Elastic surface waves at the loaded boundary of a crystal

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The existence conditions and the number of elastic surface waves on the nonfree boundary of an anisotropic medium of arbitrary symmetry are investigated. A class of elastic loadings **F** which are linear in the surface displacements **u** is considered in both the scalar ($\mathbf{F}=-\gamma\mathbf{u}$) and tensor ($\mathbf{F}=-\hat{W}\mathbf{u}$) cases, along with that in which the loading system is characterized by an eigenfrequency, which permits a resonance to occur. The analysis is based on the use of the surface impedance matrix $\hat{Z}$ of the crystal, relating the surface displacement wave to the external force by $\hat{Z}\mathbf{u}=\mathbf{F}$. By means of the general analytical properties of the eigenvalues λ_i of the matrix $\hat{Z}$ and the parameters of the force **F** at the surface, it is possible, using simple algebraic criteria admitting of a clear geometrical interpretation, to obtain exact predictions concerning the number of surface waves without going through a cumbersome search for the explicit solutions of the boundary-value problems involved. It is shown that unlike the case of a free surface, at which the existence of a single surface wave is practically always guaranteed, at a loaded boundary, depending on the elastic loading parameters, surface wave solutions may not exist at all—or there may be several of them. Examples of physical situations allowing three (or even four) surface waves to coexist are given.

1. INTRODUCTION

Surface waves in crystals differ in their propagation velocities and penetration depths, and also in the number of partial modes and their polarization. A discovery of any new type of surface wave is always an event in crystal acoustics. One of the central problems in the theory of surface waves is the number of surface waves capable of propagating in a semi-infinite anisotropic medium at specific orientations of the surface and the propagation direction. The answer depends on the physical properties of the medium and the type of boundary conditions at the surface. Thus, in dielectric crystals in any direction (except for some special orientations) no more than one surface wave can propagate.[1,2] At the same time, it is well known that in piezoelectric materials, the Bleustein–Gulyaev wave may exist[3–6] in addition to the ordinary (Rayleigh) surface wave (an analogous wave also exists in piezomagnetic materials[7,8]). These statements are valid only for the case of a mechanically free surface; on a clamped surface (when the elastic displacement on the surface is zero), either in dielectric or in piezoelectric (piezomagnetic) materials surface wave solutions do not exist in principle.[2,5,6] On the other hand, in crystals possessing piezoelectric and piezomagnetic properties simultaneously, a surface wave may propagate in a clamped surface as well[9,10] (at a free surface there may exist two waves in this case).

In the present work we consider purely elastic (dielectric) semi-infinite media and discuss the question of the possible number of surface waves, without restricting ourselves to these two limiting types of boundary conditions (clamped surface—no solutions, free surface—no more than one solution). There exist a whole series of intermediate boundary-value problems amenable to exact analysis. It is known, for example, that in the presence of variation near the surface, in addition to a Rayleigh wave a surface wave polarized transversely in the boundary plane may exist, due to the localization of the corresponding bulk shear-horizontal (SH) wave (see, e.g., monographs by Viktorov[11] and by Biryukov *et al.*[12]). References 11–13 indicate that if a two-dimensional defect exists, then at the surface of an isotropic elastic half-space as many as three surface waves may occur: a quasibulk SH wave and two Rayleigh waves polarized in the sagittal plane. The results of Refs. 11–13 were obtained by a direct analytical solution of the wave equation with the appropriate boundary conditions, the solution being possible only for particular highly symmetrical orientations.

In the present work, we would like to address the problem by assuming an arbitrary anisotropy of the medium. In this case a "brute-force" solution via the analysis of the wave equation is unfortunately impossible since, generally, even the wave velocity dispersion relation cannot be written down explicitly. In what follows we shall demonstrate an alternative approach circumventing this difficulty. The basis of this approach is the use of the surface impedance method, a technique allowing to take as a key element of the analysis the known analytical properties of the surface impedance matrix,[14,2] even though the explicit form of this latter is just as unknown as is the dispersion equation mentioned above.

Taking the elastic loading of the boundary of an anisotropic semi-infinite medium as intermediate (between free and clamped) boundary conditions, we will show how naturally the requirements for the exclusion or existence of one, two, three, or even four surface waves can be obtained within the surface impedance concept. In particular, the coexistence

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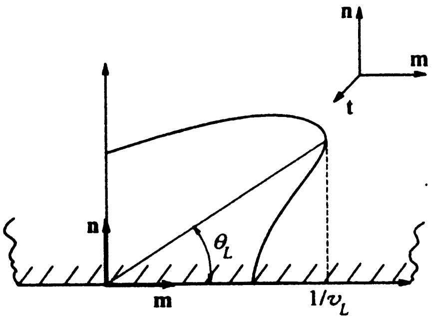

FIG. 1. Section of the outer cavity of the surface of slownesses (reciprocal phase velocities of the bulk elastic waves) by the sagittal plane $\{\mathbf{m},\mathbf{n}\}$ containing the complex wave vectors of the surface wave partial modes.

of several surface waves at a boundary "loaded" by a layer of implanted heavy atoms is predicted.

2. SURFACE IMPEDANCE MATRIX: DEFINITION AND BASIC PROPERTIES

In a semi-infinite anisotropic medium, the elastic wave field is a superposition of partial acoustic modes:

$$\mathbf{u}(\mathbf{r},t)\sum_{\alpha} b_\alpha \mathbf{A}_\alpha \exp[ik(\mathbf{mr}+p_\alpha \mathbf{nr}-vt)], \tag{1}$$

where $\mathbf{m}$ is the wave normal, $\mathbf{n}$ is the inward normal to the surface (Fig. 1), and v the phase velocity. The polarization vectors $\mathbf{A}_\alpha$, and the parameters p_α characterizing the localization of the partial waves, are given by the wave equation

$$(\hat{\Lambda}^{(\alpha)}-v^2)\mathbf{A}_\alpha=0 \tag{2}$$

and by the condition that there be nontrivial solutions to it,

$$\det(\hat{\Lambda}^{(\alpha)}-v^2)=0. \tag{3}$$

Here

$$\Lambda_{il}^{(\alpha)}=(m_j+p_\alpha n_j)c_{ijkl}(m_k+p_\alpha n_k)/\rho,$$

$\hat{c}$ is the tensor of elastic moduli, ρ the density of the medium, and summation is implied over repeated Latin indices. The condition (3) is an equation of sixth degree in the parameters p_α. For phase velocities v slower than the so-called limiting velocity v_L (Fig. 1), the roots of Eq. (3), $p_\alpha=p_\alpha(v)$, $\alpha=1$, 2, ..., 6, occur in complex conjugate pairs. Clearly, the roots with negative imaginary parts must be thrown away as unphysical, so that for $v<v_L$ the wave field $\mathbf{u}$ consists of three partial modes which decay away from the surface according to the condition $\operatorname{Im} p_\alpha>0$, $\alpha=1$, 2, 3.

Each partial wave mode at the boundary Γ of the medium gives rise to an elastic force

$$\mathbf{f}_\alpha=\hat{\sigma}_\alpha\mathbf{n}=\mathbf{J}_\alpha\exp[ik(\mathbf{mr}-vt)], \quad \mathbf{r}\in\Gamma, \tag{4}$$

where $\hat{\sigma}_\alpha$ is the tensor of the mechanical stresses associated with the given mode, and

$$\mathbf{J}_\alpha=ik\mathbf{n}\hat{c}(\mathbf{m}+p_\alpha\mathbf{n})\mathbf{A}_\alpha. \tag{5}$$

The boundary conditions at the surface of the medium can be written in the form

$$\mathbf{f}=\sum_\alpha b_\alpha\mathbf{f}_\alpha=-\mathbf{F}, \tag{6}$$

where $\mathbf{F}=\mathbf{F}_0\exp[ik(\mathbf{mr}-vt)]$ is the external force per unit area of the surface (in what follows we consider "coherent" external loadings, which are proportional to the elastic surface displacement); the sign on the right-hand side of Eq. (6) is due to the choice of the inward normal $\mathbf{n}$ in (4). From Eq. (6), the amplitudes b_α are determined.

References 14 and 2 have introduced, for $v\leqslant v_L$, the concept of a surface impedance, a matrix $\hat{Z}$ which relates an arbitrary elastic displacement in the wave field at the boundary,

$$\mathbf{u}=\sum_{\alpha=1}^{3} b_\alpha\mathbf{u}_\alpha$$

to the corresponding surface elastic force:

$$\mathbf{f}=\sum_{\alpha=1}^{3} b_\alpha\mathbf{f}_\alpha=-\hat{Z}\mathbf{u}. \tag{7}$$

Following Refs. 14 and 2, it is not difficult to get a convenient representation for the $\hat{Z}$ matrix. In fact, noting that for $v\leqslant v_L$ the vectors $\mathbf{A}_\alpha$, $\alpha=1$, 2, 3, are always linearly independent (there are no wave solutions satisfying the clamped-surface condition $\mathbf{A}=0$) and expressing the coefficients b_α in (7) in the form $b_\alpha=u_{\alpha i}^{-1}u_i$, we have for the surface impedance matrix $\hat{Z}$ the following representation:

$$Z_{ij}=-\sum_{\alpha=1}^{3} f_{i\alpha}u_{\alpha j}^{-1}=-\sum_{\alpha=1}^{3} J_{i\alpha}A_{\alpha j}^{-1}, \quad i,j=1,2,3. \tag{8}$$

With the impedance matrix thus introduced, the boundary conditions (6) can be written in the form

$$\hat{Z}=\mathbf{u}=\mathbf{F}. \tag{9}$$

It should be noted that the surface impedance matrix as a function of the velocity, $\hat{Z}=\hat{Z}(v)$, is determined only by the geometry of the problem and the elastic properties of the medium and does not depend on the particular form of the boundary conditions and the corresponding form of the wave solution [as a function of the amplitudes b_α in Eq. (1)].

The surface impedance matrix for the region $v\leqslant v_L$ has the following general properties (Refs. 2, 14–16):

1) the matrix $\hat{Z}$ is Hermitian and hence its eigenvalues λ_i, $i=1$, 2, 3, are real;

2) the matrix $\hat{Z}$ for $v=0$ is positive definite, i.e., $\lambda_i(0)>0$;

3) the matrix $\partial\hat{Z}/\partial v$ is negative definite, i.e., $\partial\lambda_i/\partial v<0$;

4) the real (symmetric) part of the matrix $\hat{Z}$ is positive definite for $v<v_L$;

5) $\mathbf{A}_L\hat{Z}(v_L)\mathbf{A}_L=0$, where $\mathbf{A}_L$ is the polarization vector of a one-partial bulk wave characterized by the velocity v_L and the parameter $p=\tan\theta_L$ (Fig. 1).

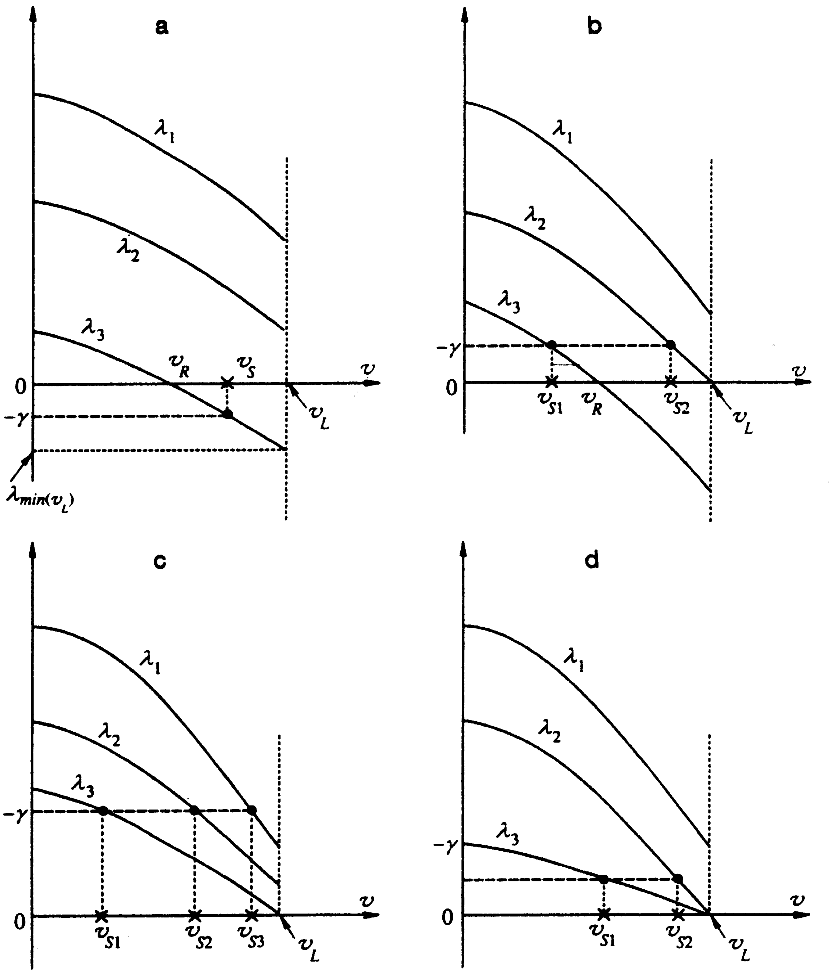

FIG. 2. Possible ways the eigenvalues of the surface impedance matrix can depend on the velocity: a) $\lambda_2(v_L)>0$, $\lambda_3(v_L)<0$; b) $\lambda_2(v_L)=0$, $\lambda_3(v_L)<0$; c) $\lambda_2(v_L)>0$, $\lambda_3(v_L)=0$; d) $\lambda_{2,3}(v_L)=0$; in all cases $\lambda_1(v_L)>0$. The dashed lines correspond to scalar loadings independent of the velocity. The coordinates of the closed circles correspond to the surface wave parameters.

The properties listed above[1)] enable one to prove[16] that the examples presented in Fig. 2 exhaust the possible ways in which the eigenvalues λ_i of the impedance matrix $\hat{Z}$ may depend on the velocity v.

In the Appendix is given, as an illustration, the explicit form of both the impedance matrix $\hat{Z}(v)$ and its eigenvalues calculated for a monoclinic medium with axis 2 orthogonal to the sagittal plane $\{\mathbf{m},\mathbf{n}\}$; and for a transversely isotropic semi-infinite medium for two different orientations of the symmetry axis 6, perpendicular to the sagittal plane of the medium and perpendicular to the surface. The corresponding curves $\lambda_i(v)$ for the case $6\perp\{\mathbf{m},\mathbf{n}\}$ are given in Fig. 3. We will also show how these results can be applied to rhombic crystals (propagation along the axes of the standard coordinate system of such crystals, in surfaces parallel to the coordinate planes).

3. MODEL EXAMPLE OF THE ELASTIC LOADING OF THE SURFACE

As an illustration of the idea we are discussing, consider the loading of the surface by the external force

$$\mathbf{F}=-\gamma\mathbf{u}, \tag{10}$$

where $\mathbf{u}$ is the displacement of the surface and γ is a scalar factor. Clearly, the boundary condition (9) for such a "scalar" loading reduces to the equation

$$\det(\hat{Z}+\gamma)=[\lambda_1(v)+\gamma]\,[\lambda_2(v)+\gamma]\,[\lambda_3(v)+\gamma]=0, \tag{11}$$

i.e., the required solutions of the boundary-value problem (surface-wave phase velocity values $v<v_L$) are determined simply as the points of intersection of the curves of the impedance matrix eigenvalues, $\lambda_i(v)$, with the horizontal intersecting the vertical axis at the corresponding value $-\gamma$ (see Fig. 2). For a mechanically free surface ($\gamma=0$), when the Rayleigh wave phase velocity $v_R<v_L$ is determined as the point of intersection of the curve $\lambda_i(v)$ with the abscissa, Fig. 2 illustrates the theorems of existence and uniqueness for the Rayleigh surface waves: a) the solution $v_R<v_L$ always exists if $\lambda_i(v_L)\neq 0$, $i=1, 2, 3$; b) the solution $v_R<v_L$ is always unique[1,2] (there cannot be two solutions present in the region $v<v_L$, not even for the case $\lambda_i(v_L)=0$).

Consider the case $\gamma>0$. Obviously, the simplest physical model for this case is a system of springs between a surface and a perfectly rigid platform, the stiffnesses of the springs being chosen so that a spring responds in the same way to the normal and tangential surface displacements, and the separation between the springs being assumed small compared to the wavelength. If a Rayleigh wave exists at the free surface, then the elastic loading (10) with a sufficiently small stiffness γ $[0<\gamma<-\lambda_{\min}(v_L)]$ leads to a change in the wave velocity ($v_R\rightarrow v_S$) at the loaded surface (Fig. 2a).

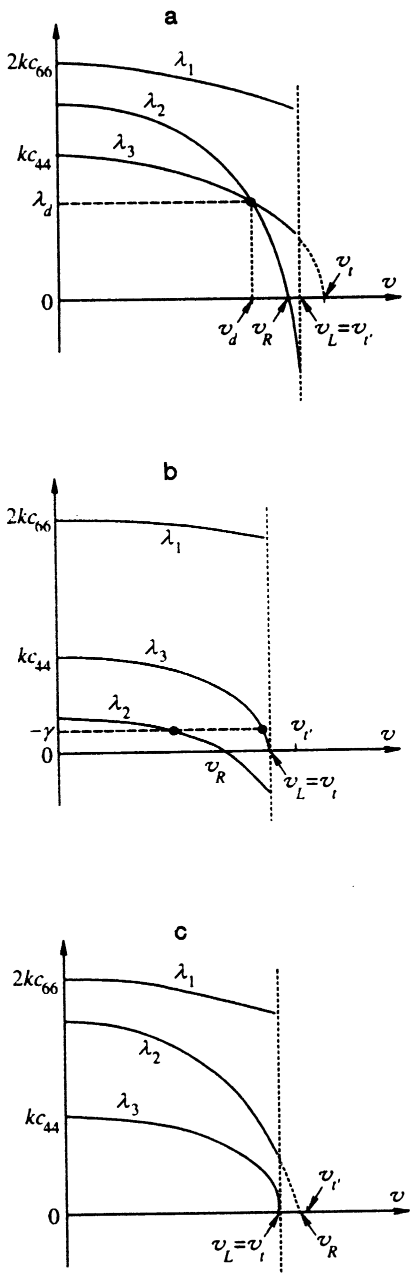

FIG. 3. Branches $\lambda_i(v)$ in hexagonal crystals: a) $\eta \equiv c_{66}/c_{11}=0.1$, $v_t=1.1v_{t'}$ [degeneracy, see Eq. (A.25)]; b) $\eta=0.75$, $v_t=0.9v_{t'}$; c) $\eta=0.1$, $v_t=0.9v_{t'}$. In the cases a, b, and c the relations for $\lambda_i(v_L)$ are the same as in Figs. 2a,b,c, respectively.

For $\gamma=-\lambda_{\min}(v_L)$, one of the partial waves of this solution becomes a bulk wave.[2)] But for $\gamma>-\lambda_{\min}(v_L)$, the existence theorem does not hold for such a loaded surface because in this case there are *a priori* no surface waves in the region $v<v_L$, even if $\lambda_i(v_L)\neq 0$, $i=1, 2, 3$. Thus we see that the surface wave solutions disappeare long before the the surface is totally clamped (for $\gamma=+\infty$) and takes places even for $\gamma=-\lambda_{\min}(v_L)$.

If we have $\gamma<0$, then for not too large values of $|\gamma|$ the existence of at least one surface wave solution is clearly guaranteed. In this case the uniqueness theorem may be invalidated for an infinitesimal loading of the surface of the elastic medium. In fact, from Figs. 2b,d and 3b it is seen how for $\gamma<0$ two surface wave solutions, v_{S1}, $v_{S2}<v_L$, are obtained; in Fig. 2d both the waves are quasibulk waves for $|\gamma|$ small (weakly decaying inward because of the small value of γ), while in Figs. 2b and 3b one of the wave solutions is quasibulk and the other is a Rayleigh wave. For a sufficiently large value of the parameter $|\gamma|$ $(\gamma<0)$and provided $\lambda_{\min}(0)>\lambda_{\max}(v_L)$, three surface waves may exist in the region $v<v_L$ (see Fig. 2c).

Note that unlike the mechanically free surface of an elastic half-space, which admits of no one-partial wave solutions for $v<v_L$ (Refs. 17 and 20), at the loaded boundary a surface wave may also be one-partial, i.e., we may have $\mathbf{u}\|\mathbf{A}_\alpha$ for $v<v_L$ [see Eq. (1)]. The localization of the horizontally polarized wave ($\mathbf{A}_3\|[\mathbf{mn}]\equiv\mathbf{t}$) may be mentioned as an example (see the Appendix and Figs. 3b,c). Note also that the presence in Fig. 3a of the velocity v_d, for which $\lambda_2(v_d)=\lambda_3(v_d)\equiv\lambda_d$ holds, implies that for the scalar loading (10) with the stiffness coefficient $\gamma=-\lambda_d<0$, there may exist, in a transversely isotropic medium, a degenerate surface wave having a velocity v_d and an arbitrary orientation of the complex elastic displacement vector $\mathbf{u}$ [provided only that $\mathbf{u}\mathbf{e}_1^*(v_d)=0$, where the vector $\mathbf{e}_1^*(v_d)$ is complex conjugate to that eigenvector of the matrix $\hat{Z}(v_d)$ corresponding to the nondegenerate eigenvalue $\lambda_1(v_d)\neq\lambda_d$].

The relevant question here is the physical realizability of the loading (10) with the coefficient γ negative, a loading from which additional wave solutions arise. Clearly, in the above spring model the value of γ (spring stiffness) is essentially positive. Nevertheless, it is readily seen that even a slight modification of the model immediately leads to a scalar loading of the form (1) with a coefficient $\gamma'<0$. In the discussion above we assumed that at the end of every one of the springs there is a heavy platform fixing the reference point for the displacement of the spring from its equilibrium position, so that the displacements of the surface of the elastic medium under study exactly determine the tension or compression of the spring. Now suppose that at the ends of the springs there are minute loads distributed with a surface density (mass per unit area) ρ_s and possessing their own degrees of freedom (displacements $\mathbf{u}'$). This loading of the crystal surface is resonant: all of the springs have the same eigenfrequencies that are determined by their stiffness and mass. Then, instead of the equation $\hat{Z}\mathbf{u}=-\gamma\mathbf{u}$ one should solve the system which determines the correlated displacements $\mathbf{u}$ and $\mathbf{u}'$ at both ends of the springs,

$$\hat{Z}\mathbf{u}=-\gamma(\mathbf{u}-\mathbf{u}'),\quad \rho_s\mathbf{u}'=-\gamma(\mathbf{u}'-\mathbf{u}). \tag{12}$$

Eliminating $\mathbf{u}'$ from (12) we arrive at the equation

$$\hat{Z}\mathbf{u}=-\gamma'\mathbf{u},\quad \gamma'=\frac{k^2\rho_s v^2}{-1+k^2\rho_s v^2/\gamma}=\gamma\frac{\omega^2}{\omega^2-\omega_r^2}, \tag{13}$$

where $\omega=vk$ is the wave frequency and $\omega_r=\sqrt{\gamma/\rho_s}$ the resonant frequency. The quantity γ' is a function of the velocity and may be either positive or negative. It is essential that the change of sign of the coefficient γ' at $v=v_r\equiv\omega_r/k$ be resonant and occur via an infinite discontinuity (see Fig. 4). As seen from Figs. 4a and 4b, for $v_r<v_L$ (i.e., for $k>\omega_r/v_L$) the existence of three surface wave solutions is guaranteed. What is more, for

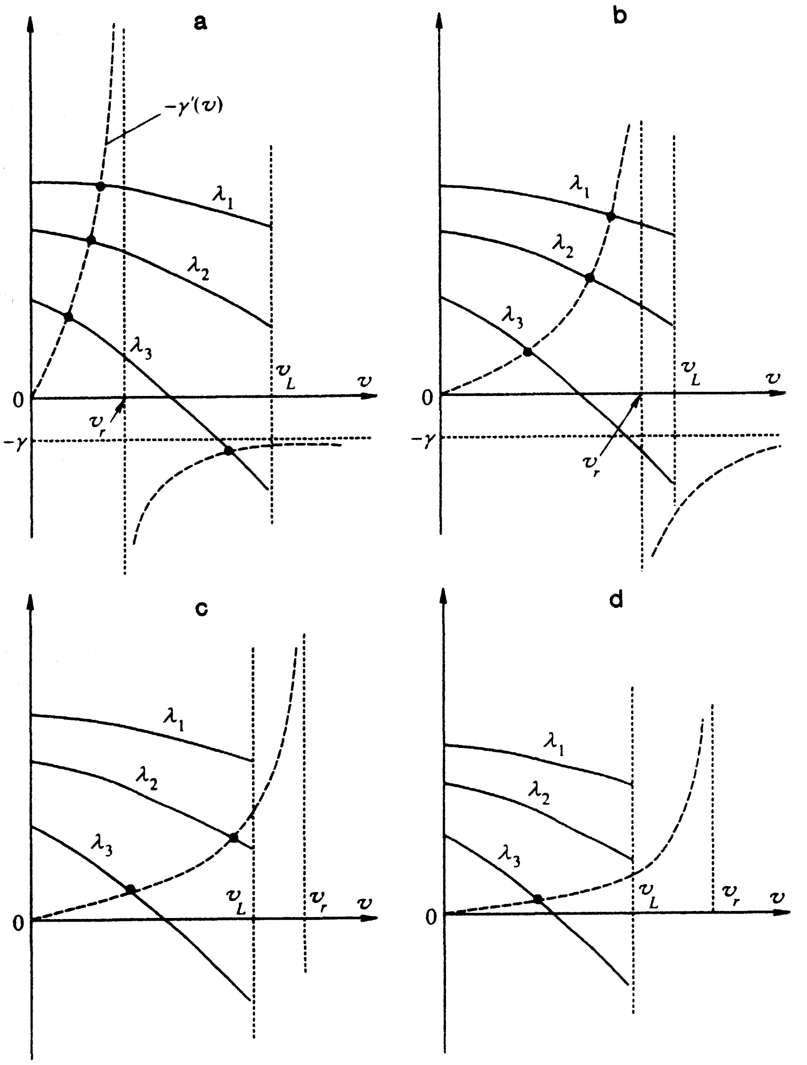

FIG. 4. Surface wave solutions in the case of a resonant loading on the surface: a, b, c and d correspond to the existence of four, three, two, and one solution.

$-\gamma'(v_L)>\lambda_{\min}(v_L)\equiv -c_L k$ ($c_L>0$ is of the order of the elasticity modulus, see the Appendix), that is, for

$$k>k_c,\quad k_c=\frac{\gamma}{2c_L}+\sqrt{\frac{\gamma^2}{4c_L^2}+\frac{\gamma}{\rho_s v_L^2}}\tag{14}$$

a fourth solution must emerge (Fig. 4a).

It is easily understood (see also Figs. 4c,d) that for $v_r>v_L$ one surface wave at least is guaranteed, but we may actually have two or three of them. The last case occurs, obviously, for $-\gamma'(v_L)>\lambda_{\max}(v_L)\equiv C_L k$ ($C_L>0$, $C_L\sim c_L$), when

$$k>K_c,\quad K_c=-\frac{\gamma}{2C_L}+\sqrt{\frac{\gamma^2}{4C_L^2}+\frac{\gamma}{\rho_s v_L^2}}.\tag{15}$$

Clearly, this is a less stringent inequality than the condition $k>\omega_r/v_L$, which is equivalent to the requirement $v_r<v_L$. In other words, as the wavelength is decreased, one, two, and, for $k>K_c$ [see Eq. (15)], three waves are successively excited at thresholds; they also persist beyond the point $k_0=\omega_r/v_L$—but only as long as $k<k_c$, Eq. (14). For $k>k_c$, four simultaneous surface waves must exist in the system, generally of different polarization.

4. ELASTIC MEDIUM VARYING NEAR THE SURFACE

In this section we will show how the surface impedance concept works in determining the number of surface waves under conditions where a semi-infinite elastic medium is characterized by nonuniformity near the surface. The nature of this nonuniformity may vary widely. Even in a perfect crystal, the elastic properties and density of the medium in a layer several interatomic separations thick near the surface differ appreciably from those in the bulk. Still greater boundary effects may be associated with "surface membranes," for example with adsorbed layers of heavy atoms or technologically treated (say, polished) surfaces. In the latter case a modified layer with irregular behavior forms, whose thickness may be of order 100 Å. The question of the boundary conditions for such a boundary has been discussed in detail in the literature (see, e.g., Refs. 12, 13, 21–26). In the "strong coupling" case, when the elastic displacement field

$\mathbf{u}$ given by Eq. (1) is considered continuous near the surface $(x_2 \to 0)$,[13] these boundary conditions may be written in the form (6) with the external force

$$\mathbf{F} = -\hat{W}\mathbf{u}, \quad W_{ij} = k^2[(g_{ab}m_a m_b - \rho_s v^2)\delta_{ij} + h_{abcd}m_b m_d \delta_{ia}\delta_{jc}], \quad a,b,c,d = 1,3; \quad i,j = 1,2,3, \tag{16}$$

where $\hat{g}$ is the residual stress tensor, ρ_s is the surface density, $\hat{h}$ is the surface elastic modulus tensor, and δ_{ij} is the Kronecker symbol.

As mentioned in the Introduction, the influence on elastic wave propagation of nonuniformity near the surface has usually been treated[11–13,23,27] for highly symmetric media (mostly isotropic or transversely isotropic) by explicitly solving an equation of the form (6), (13). The topics discussed were SH-wave localization, the change in Rayleigh wave velocity, and the occurrence of an additional Rayleigh mode. In particular, Ref. 27 shows that at short wavelengths in a model hexagonal crystal with a plane defect at the surface, two surface waves polarized in the sagittal plane may exist. Clearly, for the general case of an arbitrarily anisotropic medium, to determine the number of solutions to the boundary-value problem (16), i.e., the number of surface waves at the loaded surface, appears to be very complicated to attack head-on, by solving the boundary-value problem (6), (16) directly. It turns out, though, that the answer can be obtained in one fell swoop and with no calculation, by writing the boundary conditions with the aid of the surface impedance matrix and then using the properties of the matrix.

Using the definition (7) of the surface impedance, from the boundary condition (6), (16) we obtain the surface-wave velocity dispersion relation of the form

$$\det(\hat{Z} + \hat{W}) = 0, \tag{17}$$

where in terms of the coordinate system with $x_1 \| \mathbf{m}$, $\mathbf{x}_2 \| \mathbf{n}$, the matrix $\hat{W}$ has the form

$$\hat{W} = k^2 \begin{pmatrix} h_{11} + g_{11} - \rho_s v^2 & 0 & h_{15} \\ 0 & g_{11} - \rho_s v^2 & 0 \\ h_{15} & 0 & h_{55} + g_{11} - \rho_s v^2 \end{pmatrix} \tag{18}$$

(from now on the components of the tensor $\hat{h}$ are given in the two-index notation). Cases of different interrelations between the surface parameters will next be considered.

Clearly, if in (18) one formally sets $|g_{11}| \gg |h_{\alpha\beta}|$, $\rho_s v^2$, then Eq. (17) reduces to Eq. (11) with a wavelength-dependent scalar loading, $\gamma = k^2 g_{11}$. For $g_{11} > 0$, it was shown in the preceding section that in the region $v < v_L$ there is no more than one surface wave, whereas for $g_{11} < 0$ the number of surface waves, which is determined by the number of points where the straight line with intercept $-\gamma$ intersects the $\lambda_i(v)$ curves, may reach three (see Fig. 2c). zif we take into account the frequency dispersion [$\gamma \propto k^2$, $\lambda_i \propto k$; see Eqs. (5) and (8)], surface waves may appear or disappear in a passage through a certain threshold value of wave vector k.

However, the limiting case we have considered, which assumes the residual stresses g_{11} in the layer near the surface to exceed the elastic moduli $h_{\alpha\beta}$, is hardly realizable. At the same time, another limiting case, $\rho_s v^2 \gg |g_{11}|$, $|h_{\alpha\beta}|$ appears to be fairly realistic. It corresponds, e.g., to a layer of heavy atoms adsorbed at the surface of a light substrate (assuming the elastic moduli and residual stresses in the layer are not abnormally high). In this limit, the tensor boundary condition (13), (18) clearly reduces again to the scalar loading (10) with a constant $\gamma = \bar{\gamma}$, which depends in addition on the velocity: $\bar{\gamma}(v) = -k^2 \rho_s v^2$. Note that $\bar{\gamma}(v)$ is identical to the expression for $\gamma'(v)$, Eq. (13), if this latter is treated in the strong coupling limit ($\gamma \gg k^2 \rho_s v^2$). Figure 5 displays some examples of the graphical solution of the equations $\lambda_i(v) = -\bar{\gamma}(v)$ ($i = 1, 2, 3$) for various values of the parameter $k\rho_s$. It is seen that the number of surface wave solutions, depending on the wavelength, the density ρ_s, and the behavior of the functions $\lambda_i(v)$ at $v \leq v_L$ (see also Fig. 2), varies from one to three. In situations corresponding to Figs. 5a,c the minimum number of solutions is one, whereas in cases corresponding to Figs. 5b,d there are always at least two surface wave solutions. In all cases, for

$$k^2 \rho_s v_L^2 > \lambda_{\max}(v_L) \tag{19}$$

three surface waves can propagate.[3)]

Now consider the general case, where all the surface parameters, $\rho_s v^2$, $\hat{g}$, and $\hat{h}$, are comparable. We assume the sagittal plane to be coincident with the plane of symmetry of the monoclinic medium. We further assume that the surface defect does not break the assumed symmetry. Then in the chosen coordinate system we have $h_{15} = 0$, and for a horizontally polarized (SH) wave the dispersion equation (17) reduces to the condition

$$\lambda_3 + W_{33} = 0, \tag{20}$$

where

$$W_{33} = k^2(h_{55} + g_{11} - \rho_s v^2), \quad \lambda_3 = k c_{44} \kappa, \tag{21}$$

$$\kappa = \kappa(v) = \sqrt{\frac{\rho(v_t^2 - v^2)}{c_{44}}}, \quad v_t^2 = \frac{1}{\rho}\left(\frac{c_{55} - c_{45}^2}{c_{44}}\right) \tag{22}$$

[see Eq. (18) and Eq. (A.1) of the Appendix, respectively]. The solution of Eq. (16) is clearly the point of intersection of the curves $\lambda_3(v)$ and $-W_{33}(v)$. Such a solution, corresponding to the localization of the SH wave at a "defect" boundary, arises if

$$-\sqrt{\frac{\rho v_t^2 c_{44}}{k}} < h_{55} + g_{11} < \rho_s v_t^2. \tag{23}$$

As regards the waves polarized in the sagittal plane (coinciding with the symmetry plane), to study them within our model taking into account the component h_{11} is a more complicated problem, which does not reduce in principle to the analysis of the intersection of the curves $\lambda_1(v)$, and $\lambda_2(v)$ with the $W_{11}(v)$ and $W_{22}(v)$ curves. In fact, in the coordinate system $x_1 \| \mathbf{m}$, $x_2 \| \mathbf{n}$, in which the matrix $\hat{W}$ is diagonal, the matrix $\hat{Z}$ has a decoupled component $Z_{33} = \lambda_3$ (the eigenvalue for the real eigenvector of SH orientation, i.e., parallel to $\mathbf{t} = [\mathbf{mn}]$), and a generally nondiagonal upper 2×2 block. At the same time, one would expect that the equation $\det(Z_{ij} + W_{ij}) = 0$, $i, j = 1, 2$ [see Eq. (17)], unlike the stan-

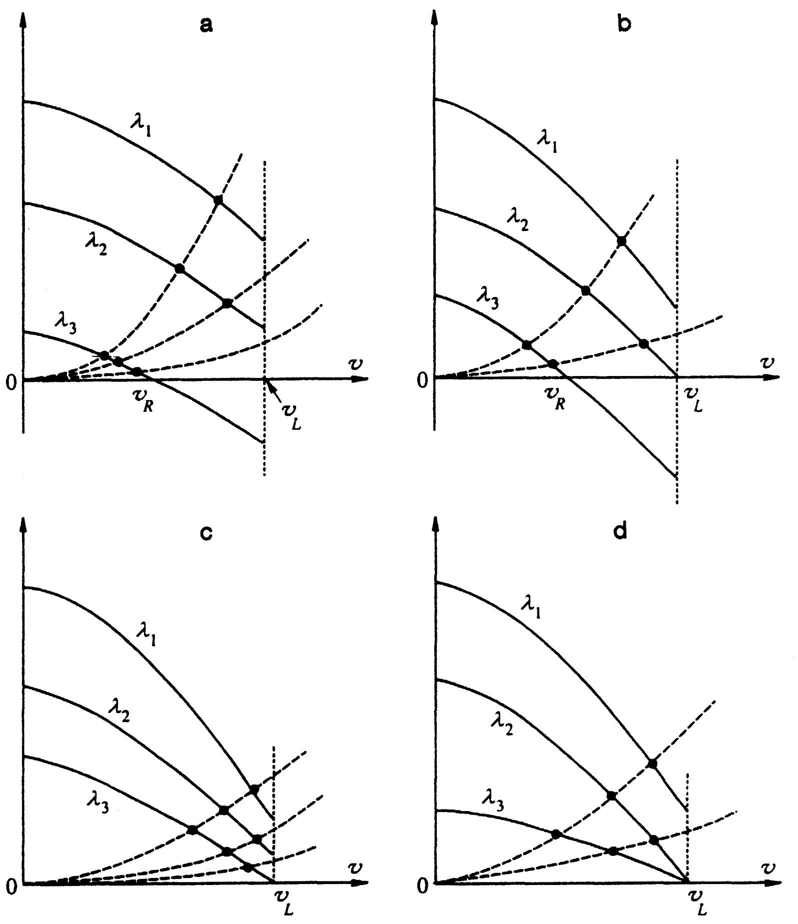

FIG. 5. Surface wave solutions for the case of loading produced by a layer of surface-absorbed heavy atoms (strong coupling approximation). Relations for $\lambda_i(v_L)$ in the cases a, b, c, and d are the same as in Figs. 2a,b,c, and d, respectively.

dard equation $\det Z_{ij}=0$ valid for the homogeneous half-space, may have more than one solution, i.e., in this case several Rayleigh surface waves with sagittal plane polarization may in principle exist.

In concluding this section we consider an extension of the above boundary-value problem to include the degrees of freedom (or displacements $\mathbf{u}'$) of the near-surface defect itself. In this case, following Ref. 13 we have instead of the strong coupling boundary condition (16), the system of equations

$$\hat{Z}\mathbf{u}=-\hat{\Gamma}(\mathbf{u}-\mathbf{u}'),\quad \hat{W}\mathbf{u}'=-\hat{\Gamma}(\mathbf{u}'-\mathbf{u}),\tag{24}$$

where the tensor $\hat{\Gamma}$ describes the elastic coupling of the defect with the substrate. Eliminating $\mathbf{u}'$ from Eq. (24) we are led to a boundary-value problem physically analogous to the boundary condition (13),

$$\hat{Z}\mathbf{u}=-\hat{W}'\mathbf{u},\quad \hat{W}'=(\hat{I}+\hat{W}\hat{\Gamma}^{-1})^{-1}\hat{W},\tag{25}$$

where $\hat{I}$ is the unit matrix. In fact, setting in the simplest case $\Gamma_{ij}=\gamma\delta_{ij}$ and returning to the limiting case $\rho_s v^2 \gg |g_{11}|$, $|h_{\alpha\beta}|$, which corresponds to the near-surface defect problem for a layer of adsorbed heavy atoms, one easily obtains that

$$W'_{ij}=\gamma'\delta_{ij},\tag{26}$$

where the parameter γ' is identical to that in Eq. (13). Clearly, the larger ρ_s, the less justified appears the strong coupling approximation $k^2\rho_s v^2 \ll \gamma$ used above. This means in particular that, for high enough frequencies, the maximum number of surface waves at a boundary with a heavy atom absorption layer on it may, in principle, reach four.

Let us estimate, for this case, the threshold frequencies ν_3 and ν_4 corresponding to the excitation thresholds for three and four waves respectively. From Eqs. (14) and (15) we have

$$\nu_{3,4}\approx\frac{\gamma}{4\pi\rho v_L}\left(\mp 1+\sqrt{1+\frac{\gamma_0}{\gamma}}\right),\tag{27}$$

where the parameter $\gamma_0 \approx 4\rho^2 v_L^2/\rho_s$. For $\gamma \gg \gamma_0$, the quantity ν_3 is independent of γ to lowest order,

$$\nu_3\approx\frac{\gamma_0}{8\pi\rho v_L}=\frac{\rho v_L}{2\pi\rho_s},\tag{28}$$

and the frequency ν_4 is linear in γ,

$$\nu_4\approx\gamma/2\pi\rho v_L.\tag{29}$$

In the opposite extreme $\gamma \ll \gamma_0$, the frequencies $\nu_{3,4}$ differ little and are proportional to $\sqrt{\gamma}$:

$$\nu_{3,4}\approx\sqrt{\frac{\gamma\gamma_0}{4\pi\rho v_L}}.\tag{30}$$

Let us make a numerical estimate for the frequencies $\nu_{3,4}$ in the intermediate case $\gamma=\gamma_0$. Setting $\rho_s \sim \rho' d$ (ρ' is the bulk density of the adsorbed layer and d its thickness), $\rho'/\rho \sim 10^2$, $d \sim 10^{-7}$ cm, and $v_L \approx 2\cdot 10^5$ cm/s in Eq. (27), we find the desired estimates $\nu_3 \approx 2$ GHz, $\nu_4 \approx 10$ GHz. Thus, in the light of this estimate the possibility for the excitation of three or even four different surface waves looks realistic in the system discussed. The appearance of new surface modes in crystals with coatings is indeed observed experimentally.[28]

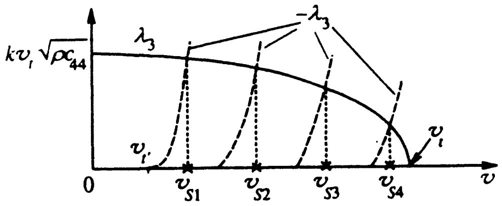

FIG. 6. Appearance of the family of transverse surface waves in the Love problem (a layer at the surface of a monoclinic crystal).

5. APPLICATION OF THE SURFACE IMPEDANCE CONCEPT TO THE LOVE PROBLEM

As another example of the elastic loading of the surface of a semi-infinite medium, let us consider the Love problem for a finite-thickness layer rigidly connected with a half-space (substrate) using the classical formulation, when it is further assumed that the sagittal plane is a plane of symmetry for both the layer and the substrate. If the layer is "softer" than the substrate (more precisely, if the transverse bulk wave in it is slower than in the substrate) then, as is well known, in such a structure an SH Love wave can propagate, which corresponds to the waveguide solution in the layer and to the surface (localized) solution in the substrate. In this section we will show that the use of the surface impedance concept yields a clear physical picture in connection with the question of the number of Love modes in the region $v<v_t$ (v_t the velocity of the transverse bulk wave in the substrate). We also demonstrate how a limiting process can be made to recover the surface defect problem above and the Stoneley wave localization problem for the interface of two half-spaces.

We introduce the scalar impedance λ' for a bipartial field of an SH wave in a monoclinic ($2\|\mathbf{t}$) layer of thickness d as minus the ratio of the elastic force to the elastic displacement at one surface of the layer ($x_2=\mathbf{n}'\mathbf{r}=0$, $\mathbf{n}'=-\mathbf{n}$ is the inward normal to a given surface) under the condition that the other boundary of the layer ($x_2=-d$) is free. After straightforward calculation we find

$$\lambda' = kc'_{44}\kappa' \tanh(kd\kappa'), \tag{31}$$

where the parameter κ' is defined by Eq. (22) with ρ, v_t, and $c_{\alpha\beta}$ primed (i.e., referred to the layer).

The continuity condition at the layer–substrate interface $x_2=0$ for the SH Love wave may be written in impedance language as

$$\lambda_3 + \lambda' = 0, \tag{32}$$

where λ_3 is the corresponding eigenvalue of the impedance matrix of the monoclinic substrates [see Eq. (21)]. Clearly, the solutions of Eq. (32) are the intersection points of the $-\lambda'(v)$ and $\lambda_3(v)$ curves (see Fig. 6), and occur only in the velocity interval $v_t'<v<v_t$, where each such solution specifies the Love wave velocity v_{Si}. It is readily seen that at least one solution always exists, and that the total number of solutions, accounting for waveguide modes of different order in the layer, is determined as the maximum number n satisfying the inequality

$$n < 1 + \frac{kd}{\pi}\sqrt{\frac{\rho'(v_t^2 - v_t'^2)}{c'_{44}}}. \tag{33}$$

We next consider the limiting cases which arise from taking the layer thickness d as the parameter. For $d \to 0$ we normally have $\lambda' \to 0$, so that the Love wave ceases to be localized and turns into a standard homogeneous SH wave, with velocity v_t, satisfying the condition of the half-space surface being mechanically free. At the same time, if for $d \to 0$ the quantities $c'_{44}d$, $\rho'd$ are taken small but finite, then $\lambda' \to W_{33}$ [cf. Eqs. (31) and (21)] and the slowest Love mode survives the limiting process in the form of a solution to Eq. (20), as a surface SH wave localized on a defect near the surface with the parameters

$$\rho_s = \rho'd, \quad h_{55} + g_{11} = (c'_{55} - c'^2_{45}/c'_{44})d \equiv \rho' d v_t'^2 > 0.$$

For $d \to \infty$, the Love problem goes over into the Stoneley problem for two half-spaces each of which may be considered as an elastic loading on the surface of the neighboring one. In this case for the SH wave in Eq. (32) $\lambda' \to \lambda_3'$ [see Eqs. (31) and (21)], where $\lambda_3' = kc'_{44}\kappa'(v)$ is the corresponding eigenvalue of the impedance matrix $\hat{Z}'$ of the half-space with an inward normal $\mathbf{n}'=-\mathbf{n}$. Since the quantities λ_3 and λ_3' are complex for $v>v_t$, v_t' and since for $v<v_t$, v_t' they have the same sign (positive), a SH Stoneley wave for $v_t \neq v_t'$ is not possible. We note that this result is a consequence of the general theory of the Stoneley waves developed in Ref. 29.

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APPENDIX

A1. SURFACE IMPEDANCE FOR MONOCLINIC CRYSTALS

Consider for simplicity the case in which the sagittal plane is coincident with the plane of symmetry of the monoclinic crystal. In this case two waves can propagate along the free surface independently of one another, a surface wave formed by the waves of the $l-$ and $t'-$ branches and a transverse (t) bulk SH wave. As a result, the general problem of evaluating the three-dimensional impedance simplifies: the three-dimensional $\hat{Z}$ matrix reduces to a two-dimensional and a one-dimensional matrix,

$$\hat{Z} = \hat{Z} + \lambda_3 \mathbf{t} \otimes \mathbf{t}, \tag{A1}$$

where $\otimes$ is the diadic product sign, $t=[\mathbf{mn}]\|x_3$, and the eigenvalue λ_3 is determined by the expressions (21) and (22). This quantity corresponds to the force produced by the split-off SH wave for which

$$\mathbf{J}_t=\lambda_3\mathbf{A}_t, \quad \mathbf{A}_t\|\mathbf{t}, \quad p_t(v)=-\frac{c_{45}}{c_{44}}+i\kappa(v), \tag{A2}$$

where the function $\kappa(v)$ is given by Eq. (22).

We now turn to the two-dimensional matrix z_{ab} [see Eq. (8)]:

$$z_{ab}=-\sum_\alpha J_{a\alpha}A^{-1}_{\alpha b} \quad (\alpha,a,b=1,2). \tag{A3}$$

In accordance with Eq. (5), the forces $\mathbf{J}_\alpha$ are expressed in terms of the polarization vectors $\mathbf{A}_\alpha$

$$\mathbf{J}_\alpha=\hat{M}^{(\alpha)}\mathbf{A}_\alpha, \tag{A4}$$

where

$$\hat{M}^{(\alpha)}=ik(\hat{M}_1+p_\alpha\hat{M}_2),$$
$$\hat{M}_1=\begin{pmatrix} c_{16} & c_{66} \\ c_{12} & c_{26}\end{pmatrix}, \quad \hat{M}_2=\begin{pmatrix} c_{66} & c_{26} \\ c_{26} & c_{22}\end{pmatrix}. \tag{A5}$$

Combining the Eqs. (A3)–(A5), it is a simple matter to represent the matrix $\hat{z}$ in the form

$$\hat{z}=\frac{1}{|\hat{A}|}(\hat{M}^{(2)}\mathbf{A}_2\otimes[\mathbf{A}_1\mathbf{t}]-\hat{M}^{(1)}\mathbf{A}_1\otimes[\mathbf{A}_2\mathbf{t}]), \tag{A6}$$

where $|\hat{A}|$ is the determinant of thematrix $A_{\alpha a}$. Knowing the matrix (A6), one can express its eigenvalues $\lambda_{1,2}$ in terms of the matrix elements,

$$\lambda_{1,2}=\frac{1}{2}[z_{11}+z_{22}\pm\sqrt{(z_{11}-z_{22})^2+4|z_{12}|^2}]. \tag{A7}$$

Here and below the upper sign refers to the first of the quantities.

The polarization vectors A_α determining the matrix $\hat{z}$ of Eq. (A6) and its eigenvalues (A7) may be expressed as functions of the parameters p_α,

$$\mathbf{A}_\alpha\|p_\alpha^2\mathbf{a}+p_\alpha\mathbf{b}+\mathbf{c}, \tag{A8}$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the following vectors:

$$\mathbf{a}=(c_{22},\ -c_{26}, \quad \mathbf{b}=(2c_{26},\ -c_{12}-c_{66}),$$
$$\mathbf{c}=(c_{66}-\rho v^2,\ c_{16}). \tag{A9}$$

We omit from (A8) the normalization factor, which is of no significance in the calculation of $\hat{z}$ [see Eqs (A3), (A4), and (A6)].

The parameters p_α in Eqs. (A5) and (A8) are determined by the condition $\operatorname{Im}p_\alpha>0$ as two of the four roots of the equation

$$p_\alpha^4+d_1p_\alpha^3+d_2p_\alpha^2+d_3p_\alpha+d_4=0. \tag{A10}$$

Here

$$d_1=\frac{2}{d}(c_{16}c_{22}-c_{26}c_{12}),$$

$$d_2=\frac{1}{d}[c_{11}c_{22}+c_{66}^2-(c_{12}+c_{66})^2+2c_{16}c_{26}$$
$$-\rho v^2(c_{22}+c_{66})],$$

$$d_3=\frac{2}{d}[c_{26}c_{11}-c_{16}c_{12}-\rho v^2(c_{16}+c_{26})], \tag{A11}$$

$$d_4=\frac{1}{d}[(c_{11}-\rho v^2)(c_{66}-\rho v^2)-c_{16}^2],$$

$$d=c_{22}c_{66}-c_{26}^2.$$

Equations (A8)–(A11) determine the dependence of the surface impedance matrix (A6) and its eigenvalues (A7) on the velocity v. Specifying this dependence, after some manipulation we have

$$\hat{z}=-ik\begin{pmatrix} c_{16}+c_{26}P_{22}+c_{66}P_{21} & c_{66}-c_{26}P_{12}-c_{66}P_{11} \\ c_{12}+c_{22}P_{22}+c_{26}P_{21} & c_{26}-c_{22}P_{12}-c_{26}P_{11}\end{pmatrix}$$
$$=-ik(\hat{M}_1+\hat{M}_2\hat{T}\hat{P}). \tag{12}$$

Here the matrices $\hat{M}_1$ and $\hat{M}_2$ are defined by Eqs. (A5), the matrix $\hat{T}$ is of the form

$$T=\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, \tag{A13}$$

and the matrix

$$\hat{P}\equiv|\hat{P}|\hat{P}^{-1}\equiv\mathrm{Sp}\,\hat{P}-\hat{P} \tag{A14}$$

is the inverse of the matrix $\hat{P}$:

$$\hat{P}\equiv\{p^2[\mathbf{a}\otimes(p^2\mathbf{a}-\mathbf{c})+\mathbf{b}\otimes(2q\mathbf{a}+\mathbf{b})]$$
$$+\mathbf{c}\otimes[(4q^2-p^2)\mathbf{a}+2q\mathbf{b}+\mathbf{c}]\}N, \tag{A15}$$

where we have introduced the notation

$$p^2=p_1p_2, \quad q=(p_1+p_2)/2, \tag{A16}$$

$$N=\{p^2[\mathbf{ab}]\mathbf{t}+2q[\mathbf{ac}]\mathbf{t}+[\mathbf{bc}]\mathbf{t}\}^{-1} \tag{A17}$$

(p is the geometric and q the arithmetic mean of the parameters p_1 and p_2).

From the hermiticity of the matrix $\hat{z}$, Eq. (A12), it follows that $z_{12}=z_{21}^*$ Explicitly, these properties as well the reality of $\lambda_{1,2}$ in Eq. (A7) and of z_{11} and z_{22} can be established by a rather lengthy procedure similar to that carried out in Ref. 30 in the analysis of the Rayleigh wave dispersion relation. The basis of the proof is the use of the Vieta theorem for the roots of Eq. (A10).

If not only the sagittal plane but as well the crystal surface turns out to be a symmetry plane, the relations obtained are simplified considerably. This may occur in rhombic, tetragonal, hexagonal, and cubic crystals. Thus, even in the case of rhombic crystals $c_{45}=c_{16}=c_{26}=0$, and Eq. (A10) becomes biquadratic [in Eq. (A11), $d_1=d_3=0$)]. Let us consider the simplest of all those cases cited above.

A2. SURFACE IMPEDANCE FOR HEXAGONAL CRYSTALS

A2.1. Sagittal plane perpendicular to the principal symmetry axis

In this case the sagittal plane is clearly the plane of transverse isotropy. The expressions for the parameters p_α and the vectors $\mathbf{A}_\alpha$ are especially simple:

$$p_\alpha = i\sqrt{\frac{1-v^2}{v_\alpha^2}} \quad (\alpha=1,2,3),$$
$$v_1^2 \equiv v_l^2 = \frac{c_{11}}{\rho}, \quad v_2^2 \equiv v_{t'}^2 = \frac{c_{66}}{\rho}, \quad v_3^2 \equiv v_t^2 = \frac{c_{44}}{\rho}, \tag{A18}$$

$$\mathbf{A}_1 || (1, p_1, v), \quad \mathbf{A}_2 || (-p_2, 1, 0), \quad \mathbf{A}_3 || (0,0,1). \tag{A19}$$

Here $_l$, $v_{t'}$, v_t is the conventional notation for the velocities corresponding, respectively, to the quasilongitudinal and quasitransverse branches of bulk elastic waves.

Using these relations we obtain

$$z_{ab} = ikc_{66}(p^2+1)^{-1} \times \begin{pmatrix} -p_1(v/v_{t'})^2 & -2(p^2+1)+(v/v_2)^2 \\ 2(p^2+1)-(v/v_2)^2 & -p_2(v/v_2)^2 \end{pmatrix}. \tag{20}$$

For the eigenvalues we then have

$$\lambda_{1,2} = kc_{66}(p^2+1)^{-1}\left[-iq\left(\frac{v}{v_{t'}}\right)^2 \pm 2\sqrt{f(v)(p^2+1) - q^2\left(\frac{v}{c_{t'}}\right)^4}\right]. \tag{A21}$$

Here

$$f(v) = (1-p_2^2)^2 + 4p^2. \tag{A22}$$

The Rayleigh velocity v_R is the root of the equation $f(v)=0$, equivalent to the condition $\lambda_2(v)=0$.[4]

Qualitatively, the character of the monotonically decreasing functions $\lambda_i(v)$ in Eqs. (A21) is clearly understood (see Fig. 3) if one has their values for $v=0$, $v_{t'}$ and v_t:

$$\lambda_1(0) = 2kc_{66}, \quad \lambda_2(0) = 2kc_{66}\frac{1-\eta}{1+\eta}, \quad \lambda_3(0) = kc_{44}, \tag{A23}$$

$$\lambda_{1,2}(v_{t'}) = \frac{kc_{66}}{2}(\sqrt{1-\eta} \pm \sqrt{5-\eta}), \quad \lambda_3(v_t) = 0. \tag{A24}$$

In these relations, $\eta \equiv c_{66}/c_{11}$. Using the relations (A23) and (A24), one can formulate the conditions for the intersection of the $\lambda_2(v)$ and $\lambda_3(v)$ branches [see Fig. 3a]:

$$A < c_{44}/c_{66} < B (A<B); \quad B < c_{44}/c_{66} < A (B<A), \tag{A25}$$

where $A = v_R^2/v_{t'}^2$, $B = 2(1-\eta)/(1+\eta)$.

A2.2. SURFACE PERPENDICULAR TO THE PRINCIPAL SYMMETRY AXIS

In this case the orientation of our coordinate system differs from that assumed everywhere before: whereas before we had $x_1 \| \mathbf{m}$, now we have $x_2 \| \mathbf{t}$, $x_3 \| \mathbf{n}$. For the split-off SH wave we have

$$\mathbf{A}_2 || (0,1,0), \quad p_2 = i\sqrt{\frac{c_{66}-\rho v^2}{c_{44}}}, \quad \lambda_2 = -kc_{44}p_2. \tag{A26}$$

The parameters p_1 and p_3 for the two remaining branches are found from the condition

$$p_{1,3}^2 = \frac{P}{2} \pm \sqrt{\frac{P^2}{4} - Q}, \tag{A27}$$

where

$$P = [c_{11}c_{33} + c_{44}^2 - (c_{13} + cc_{44})^2 - \rho v^2 \times (c_{33}+c_{44})]/c_{33}c_{44}, \tag{A28}$$

To the waves of these branches there correspond the two-dimensional surface impedance matrix z_{ab}, where $a,b=1,3$:

$$z_{ab} = ikc_{44}(c_{11} - \rho v^2 - c_{44}p^2)^{-1} \times \begin{pmatrix} 2(\rho v^2 - c_{11})q & \rho v^2 - c_{11} - c_{13}p^2 \\ -\rho v^2 + c_{11} + c_{13}p^2 & 2c_{33}p^2 q \end{pmatrix}, \tag{A29}$$

where p and q are the geometric and arithmetic means of the parameters $p_{1,3}$. The eigenvalues $\lambda_{1,3}$ of this matrix are found from Eq. (A7) in which all the '2' subscripts must be replaced by '3'.

A3. SURFACE IMPEDANCE FOR RHOMBIC CRYSTALS

The relations (A26)–(A29) are also directly applicable to the rhombic crystal for propagation along the x_1 axis in a surface parallel to the x_1 and x_2 axes. If the propagation direction is the same (x_1) but lies in another surface (x_1, x_3), the results are updated by simply renaming the coordinate axes, $x_2 \rightleftarrows x_3$, with a resulting change in the indices of the elastic constants: $c_{33} \to c_{22}$, $c_{13} \to c_{12}$, $c_{66} \to c_{55}$ (the c_{44} modulus is unchanged). Exactly the same kind of argument applies to the propagation directions x_2 and x_3 lying in two mutually orthogonal surfaces. The necessary changes actually reduce to a cyclic permutation of the indices on the elastic constants. Thus, six geometrically different cases are here covered.

[1]References 14–17 also discuss the more subtle questions of the definition of the impedance matrix in the case of degenerate parameters p_α, in particular for $v=v_L$. The properties of the surface impedance for inhomogeneous media have been analyzed in Ref. 12.

[2]Such a solution, in contrast to the case of free-surface bulk waves,[15,18,19], is not generally one-partial.

[3]The present surface wave solutions, found by direct calculation for the isotropic medium[13] and for a transversely isotropic model medium[27] are precisely of this latter type.

[4]The hermiticity of the matrix z_{ab} in (A 20) and the reality of its eigenvalues (A21) are in this case evident.

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