Effect of finite nuclear size on vacuum polarization in heavy atoms

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We obtain a simple analytical expression, which is exact in the external field strength, for variations in the induced-charge density $\delta\rho_{\rm fs}(r)$ caused by the finiteness of the size R of the nucleus at distances much greater than R. We show that in calculating $\delta\rho_{\rm fs}(r)$ in heavy atoms via a perturbation-theory expansion in powers of the deviation of the potential from the Coulomb potential it is impossible to restrict oneself to the few first terms in the perturbation series; the entire series must be summed. We also examine the effect of $\delta\rho_{\rm fs}(r)$ on the charge-density distribution inside the nucleus. This is done by using the exact Green's function of an electron in a Coulomb field. The results are in good agreement with the known numerical results of other treatments.

1. INTRODUCTION

Measuring radiative level shifts of charged particles (electrons and muons) in high-Z hydrogenlike atoms is one of the chief methods of verifying the predictions of quantum electrodynamics (see recent papers by Stöhlker *et al.*[1] and Pross *et al.*[2]). The constantly growing accuracy of spectroscopic measurements requires consideration of ever more subtle effects in theoretical calculations. The main contributions to a radiative level shift are the self-energy of the charged particle and vacuum polarization. In calculating these effects in high-Z atoms the electric potential of the nucleus must be taken into account exactly in the parameter $Z\alpha$, where $Z|e|$ is the electric charge of the nucleus, $\alpha=e^2=1/137$ the fine-structure constant, e the electron charge, and $\hbar=c=1$.

Much effort has gone into studying the effect of vacuum polarization on energy shifts in atoms. This effect plays a special role in muonic atoms (see Ref. 3 and the literature cited there). In the pioneering work of Wichmann and Kroll[3] the Laplace transform of the product $\rho(r)r^2$ was calculated exactly in $Z\alpha$ for the case of a Coulomb potential. The density $\rho(r)$ itself was first analytically deduced in Ref. 5, while in Refs. 6 and 7 the behavior of the induced-charge density at small distances was studied by operator methods. There is also a vast literature on numerical studies of the induced-charge density and the respective potential (see, e.g., Refs. 3 and 8–13). The line shifts of atoms can be calculated from the induced-charge potential. These papers take into account only the spherically symmetric part of the induced-charge distribution, but there are high-Z nuclei with large multipole moments. The fields of such nuclei induce respective moments in the vacuum. The induced vacuum magnetic dipole and electric quadrupole moments were studied in Ref. 14.

In high-Z atoms the deviation of the nuclear potential from the Coulomb potential at small distances becomes important. Analytically, the effect of a finite nucleus on the induced-charge density was studied by Brown, Cahn, and McLerran[7] at distances r much larger than R but much smaller than λ_C, where R is the radius of the nucleus, $\lambda_C=1/m$ the electron Compton wavelength, and m the electron mass. For some nuclei and arbitrary distances this problem was studied numerically in the papers cited above. The effect of a finite nucleus on the contribution of the self-energy diagram was studied recently by Mohr and Soff.[15].

In this paper we analyze for $Z\alpha<1$ the effect of finite nuclear size on the induced-charge density at distances large compared to the radius of the nucleus. We derive a simple formula describing the addition to the induced-charge density caused by the fact that the potential differs from the Coulomb form. Our treatment uses the convenient integral representation for the Green's function of an electron in a Coulomb field.

2. THE GREEN'S FUNCTION AND THE INDUCED-CHARGE DENSITY

In conformity with the ordinary rules of the diagrammatic technique, the induced-charge density is given by the following formula:

$$\delta\rho(r)=-ie\int\frac{d\epsilon}{2\pi}\mathrm{Tr}[\gamma_0 G(\mathbf{r},\mathbf{r}'|\epsilon)], \tag{1}$$

where $G(\mathbf{r},\mathbf{r}'|\epsilon)$ is the electron Green's function, which we write as

$$G(\mathbf{r},\mathbf{r}'|\epsilon)=\left\langle \mathbf{r}\left|\frac{1}{\gamma^0(\epsilon+V(r))-\boldsymbol{\gamma}\mathbf{p}-m}\right|\mathbf{r}'\right\rangle. \tag{2}$$

Here the γ_μ are the Dirac matrices, and $V(r)$ the electron potential energy. In accordance with the Feynman rules, for $Z\alpha<1$ the path of integration with respect to energy ϵ passes from $-\infty$ to ∞ below the real axis in the left half-plane of the complex variable ϵ and above the axis in the right half-plane. The Green's function G has cuts along the

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real axis from $-\infty$ to $-m$ and from m to $+\infty$ corresponding to the continuous spectrum. It also has poles lying in the $(0,m)$ interval and corresponding to the discrete spectrum. The analytic properties of the Green's function make it possible to rotate the path of integration with respect to energy ϵ in (1) by $\pi/2$, so that it coincides with the imaginary axis.

We take the potential $V(r)$ in the form

$$V(r)=-\frac{Z\alpha}{r}+U(r). \tag{3}$$

The function $U(r)$ is the difference between the potential energy of an electron in the field of a nucleus of finite radius and that of an electron in a Coulomb field. Hence it is finite at distances smaller than the radius of the nucleus, but rapidly decreases on the scale on which the nuclear charge density $\rho(r)$ decreases.

We introduce the notation $\hat{P}=\gamma^0(\epsilon+Z\alpha/r)-\gamma\mathbf{p}$. By a direct expansion in $U(r)$ we can easily verify that

$$\frac{1}{\hat{P}-\gamma^0U(r)-m}=\frac{1}{\hat{P}-m}+\frac{1}{\hat{P}-m}\Big[\gamma^0U(r)+\gamma^0U(r)\frac{1}{\hat{P}-\gamma^0U(r)-m}\gamma^0U(r)\Big]\frac{1}{\hat{P}-m}. \tag{4}$$

We represent $\delta\rho(r)$ in the form

$$\delta\rho(r)=\delta\rho_C(r)+\delta\rho_{\rm fs}(r), \tag{5}$$

where $\delta\rho_C(r)$ is the induced charge density in the Coulomb field. Substituting (4) into (2) and the result into (1), we arrive at the following representation for the addition $\delta\rho_{\rm fs}(r)$ caused by the finite nucleus:

$$\delta\rho_{\rm fs}(r)=e\int\frac{d\epsilon}{2\pi}\int\int dr'\,dr''\,{\rm Tr}\{\gamma^0G_C(\mathbf{r},\mathbf{r}'\,|\,i\epsilon)\times[\gamma^0U(r')\delta(\mathbf{r}'-\mathbf{r}'')''+\gamma^0U(r')G(\mathbf{r}',\mathbf{r}''\,|\,i\epsilon)\gamma^0U(r'')]G_C(\mathbf{r}'',\mathbf{r}\,|\,i\epsilon)\}, \tag{6}$$

where $G_C(\mathbf{r},\mathbf{r}'\,|\,i\epsilon)$ is the Green's function of an electron in the Coulomb field. It can easily be seen that at least one of the arguments of the Green's functions in (6) is small because of the presence of the functions $U(r)$. Clearly, the angular momentum $j=\frac{1}{2}$ provides the main contribution to the angular–momentum expansion of the Green's functions. In Ref. 16 a convenient integral representation was obtained for the Green's function of an electron in the Coulomb field, G_C, valid in the entire complex ϵ plane. Using Eqs. (19) and (20) of Ref. 16, we find the contribution of the angular momentum $j=\frac{1}{2}$ to the Coulomb Green's function of an imaginary energy variable (for the sake of convenience in what follows we retain the notation G_C for this contribution):

$$G_C(\mathbf{r},\mathbf{r}'\,|\,i\epsilon)=-\frac{i}{4\pi rr'k}\int_0^\infty ds\exp\Big[\frac{2iZ\alpha\epsilon s}{k}-k(r+r')\coth s\Big]\Big\{[1-(\gamma\mathbf{n})(\gamma\mathbf{n}')]\Big[(\gamma^0\epsilon-im)\frac{y}{2}I'_{2\nu}(y)-iZ\alpha I_{2\nu}(y)\gamma^0k\coth s\Big]+I_{2\nu}(y)\Big[[1+(\gamma\mathbf{n})(\gamma\mathbf{n}')](\gamma^0\epsilon-im)+mZ\alpha\gamma^0(\gamma,\mathbf{n}+\mathbf{n}')-\frac{k^2(r-r')}{2\sinh^2 s}\times(\gamma,\mathbf{n}+\mathbf{n}')-(\gamma,\mathbf{n}-\mathbf{n}')k\coth s\Big]\Big\}. \tag{7}$$

Here $I_{2\nu}(y)$ $I_{2\nu}(y)$ is the modified Bessel function of the first kind, $\mathbf{n}=\mathbf{r}/r$, $\mathbf{n}'=\mathbf{r}'/r'$, $y=2k\sqrt{rr'}/\sinh s$, $\nu=\sqrt{1-(Z\alpha)^2}$, and $k=\sqrt{m^2+\epsilon^2}$. If $\lambda_C\sim r\gg r'$, the domain of integration with respect to ϵ providing the main contribution to $\delta\rho_{\rm fs}(r)$ is $\epsilon\sim 1/r$. Then in Eq. (7) the parameter s is of order unity and the argument of the Bessel function, $y\sim r'/r$, is much less than unity, and we can use the following asymptotic formula:

$$I_{2\nu}(y)\approx\frac{1}{\Gamma(2\nu+1)}\Big(\frac{x}{2}\Big)^{2\nu}. \tag{8}$$

But if $r\gg\lambda_C\gg r'$ holds, we have $\epsilon\sim\sqrt{\lambda_C/r}$, $k\approx m$, $e^{2s}\sim(r/\lambda_C)$, and $y\sim\sqrt{r'/\lambda_C}$ is much less than unity, with the result that we can again use the asymptotic formula for the Bessel function. Also in Eq. (7) it is convenient to perform the integration with respect to s in the term proportional to $r-r'$.

For further calculations the following method is expedient. We consider the case of a spherically symmetric distribution for the charge of the nucleus. Obviously, the induced charge density is also a spherically symmetric function of $\mathbf{r}$. Hence we can multiply both sides of (6) by $d\mathbf{n}/4\pi$ and integrate over the angles of the unit vector $\mathbf{n}$. After performing these transformations we arrive at the following formula for $\delta\rho_{\rm fs}(r)$:

$$\delta\rho_{\rm fs}(r)=\frac{2e}{\pi^2\Gamma^2(2\nu+1)r^{2(1-\nu)}}\int_{-\infty}^{\infty}d\epsilon\,k^{4\nu}\times\int\int_0^\infty\frac{ds_1ds_2(\sinh}{(\sinh s_1\sinh s_2)^{2\nu}}\times\exp\Big[\frac{2iZ\alpha\epsilon(s_1+s_2)}{k}-kr(\coth s_1+\coth s_2)\Big]\Big[\coth s_1\coth s_2+\frac{2m^2}{k^2}-1\Big]F, \tag{9}$$

where

$$F=\int_0^\infty dr'\ (r')^{2\nu}U(r')$$
$$+\frac{1}{32\pi}\mathrm{Tr}\Big\{\int d\mathbf{r}'\,d\mathbf{r}''\ (r'r'')^{\nu-1}U(r')U(r'')$$
$$\times[1-(\gamma\mathbf{n}'')(\gamma\mathbf{n}')+\nu[1+(\gamma\mathbf{n}'')(\gamma\mathbf{n}')]+iZ\alpha\gamma^0$$
$$\times(\gamma,\mathbf{n}'-\mathbf{n}'')]\gamma^0 G(\mathbf{r}',\mathbf{r}''|0). \tag{10}$$

In Eq. (10) we allowed for the fact that the values of ϵ providing the main contribution to the integrals in (9) are much smaller than $1/R$, with R the radius of the nucleus. The arguments of the Green's function G in (10) satisfy the inequalities $r',r''<R\ll\lambda_C$. Hence we can set the energy ϵ of the electron equal to zero in G in (10). At distances on the order of the nuclear radius the function G is also independent of the electron mass m. Thus, we see that the factor F in (9) is independent of distance r and the parameter of integration ϵ. On the other hand, the entire dependence on the charge distribution in the nucleus is contained in F. That is, the contributions to $\delta\rho_{\mathrm{fs}}(r)$ of large and small distances factorize. All the integrals in (9) have finite values and the expression obtained does not require renormalization.

3. CALCULATING FACTOR F

Let us now calculate the factor F. As in the case of Eq. (4), we can easily show that

$$\frac{1}{\hat{P}-\gamma^0U(r)-m}=\frac{1}{\hat{P}-m}+\frac{1}{\hat{P}-m}\gamma^0U(r)\frac{1}{\hat{P}-\gamma^0U(r)-m}, \tag{11}$$

or

$$G(\mathbf{r},\mathbf{r}'|i\epsilon)=GC(\mathbf{r},\mathbf{r}'|i\epsilon)$$
$$+\int d\mathbf{r}''\,G_C(\mathbf{r},\mathbf{r}''|i\epsilon)\gamma^0U(r'')G(\mathbf{r}'',\mathbf{r}'|i\epsilon). \tag{12}$$

We must find the asymptotic behavior of the Coulomb Green's function in the limit $kr\sim kr'\ll 1$. At such distances the main contribution to the integral in (7) is provided by small $s\sim kr$. Replacing $1/\sinh s$ and $\coth s$ by $1/s$ in (7) and using relations from Ref. 17, p. 303

$$\int_0^\infty\frac{dx}{x}I_a(x)e^{-px}=\frac{1}{a(p+\sqrt{p^2-1}\,)^a},$$
$$\int_0^\infty dx\,I_a(x)e^{-px}=\frac{1}{\sqrt{p^2-1}\,(p+\sqrt{p^2-1}\,)^a}, \tag{13}$$

we arrive at an asymptotic formula for the Coulomb Green's function at small distances:

$$G_C(\mathbf{r},\mathbf{r}'|i\epsilon)\approx\frac{i}{8\pi vrr'}\left[\Theta(r-r')\left(\frac{r'}{r}\right)^\nu+\Theta(r'-r)\left(\frac{r}{r'}\right)^\nu\right]$$
$$\times\{iZ\alpha\gamma^0[1-(\gamma\mathbf{n})(\gamma\mathbf{n}')]+(\gamma,\mathbf{n}-\mathbf{n}')$$
$$+\nu\,\mathrm{sign}(r-r')(\gamma,\mathbf{n}+\mathbf{n}')\}, \tag{14}$$

where $\Theta(x)$ is the Heaviside unit function. We see that this asymptotic behavior is independent of ϵ and the electron mass m. Equation (12) shows that this is also true of the asymptotic behavior of the Green's function.

We write Eq. (12) in the form

$$G(\mathbf{r},\mathbf{r}'|i\epsilon)\approx\gamma^0[1-(\gamma\mathbf{n})(\gamma\mathbf{n}')]A_1(r,r')+\gamma^0[1+(\gamma\mathbf{n})$$
$$\times(\gamma\mathbf{n}')]A_2(r,r')$$
$$+(\gamma,\mathbf{n}-\mathbf{n}')iZ\alpha A_3(r,r')$$
$$+(\gamma,\mathbf{n}+\mathbf{n}')iZ\alpha A_4(r,r'), \tag{15}$$

where A_1, A_2, A_3, and A_4 are certain functions. Substituting representation (15) into (12) and equating the coefficients of the respective matrix structures, we arrive at a system of linear integral equations for the functions A_1, A_2, A_3, and A_4. From (15) and (10) it follows that the factor F can be expressed in terms of the coefficient functions as follows:

$$F=\int_0^\infty dr\,r^{2\nu}U(r)$$
$$+4\pi\int\int_0^\infty dr\,dr'\ (rr')^{\nu+1}U(r)U(r')[A_1(r,r')$$
$$+\nu A_2(r',r')-(Z\alpha)^2A_3(r,r')]. \tag{16}$$

This expression shows that for our purposes it is sufficient to find the functions

$$a_i(r)=4\pi r^{\nu+1}\int_0^\infty dr'\ (r')^{\nu+1}U(r')A_i(r,r'),\quad i=1,2,3,4, \tag{17}$$

the equations for which can be set up by multiplying the left- and right-hand sides of the system of equations for the functions $A_i(r,r')$ by $4\pi(rr')^{\nu+1}U(r')$ and integrating with respect to dr'. The system of equations for the $a_i(r)$ can be conveniently written in the following form:

$$a_1(r)\mp\nu a_2(r)+(Z\alpha)^2a_3(r)=Z\alpha\int_0^\infty dx\ U(x)\theta_\pm[a_2(x)$$
$$\pm\nu a_3(x)-a_4(x)],$$
$$a_1(r)+a_3(r)\pm\nu a_4(a)=\frac{1}{Z\alpha}\int_0^\infty dx\ U(x)\theta_\pm[\pm\nu a_1(x)$$
$$+a_2(x)-(Z\alpha)^2a_4(x)\pm\nu x^{2\nu}], \tag{18}$$

where $\theta_+=\Theta(r-x)$, and $\theta_-=-(r/x)^{2\nu}\Theta(x-r)$. Combining Eqs. (18) and Eq. (16), we arrive at the following relations:

$$F=2\nu Z\alpha a_3(\infty),$$
$$a_1(\infty)+\nu a_2(\infty)=-(Z\alpha)^2 a_3(\infty), \tag{19}$$
$$a_1(\infty)+a_3(\infty)=\nu a_4(\infty).$$

Actually, since the integrals rapidly converge at the upper limit, the "infinity" in (18) and (19) means a distance greater than the size of the nucleus. Let us consider the functions

$$L(r)=\nu a_1(r)+a_2(r)+(Z\alpha)^2 a_4(r),$$
$$M(r)=Z\alpha[a_2(r)+\nu a_3(r)+a_4(r)], \tag{20}$$

for which simple differentiation of (18) yields

$$\nu\frac{d}{dr}M(r)=U(r)[L(r)+\nu r^{2\nu}-Z\alpha M(r)], \tag{21}$$
$$\nu\frac{d}{dr}r^{-2\nu}L(r)=U(r)r^{-2\nu}[Z\alpha(L(r)+\nu r^{2\nu})-M(r)].$$

Employing (19), we find that $M(\infty)=F$ and $L(\infty)=0$. Hence to calculate F we need only find the function $f(r)$:

$$f(r)=\frac{\nu r^{2\nu}M(r)}{L(r)+\nu r^{2\nu}}, \tag{22}$$

since $F=f(\infty)$. The function $f(r)$ satisfies a closed equation obtainable directly from (21):

$$\frac{d}{dr}f(r)=U(r)\left[r^{2\nu}-\frac{2Z\alpha f(r)}{\nu}+\frac{f^2(r)}{\nu^2 r^{2\nu}}\right]. \tag{23}$$

It is convenient to solve this equation by substituting $\nu r^{2\nu}H(r)$ for $f(r)$. It remains to establish the boundary condition imposed on the function $H(r)$ at $r=0$. For this we must go back to Eqs. (21) and establish the asymptotic behavior of the functions $M(r)$ and $L(r)$ as $r\to 0$. To find this behavior, we must leave in $U(r)$ the part that is singular as $r\to 0$, that is, $Z\alpha/r$. Then the solution of Eqs. (21) can be found in the form $M(r)=br^{\gamma}$ and $L(r)=cr^{\gamma}-\nu r^{2\nu}$, where b and c are constants. Substituting this representation into (21), we find that $\gamma=\nu\pm 1$. However, the solution with $\gamma=\nu-1$ does not satisfy the system of equations for $M(r)$ and $L(r)$ written in integral form because it makes the integral with respect to r divergent at the lower limit. The final result is $\gamma=\nu+1$ and $H(0)=b/c=Z\alpha/(1+\nu)$.

If with $Z\alpha\ll 1$ we solve Eq. (23) by iterations in $U(r)$, we obtain the function

$$f^1(r)=\int_0^r U(x)x^{2\nu}dx, \tag{24}$$

which for small r is approximately $f^1(r)\approx Z\alpha r^{2\nu}$ and is consistent with the asymptotic behavior of $f(r)\approx Z\alpha\nu r^{2\nu}/(1+\nu)$; the latter follows from the boundary condition on $H(0)$, but only when $Z\alpha\to 0$.

For different charge-density distributions $\rho(r)$ Fig. 1 depicts the Z-dependence of $F/\langle r^2\rangle^{\nu}$, where $\langle r^2\rangle=(Z|e|)^{-1}\int\rho(r)r^2 d\mathbf{r}$ is the mean-square charge radius of the nucleus. (The charge-density distributions are that of a uniformly charged ball and a uniformly charged sphere and the one used in Ref. 7, which is in good agreement with the distribution determined from electron-scattering experiments.) The parameters were chosen in such a way that in addition to the total charge Z being the same for all distributions, the mean-square radius $\langle r\rangle$ was also the same. It is clear that in all cases considered the functions $F(Z\alpha)$ are extremely close. To within a few percent they are described by the formula

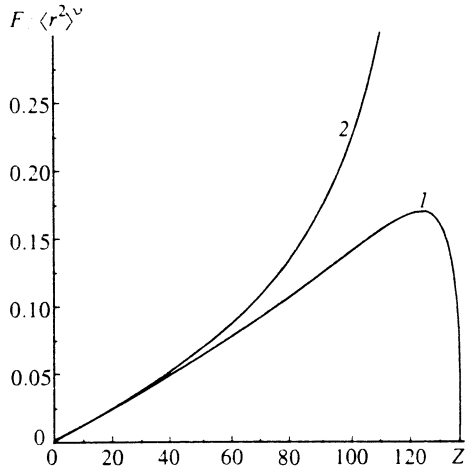

FIG. 1. Dependence of F (curve *1*) and factor F^1 calculated to first order in $U(r)$ (curve *2*) on nuclear charge Z.

$$F=\frac{Z\alpha\nu}{1+\nu}\left(\frac{\langle r^2\rangle}{3}\right)^{\nu}. \tag{25}$$

For comparison, Fig. 1 also shows the Z-dependence of $F^1/\langle r^2\rangle^{\nu}$, with $F^1=f^1(\infty)$ [see Eq. (24)]. Clearly, for $Z\alpha\sim 1$ the function F differs considerably from F^1 calculated to first order in $U(r)$.

Brown, Cahn, and McLerran[7] obtained an analytical expression for $\delta\rho_{\rm fs}(r)$ at a distance r much greater than R but much smaller than λ_C. The method that they used differs considerably from the one developed by us. For such distances we can deduce the asymptotic behavior of (9) and compare it with the results of Ref. 7. With allowance for factorization of large and small distances, the function F must coincide with the respective factor in Ref. 7. Anticipating what follows, we can say that these results coincide if in the right-hand side of Eq. (30) of Ref. 7, written for $H(r)$, we allow for an obviously omitted factor $r^{2\nu}$ (the factor is present in Eq. (31) of Ref. 7 written for the asymptotic part H^1).

4. ASYMPTOTIC BEHAVIOR

Let us study the behavior of the function $\delta\rho_{\rm fs}(r)$ at large and small distances (compared to the electron Compton wavelength). For $r\gg\lambda_C$ and $Z\alpha\sim 1$ the main contribution to the integrals in (9) is provided by the following range of variables: $s_{1,2}\sim\ln(mr)\gg 1$ and $|\epsilon|/m\sim(mr)^{-1/2}\ll 1$. Performing the necessary expansions and evaluating the elementary integrals, for $mr\gg 1$ we find

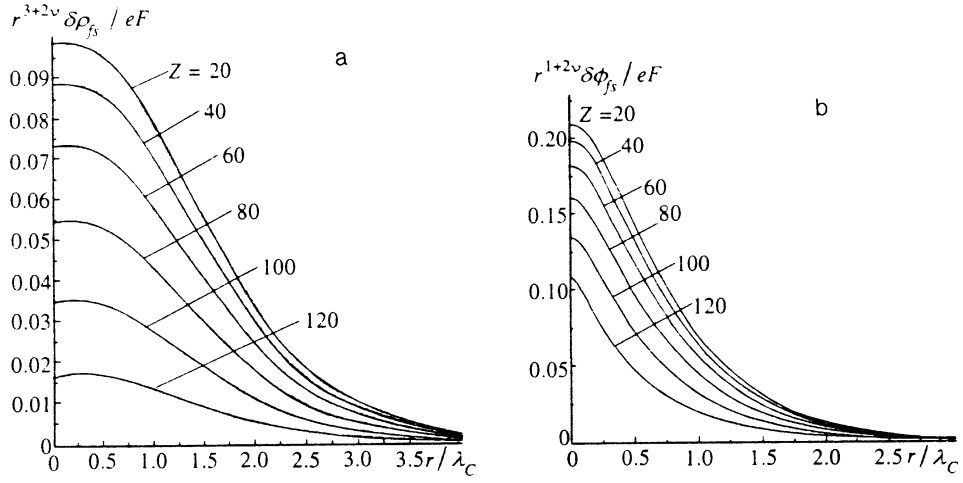

FIG. 2. (a) The r-dependence of $r^{3+2\nu}\delta\rho_{fs}(r)/eF(Z\alpha)$, and (b) the r-dependence of $r^{1+2\nu}\delta\phi_{fs}(r)/eF(Z\alpha)$ at various Z.

$$\delta\rho_{fs}(r)=\frac{em^{3+2\nu}F}{\sqrt{\pi}\,4^{\nu}\Gamma^2(\nu+1/2)}\frac{e^{-2mr}}{(mr)^{5/2}}\int_0^\infty dx\,x\exp\left[-\nu x-x\frac{x^2}{4mr}\right]. \tag{26}$$

For $\nu^2\gg 1/mr$ the integral in (26) is equal to $1/\nu^2$, and for $\nu^2\ll 1/mr$ the integral is equal to $2mr$.

At small distances $R\ll r\ll\lambda_C$ the main contribution to the integral with respect to ϵ is provided by the domain of integration $|\epsilon|\sim 1/r\gg m$. Replacing k with $|\epsilon|$ in (9), evaluating the elementary integral with respect to ϵ, and shifting to the variables $T=s_1+s_2$ and $\tau=s_1-s_2$, we obtain

$$\delta\rho_{fs}(r)=\left(\frac{eF}{r^{2\nu+3}}\right)\frac{\Gamma(4\nu+1)}{\pi^2 2^{2\nu}\Gamma^2(2\nu+1)}\int_0^\infty dT\int_0^T \times d\tau\,\frac{\cos(2Z\alpha T)\coth\tau\,(\coth T-\coth\tau)^{2\nu}}{(\sinh T)^{4\nu+1}}. \tag{27}$$

It is convenient to evaluate the integrals in (27) as follows. First we go from the variable τ to a new variable x via the substitution $\sinh x\sinh\tau=\sinh T/(\coth x+\coth T)$. Then we evaluate the integral with respect to T. The integral is expressed in terms of the Legendre polynomial of the first kind $P^{-2\nu-1/2}_{2iZ\alpha-1/2}(\coth z)$ and its derivative (see Eq. (8.713(3)) in Ref. 18), and then evaluated with respect to x via Eq. (7.132(2)) of Ref. 18. We arrive at the following asymptotic behavior of $\delta\rho_{fs}(r)$ at small distances:

$$\delta\rho_{fs}(r)=\left(\frac{eF}{r^{2\nu+3}}\right)\frac{2\nu\Gamma(4\nu+1)}{\pi^2(2\nu+1)}\left|\frac{\Gamma(\nu+iZ\alpha)}{\Gamma(2\nu+1)}\right|^4. \tag{28}$$

This result agrees with that of Ref. 7.

Now let us examine the limit of $Z\alpha\to 0$. Setting $Z\alpha=0$ in (9), we first evaluate the integrals with respect to s_1 and s_2 and then with respect to ϵ. Simple calculations lead us to

$$\delta\rho^{(0)}_{fs}(r)=\frac{2eF^{(0)}m^2}{\pi^2 r^3}\left[K_0(2mr)+\left(mr+\frac{1}{mr}\right)K_1(2mr)\right], \tag{29}$$

where $K_{0,1}(x)$ are modified Bessel functions of the second kind (modified Hankel functions), and $F^{(0)}=\int_0^\infty r^2U(r)dr$. This coincides with the result obtained from the ordinary relation in the momentum representation between the induced-charge density $\delta\rho(\mathbf{k})$, calculated in the lowest order, and the renormalized polarization operator $P(-\mathbf{k}^2)$ (see Sec. 114 of Ref. 19):

$$\delta\rho(\mathbf{k})=-V(\mathbf{k})P(-\mathbf{k}^2), \tag{30}$$

where $V(\mathbf{k})$ is the potential of the nucleus in the momentum representation. In (3) we replace $\delta\rho(\mathbf{k})$ and $V(\mathbf{k})$ with $\delta\rho_{fs}(\mathbf{k})$ and $U(\mathbf{k})$, respectively, and perform the inverse Fourier transformation. Since $U(r)$ is nonzero only at small distances of the order of the size of the nucleus, we can replace $U(\mathbf{k})$ with $U(\mathbf{k}=0)$ in the integral with respect to $\mathbf{k}$. After simple computations we arrive at (29).

5. CONCLUSION

We have obtained a simple analytical expression (9) that describes the induced-charge density at distances greater than the size of the nucleus and is exact in parameter $Z\alpha$. The expression is extremely convenient for tabulating. Comparison of our results with those of Gyilassy[9] shows that Eq. (9) is meaningful starting from distances $r\sim 8R$. Fig. 2a depicts the r-dependence of the induced-charge density $\delta\rho_{fs}(r)$ for different values of Z. Clearly, the dependence on Z is highly essential. If we allow for the fact that the net induced charge is zero, we can write the following expression linking the polarization potential $\delta\phi_{fs}(r)$ with $\delta\rho_{fs}(r)$:

$$\delta\phi_{fs}(r)=4\pi\int_r^\infty r'\left(1-\frac{r'}{r}\right)\delta\rho_{fs}(r'). \tag{31}$$

Figure 2b depicts $\delta\phi_{fs}(r)$ as a function of Z. Note once more that the entire dependence on the charge density of the nucleus is contained in the factor F [Eq. (10)], and for realistic densities $\rho(r)$ this factor is determined by the magnitude of the mean-square radius $\langle r^2\rangle$ [see Eq. (25)].

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