A positronium atom in the self-field of annihilation photons

O. N. Gadomskiĭ

Department of Theoretical Physics, Ulyanovsk Branch of the M. V. Lomonosov Moscow State University, 432700 Ulyanovsk, Russia

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In this paper the lifetime, energy shift, and phase-relaxation time of a parapositronium atom in a two-photon annihilation decay process is calculated using the Heisenberg equations of motion for atomic and photon operators, without resorting to perturbation techniques.

1. INTRODUCTION

Positronium (Ps) is a two-particle lepton system possessing annihilation channels with different lifetimes. The system has proved ideal for studying relativistic two-particle interactions. Hence interest in it has been unflagging ever since the possibility of formation of an electron-positron bound state, a positronium atom, was predicted theoretically. More than that, recent years have seen an increasing interest in the positronium problem in connection with the possibility of using a system of such atoms to build an annihilation gamma-ray laser.[1–6]

Studies of the relativistic corrections to the energy of a Ps atom in different states occupy an important place in the physics of the positronium atom. For instance, the result of well-known calculations can be expressed in the following manner:[7]

$$W=W_0+W_1+W_2+\cdots,$$

$$W_0=\frac{\alpha^4mc^2}{4}\frac{7}{3},\quad W_1=\frac{\alpha^4mc^2}{4}\frac{\alpha}{\pi}\left(\frac{32}{9}+2\ln 2\right), \tag{1.1}$$

where $\alpha^2=e^2/\hbar c$ is the fine-structure constant, and W_0 the correction to the energy of the Ps atom caused by QED second-order effects, which include virtual-photon exchange between the electron and the positron and virtual creation and annihilation of Ps (the correction is of order 203 000 MHz). The term W_1 of order α^5mc^2 (1000 MHz) is the first correction to the main interaction and incorporates vacuum polarization. Karshenboĭm[8] also found the main logarithmic contributions to hyperfine splitting of positronium with the relative order of $\alpha^3\ln^2(1/\alpha)$. All these corrections can be regarded as nonradiative electron–positron interactions since they do not allow for processes of emission and absorption of real photons.

This paper calculates the energy of a parapositronium atom and the lifetime in the event of two-photon decay without employing perturbation-theory techniques but by using the simultaneous system of Heisenberg equations of motion for photon and atomic operators. We will show that the proposed theory of the Ps atom allows for calculating the energy shift and the lifetime by using the adiabatic and rotating wave approximations. The meaning of these approximations is clarified as we develop the theory.

2. THE EQUATIONS OF MOTION

The Feynman diagram representing the process of two-photon annihilation of the Ps atom is depicted in Fig. 1. Hence, the effective energy of interaction of Ps with photons can be found from the following matrix element:

$$(\bar\psi\hat A\psi)(\bar\psi\hat A\psi)\to\left(\sum_r a_r^+\,\bar\psi_r^{(+)}+\sum_r b_r\,\bar\psi_r^{(-)}\right)\left(\sum_{r'} a_{r'}\psi_{r'}^{(+)}+\sum_{r'} b_{r'}^+\psi_{r'}^{(-)}\right)\sum_{\mathbf{k}\lambda}\sum_{\mathbf{k}'\lambda'} g_{\mathbf{k}\lambda}g_{\mathbf{k}'\lambda'}(c_{\mathbf{k}\lambda}e^{ikr}+c_{\mathbf{k}\lambda}^+e^{-ikr})(c_{\mathbf{k}'\lambda'}e^{i\mathbf{k}'r}+c_{\mathbf{k}'\lambda'}^+e^{-i\mathbf{k}'r}), \tag{2.1}$$

where $\psi_r^{(\pm)}$ and $\bar\psi_r^{(\pm)}$ are the solutions to the Dirac equation for a bound electron (positron); a_r^+, a_r, b_r^+, b_r, $c_{\mathbf{k}\lambda}^+$, and $c_{\mathbf{k}\lambda}$ are the respective creation and annihilation operators of electrons, positrons, and photons; $g_{\mathbf{k}\lambda}=\sqrt{2\pi\hbar c^2/V_R\omega_{\mathbf{k}}}$, V_R is the electromagnetic-field quantization volume, and $\omega_{\mathbf{k}}$ and $\mathbf{k}$ the frequency and wave vector of photons, respectively.

The Hamiltonian of an electron–positron system interacting with photons and having fixed electron and positron states can be written

$$H=\hbar\omega^{(+)}a^+a+\hbar\omega^{(-)}b^+b+\sum_{\mathbf{k}\lambda}\hbar\omega_{\mathbf{k}}c_{\mathbf{k}\lambda}^+c_{\mathbf{k}\lambda}+Ub^+a^+ab+(S_{\mu\nu}ba+S_{\mu\nu}^*a^+b^+)A_{1\mu}A_{2\nu}, \tag{2.2}$$

where the vector potential operators

$$A_{1\mu}=g_{\mathbf{k}_1\lambda_1}e_\mu^{\lambda_1}(c_1e^{i\mathbf{k}_1\mathbf{r}}+c_1^+e^{-i\mathbf{k}_1\mathbf{r}}),$$

$$A_{2\nu}=g_{\mathbf{k}_2\lambda_2}e_\nu^{\lambda_2}(c_2e^{i\mathbf{k}_2\mathbf{r}}+c_2^+e^{-i\mathbf{k}_2\mathbf{r}}) \tag{2.3}$$

correspond to the annihilation photons, e^{λ_1} and e^{λ_2} are the photon unit polarization vectors, $\mu,\nu=x,y,z$, and $\lambda_1,\lambda_2=1,2$. The matrix U in (2.2) reflects the QED second-order effects, which correspond to the term W_0 in Eq. (1.1). In operator (2.2) summation over repeated in-

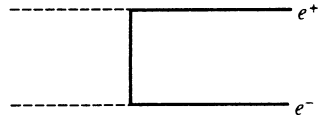

FIG. 1.

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dices is assumed, as is summation over the different modes of the annihilation photons in accordance with the laws of energy-momentum conservation in annihilation decay.

The Heisenberg representation requires using the commutation relations[9]

$$[A_\mu(x),A_\nu(x')]|_{t=t'}=0, \quad \left[A_\mu(x),\frac{\partial}{\partial t'}A_\nu(x')\right]\Big|_{t=t'}$$
$$=i\delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}'),$$
$$\{\psi_\alpha(x),\psi_\beta^+(x')\}]|_{t=t'}=\delta_{\alpha\beta}(\mathbf{r}-\mathbf{r}'),$$
$$\{\psi_\alpha(x),\psi_\beta(x')\}|_{t=t'}=0,$$
$$[\psi_\alpha(x),A_\mu(x')]_{t=t'}=0 \tag{2.4}$$

and the Heisenberg equation for an operator F:

$$-i\hbar\frac{\partial F}{\partial t}=[H,F]. \tag{2.5}$$

After applying the commutation relations (2.4) and performing certain calculations we arrive at the following operator equations, assuming that $P\equiv ab$, $P\equiv b^+a^+$, $n\equiv a^+a+b^+b-1$, and $\Omega_0=\omega^{(+)}+\omega^{(-)}$:

$$\dot P=-i\Omega_0 P+\frac{i}{\hbar}UP-\frac{i}{\hbar}S^*_{\mu\nu}A_{1\mu}A_{2\nu}n, \tag{2.6a}$$

$$\dot P^+=i\Omega_0 P^+-\frac{i}{\hbar}UP^++\frac{i}{\hbar}S_{\mu\nu}A_{1\mu}A_{2\nu}n, \tag{2.6b}$$

$$\dot n=-\frac{2i}{\hbar}S_{\mu\nu}A_{1\mu}A_{2\nu}P+\frac{2i}{\hbar}S^*_{\mu\nu}A_{1\mu}A_{2\nu}n. \tag{2.6c}$$

These equations for the atomic operators must be augmented by appropriate equations for the photon operators:

$$\dot c_1=-i\omega_1c_1+\frac{i}{\hbar}S(\mathbf{k}_1,\mathbf{k}_2)c_2^+P+\frac{i}{\hbar}S(\mathbf{k}_1,-\mathbf{k}_2)c_2P$$
$$+\frac{i}{\hbar}S^*(\mathbf{k}_1,\mathbf{k}_2)c_2^+P^++\frac{i}{\hbar}S^*(\mathbf{k}_1,-\mathbf{k}_2)P^+c_2, \tag{2.7a}$$

$$\dot c_2=-i\omega_2c_2+\frac{i}{\hbar}S(\mathbf{k}_1,\mathbf{k}_2)c_1^+P+\frac{i}{\hbar}S(-\mathbf{k}_1,\mathbf{k}_2)c_1P$$
$$+\frac{i}{\hbar}S^*(\mathbf{k}_1,\mathbf{k}_2)c_1^+P^++\frac{i}{\hbar}S^*(-\mathbf{k}_1,\mathbf{k}_2)P^+c_1, \tag{2.7b}$$

where normal ordering of operators is employed, and

$$S(\mathbf{k}_1,\mathbf{k}_2)=S_{\mu\nu}g_{\mathbf{k}_1\lambda_1}g_{\mathbf{k}_2\lambda_2}e_\mu^{\lambda_1}e_\nu^{\lambda_2}e^{-i\mathbf{k}_1\mathbf{r}}e^{-i\mathbf{k}_2\mathbf{r}}. \tag{2.8}$$

2.1. Conservation laws

From Eq. (2.6) we can derive the conservation law:

$$\frac{d}{dt}\left(P^+P+PP^++\frac{1}{2}n^2\right)=0. \tag{2.9}$$

Hence, we have the following integral of motion:

$$P^+P+PP^++\frac{1}{2}n^2=\text{const}=n_0. \tag{2.9a}$$

Let us now determine the corresponding integral of motion for the photon operators. Such integrals of motion are represented in the simplest form if we use the rotating wave approximation (RWA), in which the Hamiltonian (2.2) is

$$H=\hbar\omega^{(+)}a^+a+\hbar\omega^{(-)}b^+b+\hbar\omega_1c_1^+c_1+\hbar\omega_2c_2^+c_2$$
$$+Ub^+a^+ab+S(\mathbf{k}_1,\mathbf{k}_2)c_1^+c_2^+ba$$
$$+S^*(-\mathbf{k}_1,-\mathbf{k}_2)a^+b^+c_1c_2. \tag{2.10}$$

Then from (2.6) and (2.7) we find

$$\frac{d}{dt}(c_1^+c_1+c_2^+c_2+n)=0 \tag{2.11}$$

and

$$\frac{d}{dt}(c_1^+c_1-c_2^+c_2)=0. \tag{2.12}$$

2.2. Modified system of equations of motion

We apply the adiabatic approximation, with

$$P(t)=P_0(t)e^{-i\Omega t}, \quad c_{1(2)}(t)=c_{1(2)}^{(0)}e^{-i\omega t_{1(2)}t},$$
$$P^+=P_0^+(t)e^{i\Omega t}, \quad c_{1(2)}^+(t)=c_{1(2)}^{(0)+}(t)e^{i\omega_{1(2)}t}, \tag{2.13}$$

where P_0, P_0^+, $c_{1(2)}^{(0)}$, and $c_{1(2)}^{0+}$ are slowly varying amplitudes, and $\Omega=\Omega_0-\hbar^{-1}U$ is the Ps frequency. Then Eq. (2.7a) yields the following equation:

$$c_1(t)=c_{1v}(t)+\frac{i}{\hbar}\int_0^\infty dt'[S(\mathbf{k}_1,\mathbf{k}_2)c_2^+(t')P(t')$$
$$+S(\mathbf{k}_1,\mathbf{k}_2)c_2(t')P(t')$$
$$+S^*(\mathbf{k}_1,\mathbf{k}_2)c_2^+(t')P^+(t')$$
$$+S^*(\mathbf{k}_1,-\mathbf{k}_2)P^+(t')c_2(t')]G_1(t-t'), \tag{2.14}$$

where $c_{1v}(t)=c_{1v}(0)\exp(-i\omega_1t)$ is the vacuum field operator, and

$$G_1(t-t')=\lim_{\varepsilon\to+0}\begin{cases}0, & t>t',\\ \exp[-i(\omega_1-i\varepsilon)(t-t') & t>t'\end{cases} \tag{2.15}$$

is the retarded Green's function. Thus, in the photon field operators we can isolate the vacuum part and the part representing the effect of the photon self-field.

Using (2.13), we can write the following operator equation instead of (2.14):

$$c_1(t)=c_{1v}(t)+d_1(t)c_2^+(t)+d_2(t)c_2(t), \tag{2.16}$$

where

$$\mathrm{d1(t)=1\hbar\,S(k1,k2)\zeta*(-\Omega+\omega2+\omega1)P(t)}$$

$$+\frac{1}{\hbar}S^*(\mathbf{k}_1,\mathbf{k}_2)\zeta^*(\Omega+\omega_2+\omega_1)P^+(t), \quad (2.16a)$$

$$d_2(t)=\frac{1}{\hbar}S(\mathbf{k}_1,-\mathbf{k}_2)\zeta^*(-\Omega-\omega_2+\omega_1)P(t)$$

$$+\frac{1}{\hbar}S^*(\mathbf{k}_1,-\mathbf{k}_2)\zeta^*(\Omega-\omega_2+\omega_1)P^+(t), \quad (2.16b)$$

with the zeta function $\zeta(x)$ defined following Heitler:[10]

$$\zeta(x)=\frac{\mathscr{P}}{x}-i\pi\delta(x).$$

In the same manner we obtain the second equation:

$$c_2(t)=c_{2v}(t)+d_3(t)c_1^+(t)+d_4(t)c_1(t), \quad (2.17)$$

where

$$d_3(t)=\frac{1}{\hbar}S(\mathbf{k}_1,\mathbf{k}_2)\zeta^*(-\Omega+\omega_1+\omega_2)P(t)$$

$$+\frac{1}{\hbar}S^*(\mathbf{k}_1,\mathbf{k}_2)\zeta^*(\Omega+\omega_1+\omega_2)P^+(t), \quad (2.17a)$$

$$d_4(t)=\frac{1}{\hbar}S(-\mathbf{k}_1,\mathbf{k}_2)\zeta^*(-\Omega-\omega_1+\omega_2)P(t)+\frac{1}{\hbar}S^*$$

$$\times(-\mathbf{k}_1,\mathbf{k}_2)\zeta^*(\Omega-\omega_1+\omega_2)P^+(t). \quad (2.17b)$$

From Eqs. (2.16) and (2.17) and from the respective Hermitian conjugate equations for c_1^+ and c_2^+ we can express the photon operators in terms of the atomic operators P and P^+. Let us make several simplifying assumptions.

1. The positronium atom is immobile. Then momentum conservation yields $\omega_1=\omega_2$ and, hence $d_1=d_3$.

2. We select the phases of the wave functions in the matrix element $S_{\mu\nu}$ in such a way that $S_{\mu\nu}=-i|S_{\mu\nu}|$.

3. We employ the rotating wave approximation, in which the Hamiltonian operator of the system has the form (2.10).

With these assumptions in mind, we obtain

$$d_1(t)=d_0P(t), \quad d_2=0, \quad d_3=d_1, \quad d_4=0,$$

$$d_0=-\frac{i}{\hbar}|S|\zeta^*(-\Omega+\omega_2+\omega_1). \quad (2.18)$$

This makes it possible to easily determine the following operators:

$$c_1=\frac{c_{1v}+c_{2v}^+d_1}{1-d_1^*d_1}, \quad c_1^+=c_1^*,$$

$$c_2=\frac{c_{2v}+c_{1v}^+d_1}{1-d_1^*d_1}, \quad c_2^+=c_2^*. \quad (2.19)$$

We now transform the operator $(1-d_1^*d_1)^{-1}$ using its series expansion,

$$\frac{1}{1-d_1^*d_1}=1+d_1^*d_1+(d_1^*d_1)^2+\cdots=1+p_0P^+P, \quad (2.20)$$

where

$$p_0=\frac{|d_0|^2}{1-|d_0|^2}. \quad (2.21)$$

In finding this representation of the operator we have employed the ordinary properties of fermion operators:

$$a^2=b^2=0, \quad (a^+a)^2=a^+a, \quad (b^+b)^2=b^+b.$$

Thus, the origin of the parameter p_0 in our treatment lies in the fact of photon mode mixing in the self-field of the annihilation photons. More than that, calculation of the photon self-field of a parapositronium atom is done with allowance for arbitrary powers of the fine-structure constant α (or the matrix S), that is, as expansion (2.20) shows, we do not use any of the restrictions imposed by the perturbation-theory expansion.

Substituting the operators (2.19) into the equations of motion of the atomic operators (2.6), we can transform these equations. To this end we express the atomic-field operator in Eq. (2.6c) in the form

$$-\frac{2}{\hbar}|S|(c_1c_2+c_2^+c_1+c_1^+c_2+c_1^+c_2)(P+P^+)\rightarrow$$

$$-\frac{2}{\hbar}|S|\{P^+(c_{1v}+c_{2v}^+d_0P)(1+p_0P^+P)(c_{2v}$$

$$+c_{1v}^+d_0P)(1+p_0P^+P)+(1+p_0P^+P)(c_{1v}^+$$

$$+d_0^*P^+c_{2v})(1+p_0P^+P)(c_{2v}^++d_0^*P^+c_{1v})P\}. \quad (2.22)$$

We transform this operator by reasoning as follows. For the parameters estimating the contributions of various terms of this operator we select d_0 and p_0.

We assume that the free (vacuum) field contains no initial photons of modes 1 and 2. This means that the following relations hold true:

$$c_{\lambda v}|\text{vacuum}\rangle=0, \quad \langle\text{vacuum}|c_{\lambda v}^+=0,$$

$$c_{\lambda v}^+c_{\lambda v}|\text{vacuum}\rangle=0, \quad \lambda=1,2, \quad (2.23)$$

where $|\text{vacuum}\rangle$ is the wave function of the vacuum field in the photon-state occupation-number representation. In addition we use the following identities:

$$[P,P^+]=-n, \quad [P,n]=2P, \quad [P^+,n]=-2P,$$

$$P^+P=\tfrac{1}{3}n_0+\tfrac{1}{2}n, \quad (P^+P)^2=\tfrac{1}{9}n_0^2+\tfrac{1}{3}n_0n+\tfrac{1}{6}n_0,$$

$$n^2=\tfrac{2}{3}n_0, \quad P^+n^2P=\frac{1}{2}n+\frac{1}{3}n_0,$$

$$-nP=P, \quad PP^+=\frac{1}{3}n_0-\frac{1}{2}n. \quad (2.24)$$

Summing all terms in the operator (2.22), instead of Eq. (2.6c) we have the following equation:

$$\dot{n}=-\frac{2}{\hbar}|S|\{F_1n+F_0\}, \quad (2.25)$$

where

$$F_1=\frac{1}{2}(d_0+d_0^*)+\frac{1}{3}p_0(d_0+d_0^*)n_0,$$

$$F_0=\frac{1}{3}n_0(d_0+d_0^*)+p_0(d_0+d_0^*)\left(\frac{1}{9}n_0^2+\frac{1}{6}n_0\right)$$

are real quantities.

The atomic-field operator in Eq. (2.6a) is transformed similarly:

$$-\frac{i}{\hbar}S^*_{\mu\nu}A_{1\mu}A_{2\mu}n\to\frac{1}{\hbar}|S|n(c_{1v}+c_{2v}^+d_0P)(1+p_0P^+P)\times(c_{2v}+c_{1v}^+d_0P)(1+p_0P^+P). \tag{2.26}$$

After performing the necessary calculations, instead of Eq. (2.6a) we have the following equation:

$$P=-i\left(\Omega_0-\frac{1}{\hbar}U-\frac{1}{\hbar^2}|S|^2\frac{\mathscr{P}}{-\Omega+\omega_1+\omega_2}\times\left[1+p_0\left(\frac{1}{3}n_0+\frac{1}{2}\right)\right]\right)P-\frac{1}{\hbar^2}|S|^2P\times\left[1+p_0\left(\frac{1}{3}n_0+\frac{1}{2}\right)\right]\pi\delta(-\Omega+\omega_1+\omega_2). \tag{2.27}$$

3. THE LIFETIME OF A PARAPOSITRONIUM ATOM

Let us find the lifetime of a parapositronium atom from Eq. (2.25) by averaging this equation weighted by a function $|\Phi\rangle$ in the occupation-number representation for electron-positron states. To this end we perform several preliminary calculations. We use the conservation law (2.9a) to determine n_0. Assuming that initially $\langle P^+P\rangle=0$ and $\langle n\rangle=1$, we find that $n_0=3/2$. This yields

$$F_1=F_0=(1+p_0)(1/\hbar)|S|\pi\delta(-\Omega+\omega_1+\omega_2).$$

We allow for the multimode nature of annihilation decay when F_1 and F_2 resemble delta functions and also when the first simplifying assumption in Sec. 2 is true. Then the solution to Eq. (2.25) assumes the form

$$n(t)=-1+2\exp\left(-\frac{2}{\hbar}|S|F_1t\right), \tag{3.1}$$

where the inverse lifetime of the parapositronium atom is determined after integration with respect to frequency by the formula:

$$\tau_{2\gamma}^{-1}=\frac{V_R}{(2\pi c)^3}\frac{2\pi}{\hbar^2}\int|S|^2(1+p_0)(\Omega-\omega_2)^2d\Omega_1\equiv W_{2\gamma}. \tag{3.2}$$

Here $\Omega-\omega_2$ replaces ω_1 in S and p_0 after integration with respect to ω_1, and $d\Omega_1$ is the element of the solid angle in the direction of emission of an annihilation photon of mode 1.

The value of $|S|$ can be calculated by taking into account the definitions (2.1), (2.2), and (2.8). The matrix of the effective energy of interaction between a parapositronium atom and the photon field corresponding to the Feynman diagram in Fig. 1 can be represented as follows:

$$S_{ij}=-\alpha\pi\frac{c^3}{V_R}(V_i+V_j)\sum_{\mu\nu}\sum_{l\pm}\sum_{\lambda_1\lambda_2}e_\mu^{\lambda_1}e_\mu^{\lambda_2}\int d\mathbf{r}\bar\psi_i^{(-)}(\mathbf{r})\gamma_\mu\times e^{-i\mathbf{k}_1\mathbf{r}}\psi_l(\mathbf{r})\bar\psi_l(\mathbf{r})\gamma_\nu e^{-i\mathbf{k}_2\mathbf{r}}\psi_j^{(+)}\times\frac{\hbar}{\omega_l-\omega_1+\omega_i}\frac{1}{(\omega_1\omega_2)^{1/2}}e^{-i(\mathbf{k}_1-\mathbf{k}_2)R},$$

where we have set $V_{i(j)}=4/3\pi a_{i(j)}^3$, $a_{i(j)}$ is the characteristic radius of the positronium atom in the state $i(j)$ with a set of quantum numbers n, l, m, m_s, and $\mathbf{R}$ is the radius vector of the center of inertia of the Ps atom. Summation in the above expression is done over intermediate states with positive and negative energies. For fixed states of the electron and positron in the Ps atom we have

$$S_j=-2\pi i\alpha c^3\frac{V_j}{V_R}\frac{\hbar}{(\omega_1\omega_2)^{1/2}}M_j(\omega_1,\omega_2), \tag{3.3}$$

where the real quantity $M_j(\omega_1,\omega_2)$ is given by the following relation:

$$M_j(\omega_1,\omega_2)=-i\sum_{\mu\nu}\sum_{l\pm}\sum_{\lambda_1\lambda_2}\int d\mathbf{r}\bar\psi_j^{(-)}\times(\mathbf{r})\gamma_\mu\psi_l(\mathbf{r})\bar\psi_l(\mathbf{r})\gamma_\nu\psi_j\frac{1}{\omega_l-\omega_1-\omega_j}e_\mu^{\lambda_1}e_\nu^{\lambda_2}. \tag{3.4}$$

At $(\mathbf{k}_1-\mathbf{k}_2)\mathbf{r}=0$ and $(\mathbf{k}_1-\mathbf{k}_2)\mathbf{R}=0$ the quantity specified by (3.4) can be calculated using hydrogenic wave functions.[9]

Let us find $|S|$ in Eq. (3.2) when $p_0\to0$. Such a situation corresponds to annihilation decay of a positronium atom into two photons without allowing for the reaction of photons on the annihilation process. By $W_{2\gamma}^{(0)}(\Delta\Omega_1)$ we denote the rate (probability per unit time) of "free" annihilation decay into the solid-angle element $\Delta\Omega_1$ (or $\Delta\Omega_2$). The directions of emission of photons of modes 1 and 2 in the solid angles $\Delta\Omega_1$ and $\Delta\Omega_2$ are related to each other through the law of momentum conservation and assumption 1 of Sec. 2. Then[5,6]

$$|S|=\left(\frac{W_{2\gamma}^{(0)}(\Delta\Omega_1)\hbar^2(2\pi c)^3}{2\pi V_R\omega_1^2\Delta\Omega_1}\right)^{1/2}. \tag{3.5}$$

For an isotropic decay we have $W_{2\gamma}^{(0)}(\Delta\Omega_1)=(4\pi)^{-1}W_{2\gamma}^{(0)}\Delta\Omega_1$, and in this case $|S|$ is independent of the solid-angle element. Here $|S|$ is the energy of interaction between the positronium atom and the field of free photons.

Now we need only calculate the parameter p_0 to estimate the variation of the lifetime of the positronium atom caused by the reaction of the photon field. From (3.2) we find the rate $W_{2\gamma}$ of annihilation decay as a function of the respective quantity $W_{2\gamma}^{(0)}$ for "free" decay:

$$W_{2\gamma}=\frac{W_{2\gamma}^{(0)}}{1-f(\tau)W_{2\gamma}^{(0)}}, \tag{3.6}$$

where

$$f(\tau)=\frac{1}{2}\tau^2\frac{(2\pi c)^3}{2\pi V_R\omega_1^2}, \tag{3.7}$$

with τ the characteristic observation time of the annihilation decay process. The appearance of this parameter in (3.6) is caused, according to (2.18), by the squaring of the absolute value of the zeta function at the point $\omega_1+\omega_2-\Omega=0$. On the other hand, from Eqs. (3.7) and (3.5) we find that

$$W_{2\gamma}^{(0)}f(\tau)=\frac{1}{2}\tau^2\frac{|S|^2}{\hbar^2}. \tag{3.8}$$

This quantity can be interpreted in the following way. In conditions of annihilation decay we can fix two moments in time, one corresponding to the initial value of the energy of the positronium atom, and the other to the state of the electromagnetic field with two photons. The energy of interaction between the field and the Ps atom is determined by $|S|$. According to the uncertainty relation for energy,[11] the energy of a system capable of decaying in a quasistationary state can be determined to within $\hbar/\tau$, where $\hbar/\tau$ is the width of a level of the Ps atom in relation to annihilation decay, that is, $\tau\equiv\tau_{2\gamma}$. Thus, from Eq. (3.6) we get

$$W_{2\gamma}=\frac{W_{2\gamma}^{(0)}}{1-a^2/2}, \tag{3.9}$$

where a is a parameter of the order of unity that determines the accuracy with which the energy of the positronium atom is measured. Thus, we have a parameter a that makes it possible to estimate the limits within which $W_{2\gamma}$ varies in relation to the respective value of the rate of free annihilation decay.

Let us now return to Eq. (3.6) and assume that $\tau=\tau_{2\gamma}\equiv W_{2\gamma}^{-1}$. After performing the necessary calculations, we find

$$W_{2\gamma}=\frac{1}{2}W_{2\gamma}^{(0)}\pm\left[\frac{1}{4}(W_{2\gamma}^{(0)})^2+\frac{1}{2}bW_{2\gamma}^{(0)}\right]^{1/2}, \tag{3.10}$$

where

$$b=\frac{(2\pi c)^3}{2\pi V_R\omega_1^2}.$$

In accordance with (3.9), we leave the upper sign in Eq. (3.10) and write it as follows:

$$W_{2\gamma}=\frac{1}{2}W_{2\gamma}^{(0)}+\frac{1}{2}W_{2\gamma}^{(0)}\left[1+\frac{2b}{W_{2\gamma}^{(0)}}\right]^{1/2}. \tag{3.11}$$

To calculate the variation of the decay rate (3.11), we must determine the field quantization volume V_R. For this we note that the characteristic parapositronium frequency in the $1s$ state must be "allowed" by an appropriate choice of the electromagnetic-field quantization volume. We then find that

$$V_R=\left(\frac{8\pi c\hbar^3}{me^4}\right)^3=\left(\frac{8\pi\hbar c}{mc^2L^2}\right)^3, \tag{3.12}$$

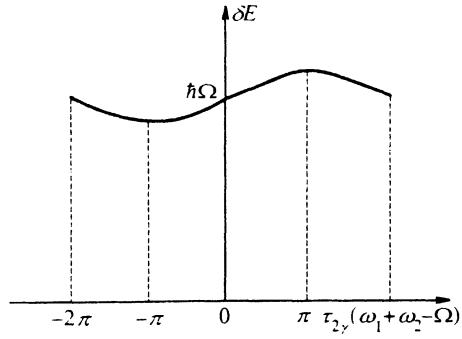

FIG. 2.

and the final expression for the rate of annihilation decay of a parapositronium atom into two photons with allowance for the retroaction of photons is

$$W_{2\gamma}=\frac{1}{2}W_{2\gamma}^{(0)}\left[1+\left(1+\frac{\alpha}{32\pi}\right)^{1/2}\right], \tag{3.13}$$

where we have used the value $W_{2\gamma}^{(0)}=(\alpha^5mc^2/2\hbar)$ for the respective "free" decay rate.[12,7]

Let us now calculate the characteristic frequency of the effective interaction of the parapositronium atom with the photon field. Combining (3.5) and (3.12), we obtain

$$\frac{|S|}{\hbar}=\frac{\alpha^5\sqrt{\alpha}c^2m}{2\sqrt{2\pi\hbar}}, \tag{3.14}$$

which agrees with the uncertainty relation for energy:

$$(|S|/\hbar)(W_{2\gamma}^{(0)})^{-1}=\sqrt{\alpha/2\pi}. \tag{3.15}$$

4. THE SHIFT OF THE ENERGY OF A POSITRONIUM ATOM IN THE PHOTON FIELD

Let us use Eq. (27) to calculate the energy of a parapositronium atom in the field proper of annihilation photons. Equation (2.27) yields

$$\Delta E=-\int\frac{|S|^2}{\hbar^2}\frac{V_R}{(2\pi c)^3}\omega_1^2d\omega_1d\Omega_1\frac{1-\cos(-\Omega\omega_1+\omega_2)\tau}{-\Omega\omega_1+\omega_2}$$
$$\times\frac{1}{1-(|S|/\hbar)^2F(\omega_1+\omega_2-\Omega)}, \tag{4.1}$$

where we have introduced the notation

$$F(\omega_1+\omega_2-\Omega)=\frac{1-\cos[(\omega_1+\omega_2-\Omega)\tau]}{(\omega_1+\omega_2-\Omega)^2}, \tag{4.2}$$

and, according to Sec. 3, τ has the meaning of the lifetime of a parapositronium atom with allowance for the retroaction of photons, that is $\tau=W_{2\gamma}^{-1}$. The properties of the function (4.2) are known: its maximum value is $(1/2)\tau^2$ at point $\omega_1+\omega_2-\Omega=0$, and the integral (4.1) at this point must be interpreted in the principal-value sense.

Figure 2 depicts the frequency dependence of the energy of a positronium atom, $\delta E=\hbar\Omega+\delta\omega_1 d(\Delta E)/d\omega_1$, in

the neighborhood of frequency Ω, with use made of Eqs. (3.12)–(3.14). The differential value of the energy shift (4.1) has the form

$$\frac{d(\Delta E)}{d\omega_1}=-2W_{2\gamma}^{(0)}\hbar f_1(\omega_1+\omega_2-\Omega)\times\frac{1}{1-(|S|/\hbar^2)^2F(\omega_1+\omega_2-\Omega)}, \tag{4.3}$$

where $\omega_1=\omega_2=(mc^2/\hbar)+\delta\omega_1$, $\Omega=(2mc^2/\hbar)-me^4/4\hbar^3$, and

$$f_1=\frac{1-\cos(\omega_1+\omega_2-\Omega)\tau_{2\gamma}}{\omega_1+\omega_2-\Omega}. \tag{4.4}$$

As formula (4.3) shows, in exact resonance ($\omega_1+\omega_2-\Omega=0$) the function f_1 vanishes and the energy of the positronium atom is $\hbar\Omega$. According to the uncertainty relation (3.15) for energy, the allowed values of frequency ω_1 are determined by the interval $\tau_{2\gamma}^{-1}$. Hence, the energy of the positronium atom is determined by the interval $(2\pi,-2\pi)$ of variation of the argument of f_1 and F. The maximum variation of the energy of the positronium atom caused by the retroaction of the photon field proper occurs at $(\omega_1+\omega_2-\Omega)\tau_{2\gamma}=\pm\pi$, and according to (4.4) we have

$$-2W_{2\gamma}^{(0)}\hbar\frac{1}{1-(|S|/\hbar)^2(\tau_{2\gamma}/\pi)^2}<\delta E<2W_{2\gamma}^{(0)}\hbar\frac{1}{1-(|S|/\hbar)^2(\tau_{2\gamma}/\pi)^2}, \tag{4.5}$$

where $|S|$, according to Eq. (3.5), is weakly dependent on frequency $\delta\omega_1$. Thus, the photon reaction changes the frequency of a positronium atom by tens of gigahertz, which is much greater than the value W_1 in (1.1) caused by nonradiative interactions.

Let us now examine the role of the second term on the right-hand side of Eq. (2.27), which resembles a delta function. To this end we calculate the coefficient standing in front of the operator, just as we did in Sec. 3 when calculating $\tau_{2\gamma}^{-1}$. We then find that

$$\frac{1}{T_{2\gamma}}=\frac{1}{2}\tau_{2\gamma}^{-1}. \tag{4.6}$$

Here $T_{2\gamma}$ is a quantity similar to the phase relaxation time T_2' in coherent resonant optics.[13] The physical meaning of this quantity is as follows: $T_{2\gamma}$ is the time during which the mean value $\langle P\rangle$ tends to zero according to an exponential law owing to random phase factors determining the wave functions of the electron and positron in a positronium atom. This explains why in calculating n_0 in the conservation law (2.9a) we assumed $\langle P^+P\rangle=0$ initially. As a result of the interaction between the positronium atom and the photons a certain coherence sets in the system consisting of the positronium atom and the photon field, a coherence determined by the finite values of the means $\langle P\rangle$ and $\langle P^+\rangle$ and, hence, a relaxation time $T_{2\gamma}$.

Thus, this paper gives a fairly complete description of the interaction of the parapositronium atom with the photon field that makes it possible to determine the necessary characteristics. The proposed method of description also allows examining the three-photon annihilation decay of an orthopositronium atom. This problem will be examined in subsequent research.

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Translated by Eugene Yankovsky

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