)_\sigma \beta_1^{\lambda\nu\mu}k_\lambda k_\nu k_\mu-\left(\frac{\partial P}{\partial\rho}\right)_\sigma \beta_2^{\lambda\nu\mu}k_\lambda k_\nu k_\mu\right]k_l^{-2}, \tag{16}$$

we obtain

$$\left[\left(\frac{\partial\varepsilon}{\partial\sigma}\right)_\rho \kappa^2-\left(\frac{\partial P}{\partial\sigma}\right)_\rho -i\tau_{11}\right]\sigma_{10}$$
$$+\left[\left(\frac{\partial\varepsilon}{\partial\rho}\right)_\sigma \kappa^2-\left(\frac{\partial P}{\partial\rho}\right)_\sigma -i\tau_{12}\right]\rho_{10}=0, \tag{17}$$

$$\left[\left(\frac{\partial\rho}{\partial P}\right)_\sigma \kappa^2+\alpha_1\left(\frac{\partial P}{\partial\sigma}\right)_\rho -\alpha_2\left(\frac{\partial T}{\partial\sigma}\right)_\rho -i\tau_{21}\right]\sigma_{10}$$
$$+\left[\alpha_1\left(\frac{\partial P}{\partial\rho}\right)_\sigma -\alpha_2\left(\frac{\partial T}{\partial\rho}\right)_\sigma -i\tau_{22}\right]=0. \tag{18}$$

Setting the determinant of the system of equations (17) and (18) to zero, we obtain a dispersion relation with which we can determine the velocities of the sounds, which, allowing for the thermodynamic inequality

$$\alpha_1\left(\frac{\partial P}{\partial \sigma}\right)_\varepsilon \ll \alpha_2\left(\frac{\partial T}{\partial \sigma}\right)_\varepsilon$$

derived in Ref. 2 with the assumptions

$$\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \ll 1, \frac{T}{\mu} \ll 1,$$

we write as follows:

$$\kappa^4-\kappa^2\left[\alpha_2\left(\frac{\partial T}{\partial \sigma}\right)_\varepsilon+i\tau_{21}+\left(\frac{\partial P}{\partial \varepsilon}\right)_\sigma+i\tau_{12}\right]+\left[\alpha_2\left(\frac{\partial T}{\partial \sigma}\right)_\varepsilon+i\tau_{21}\right]\left[\left(\frac{\partial P}{\partial \varepsilon}\right)_\sigma+i\tau_{12}\right]=0. \tag{19}$$

The roots of this equation yield the velocities of the sounds with allowance for damping:

$$\kappa_1^2=\left(\frac{\partial P}{\partial \varepsilon}\right)_\sigma+i\tau_{12} \tag{20}$$

is the square of the velocity of first sound, and

$$\kappa_2^2=\frac{\sigma^2\rho_s(\varepsilon+P)}{\mu\rho\omega_n(\partial\sigma/\partial T)_\varepsilon}+i\tau_{21} \tag{21}$$

is the square of the velocity of second sound. As Eqs. (20) and (21) show, the velocity of each sound is a complex quantity, with the imaginary part determining absorption. In the nonrelativistic limit ($\mu\approx\mu_{\mathrm{nonrel}}+mc^2$, $c\to\infty$), the real parts of the above expressions reduce to the corresponding formulas for the velocity of first and second sound in superfluid helium.[4] Studying the relations between the thermodynamic variables in the sound waves makes it possible to determine the nature of the vibrations in these waves: first-sound waves constitute small oscillations in density and pressure, and in second-sound waves both temperature and entropy oscillate, but pressure does not. In a first-sound wave, the superfluid liquid oscillates as a whole and the normal and superfluid components move together, while in a second-sound wave the superfluid and normal components move in opposition. The oscillations in the sound modes found here behave in the same way as the oscillations in the respective sounds in superfluid helium.[4]

As shown in Ref. 5, the expression for the velocity of fourth sound without dissipation is

$$\kappa^2=\frac{\rho_s\kappa_1^2+\rho_n\kappa_2^2}{\rho}, \tag{22}$$

where

$$\kappa_1^2=\left(\frac{\partial P}{\partial \varepsilon}\right)_\sigma$$

is the square of the velocity of first sound, and

$$\kappa_2^2=\frac{\sigma^2\rho_s(\varepsilon+P)}{\mu\rho\omega_n(\partial\sigma/\partial T)_\varepsilon}$$

is the square of the second-sound velocity.

In the nonrelativistic limit Eq. (22) reduces to the expression for the velocity of fourth sound in superfluid helium.[4,6]

Allowance for dissipation yields an expression for the velocity of fourth sound that has the same form as (22) but with $\kappa_1^2$ and $\kappa_2^2$ given by Eqs. (20) and (21). The dependence between the thermodynamic variables suggests that in relativistic superfluid systems, fourth-sound waves are oscillations in density, pressure, temperature, and entropy in a situation in which the normal component is "squeezed."

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