Stokes line widths, the uncertainty relation, and the region within which the exponentially weak wave superposed on the strong one develops

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(Submitted 11 January 1994)
Zh. Eksp. Teor. Fiz. **105**, 769–795 (April 1994)

A discussion is given of two Stokes linewidth conventions developed by Berry and by the authors for different asymptotic representations of the phase integral as a sum of the dominant and subdominant (recessive) waves. The narrower natural line width defined by the convention of the present work is explained by the more coherent composition of the main dominant wave; it is given by the uncertainty relation and appears as the zone of formation of the exponentially weak (recessive) wave which, within a power-law accuracy, possesses its own phase and which has a time parity opposite to that of the dominant wave. The present asymptotic representation, based on a specially devised separation of the high- and low-saddle-point contributions to the phase integral, is modified and, apart from the natural line width, allows one to find another—larger—Stokes strip width outside of which the recessive wave gets rid of its dominant-wave properties not with the power-law accuracy but rather with an exponential accuracy with which the dominant wave is separated from the recessive one outside the natural width. The modified representation is employed for the analysis of the solutions of the wave equation with a static uniform electric field, the dominant and recessive wave in these solutions describing respectively the propagation of a charged particle and the creation or absorption of a pair of particles by the field. The above-mentioned widths of the formation zone of the latter process turn out to be the Compton length m^{-1} and the barrier length $m/e\varepsilon$ (m and e are the mass and charge of the particle and ε is the electric field intensity).

1. INTRODUCTION AND DISCUSSION OF THE PROBLEM

Many physical quantities are represented by contour integrals of the form

$$F(\nu,\alpha)=\int_C dt\varphi(t)e^{f(t,\nu,\alpha)}, \tag{1}$$

dependent on two or more real parameters. Of particular interest are the asymptotic representations of these integrals when one of the parameters, ν, tends to infinity (where the integral has an essential singularity), and the other, α, is near the Stokes line. Then, restricting ourselves to the case of only two saddle points in $f(t)$, there are two terms in the asymptotic representation of the integral, one of which, the dominant one, is proportional to e^{f_2}, and the other, the subdominant (referred to in the present work as "recessive"), is exponentially small relative to the first and proportional to ige^{f_1}, where $\mathrm{Re}(f_2-f_1)\gg 1$ for $\nu\gg 1$. Here f_1 and f_2 are the values of $f(t)$ at the saddle points t_1 and t_2, respectively, and g switches on the recessive term. As a rule, $f_{1,2}$ are simple elementary functions of the parameters ν and α.

Qualitatively, the difference between the dominant and recessive terms lies in the different dependence of their phase $\mathrm{Im}\, f_{1,2}(\alpha)$ on α. It is therefore convenient to call them the dominant and recessive waves. Moreover, unlike the dominant term, the recessive one appears (or disappears) when α crosses the Stokes line $\alpha=0$—not abruptly, as suggested by Stokes,[1] but rather by a smooth change of the function $g(\nu,\alpha)$ from 0 to 1 (or from 1 to 0) over a certain effective interval $\Delta\alpha$ which may be referred to as the width of the Stokes line.[2–6]

In Ref. 6 two types of asymptotic representation were considered:

1. The usual asymptotic expansion in inverse powers of ν truncated after its smallest term, plus a residual term,[2–4]

2. A special division of the contour integral into contributions from the high and the low saddle points.

It was shown that the switching function of the recessive term is just the error function with an argument which is different for the two representations mentioned.[6] For type 2 representations this argument is $w = \sqrt{i\,\mathrm{Im}(f_1-f_2)}$, which leads to a natural Stokes line width determined by the interval over which the phase difference between the recessive and the dominant terms becomes of order unity. This implies the uncertainty relation $\Delta\alpha\Delta\omega\sim 1$ for the natural Stokes line width $\Delta\alpha$ and the difference $\Delta\omega=\omega_2(0)-\omega_1(0)$ of the phase change rates $\omega_{1,2}(\alpha)=-\partial\,\mathrm{Im} f_{1,2}(\alpha)/\partial\alpha$ near the Stokes line.

For type 1 representations, the argument of the switching function is $\xi=\mathrm{Im}(f_1-f_2)/\sqrt{2\,\mathrm{Re}(f_2-f_1)}$, so that the Stokes linewidth is $\sqrt{2\,\mathrm{Re}(f_2-f_1)}$ times the natural width. This means that the absolute value of the recessive terms varies much more slowly than the phase difference between the recessive and the dominant terms.

In order to understand why the Stokes line width for type 1 asymptotic representations is considerably larger than the natural width, let us turn to the integrals $F(z)$

discussed in Ref. 6, in which the parameters ν and α are the magnitude and phase of one complex parameter $z=|z|e^{i(\alpha+\arg z_s)}$ or, more precisely, $\nu=|z|$, and $\alpha=\arg z-\arg z_s$, the deviation of the argument of z from that of the Stokes line. For $|z|\gg 1$, in the type 1 asymptotic representation $F=S_m+R_m$ the dominant term S_m was the sum of the terms of the asymptotic inverse-power series in z down to the minimal, mth term, with coefficients growing as the Γ-function of their index,

$$S_m=d_0\sum_{k=0}^{m-1}\Gamma(ak+b)(cz)^{-dk},\quad a,b,c,d\sim 1\,. \tag{2}$$

Here $d_0\sim e^{f_2}$ and it is assumed that $\arg z_s=0$. For $|z|\gg 1$, the remote terms of this sum, within $|k-m|\lesssim\sqrt{2m/a}$ of the minimum term with number $k=m=a_{-1}(c|z|)^{d/a}\gg 1$, are described by

$$\frac{\Gamma(ak+b)}{(cz)^{dk}}\approx\sqrt{2\pi}\left(\frac{am}{e}\right)^{b-1/2}\exp\left(-am+\frac{a\delta^2}{2m}-idk\alpha\right),$$

$$\delta=k-m. \tag{3}$$

It is seen that on the order of $\sqrt{2m/a}$ terms near the minimal one are approximately of the same magnitude $\sim e^{-am}$ and that close to the Stokes line, when $d\sqrt{2m/a}|\alpha|\lesssim 1$ holds, they also have the same phase $\approx -dm\alpha$ and so may be considered coherent. Comparison with the recessive wave R_m shows that close to the Stokes line the sum of remote coherent terms in S_m has the same phase as R_m (shifted by $\pm\pi/2$) and that its magnitude is $\sim|R_m|$. This means that near the Stokes line recessive properties are present in the dominant wave S_m to the same degree as they are in the recessive wave R_m, i.e., they are poorly distinguished from the dominant properties. In both directions away from the Stokes line, the coherence of the remote terms of S_m is getting destroyed and as one gets outside the line width, the recessive properties either became fully concentrated in R_m or, together with R_m, disappear altogether.

In fact, the behavior of the sum of the remote coherent terms is accurately given by a model obtained by neglecting variation in the magnitude of the terms,

$$\begin{aligned}
&d_0\sum_{k=m-r}^{m-1}\Gamma(ak+b)(cz)^{-dk}\\
&\approx d_0\sqrt{2\pi}\left(\frac{am}{e}\right)^{b-1/2}e^{-am}\sum_{k=m-r}^{m-1}e^{-ik\gamma}\\
&=d_0\sqrt{2\pi}\left(\frac{am}{e}\right)^{b-1/2}e^{-am-im\gamma+i\frac{r+1}{2}\gamma}\cdot\frac{\sin(r\gamma/2)}{\sin(\gamma/2)},
\end{aligned} \tag{4}$$

where $r\approx\sqrt{2m/a}$, $\gamma=d\alpha$. From Ref. 6 it follows that $am+im\gamma=f_2-f_1$. It is seen that the sum (4) is in phase with the recessive wave and that its magnitude vanishes for $|\gamma|\gtrsim 2/r\approx\sqrt{2a/m}$, or

$$|\mathrm{Im}(f_1-f_2)|\gtrsim\sqrt{2am}=\sqrt{2\,\mathrm{Re}(f_2-f_1)}\,, \tag{5}$$

which is just the condition for getting outside the Stokes line width. Thus, until the condition (5) is fulfilled, the dominant term S_m does indeed have a term whose magnitude and phase are characteristic of the recessive term.

The type 2 asymptotic representation is based on the representation of $F(z)$ by the contour integral

$$F(z)=A\int_c dt\,e^{f(t,z)}\,, \tag{6}$$

[with $\mathrm{Re}\,f(t,z)=-\infty$ at the ends $t=t'_\infty,t''_\infty$ of the integration contour] and employs the saddle point method; z is one complex parameter or a pair of two real parameters. The dominant (D) and recessive (R) terms in the representation $F(z)=D+R$ are the contributions from the high and the low saddle points located at points t_1 and t_2 on the complex-t plane. When the parameter z is close to the Stokes line, the line of steepest descent (LDS) from the high saddle point t_2 passes near the low saddle point t_1 crossing the level line of the latter at point $t=t_*(z)$. By definition,[6]

$$D(z)=A\int_{t'_\infty}^{t_*}dt\,e^{f(t,z)},\quad R(z)=A\int_{t_*}^{t''_\infty}dt\,e^{f(t,z)}\,. \tag{7}$$

Tracing the contour of integration in D along the line of steepest descent from the saddle point t_2 (LDS_2) and using the fact that $\mathrm{Im}\,f(t,z)=\mathrm{Im}\,f(t_2,z)$ along this line we obtain

$$D(z)=Ae^{i\,\mathrm{Im}\,f_2}\cdot\int_{t'_\infty}^{t_*}dt\,e^{\mathrm{Re}\,f(t,z)}\approx A\sqrt{-\frac{2\pi}{f''_2}}e^{f_2}\,. \tag{8}$$

Here and below $f_{1,2}\equiv f(t_{1,2},z)$ are functions of z. The expression on the right is obtained by the saddle-point method under the assumption that t_* is far outside of the domain of influence of the saddle point t_2. At the same time, using the proximity of t_* to the saddle point t_1, the same method yields[6]

$$R(z)\approx A\sqrt{-\frac{\pi}{2f''_1}}e^{f_1}\mathrm{erfc}(w),$$

$$w=w_{L,R}(z)=\pm e^{i\pi/4}\sqrt{\mathrm{Im}(f_1-f_2)} \tag{9}$$

with the switching function $g=1/2\,\mathrm{erfc}(w)$, where $\mathrm{erfc}(w)$ is the complementary error function. Since close to the Stokes line $\mathrm{Im}(f_1-f_2)$ is an odd function of α, we have $w(\alpha)=-w^*(-\alpha)$ and the switching function has the property

$$g(\alpha)+g^*(-\alpha)=1\,. \tag{10}$$

Since the integrand of (8) is real along the LDS, the phase of the integral differs from $\arg(-2\pi/f''_2)^{1/2}$ only because of the turn the path of integration makes relative the direction it has at the saddle point t_2. Now for all the cases of which we are aware, the turning angle of the LDS_2 in its course from the saddle point t_2 to the point t_* does not exceed $\pi/2$. Hence even the exponentially small contributions $\sim e^{\mathrm{Re}\,f_1}$ in the exact dominant integral D have a difference in phase from D of no more than $\pi/2$, thus suggesting that D consists of coherent contributions. This means that when because of the variation of α point t_* leaves the domain of influence of the low saddle point and the phase difference between the recessive and dominant waves becomes appreciably larger than $\pi/2$,

$$|\mathrm{Im}(f_1-f_2)|\gtrsim\pi/2\,, \tag{11}$$

then the dominant wave will no longer have any contribution with phase and magnitude characteristic of the leading term of the recessive wave; [see Eq. (17) for Re $w\ll-1$].

In fact, in this case

$$D\approx F,\quad |R|\ll|R_S|,\quad R_S\equiv R(z_S)\,, \tag{12}$$

if α is before the broadened Stokes line (i.e., the Stokes strip) and

$$D\approx F_2=A\int_{t'_\infty}^{t_\infty}dte^{f(t,z)},\quad R\approx F_1=A\int_{t_\infty}^{t''_\infty}dte^{f(t,z)}\,, \tag{13}$$

if α is behind the Stokes strip. Here t_∞ is the infinite point into which the end of the LDS_2 is relocated when the parameter α crosses the Stokes line.

On the other hand, inside the Stokes strip, i.e., for $|w|\lesssim1$, the exact dominant integral D will contain an exponentially small contribution of order R, whose phase, at $\alpha=0$, differs from that of R by the angle $\delta_{L,R}$ of the left (or right) turn of the LDS_2 at the saddle point t_1 ($\delta_{L,R}=\pm\pi/2$ for a simple saddle and $\pm2\pi/3$ for a second-order saddle). This contribution may be manifest in certain quantities of the form $F^*\partial F/\partial\alpha-F\partial F^*/\partial\alpha$, sensitive to the interference between the D and R waves. Therefore when calculating such quantities inside the Stokes strip by approximating R by (9), it is not sufficient to use for D the approximation (8). However, near the center of the Stokes strip the contribution to D may be transferred to the recessive term by defining this, as well as the dominant term, as the half-sum of analytically continued expressions valid for R and D on both sides of, and far away from, the Stokes strip:

$$\tilde{R}=\frac{1}{2}F_1,\quad \tilde{D}=\frac{1}{2}(F+F_1) \tag{14}$$

[see Eqs. (12) and (13)]. In this case, $\tilde{D}$ will again by approximated by (8) and $\tilde{R}$, by (9) with $\mathrm{erfc}(w)\to1$. If the saddle point t_1 is of order higher than unity, then the ambiguity as to the choice of a common terminal point t_∞ for the contours of the F_1 and F_2 integrals is eliminated by the requirement that the corresponding LDS_1 near t_1 (for $\alpha=0$) be symmetric with respect to the LDS_2 descending to t_1.

2. QUALITATIVE CHARACTERISTICS OF THE DOMINANT AND RECESSIVE WAVES

The existence of the recessive wave can be detected from its interference with the dominant wave, i.e., from the rate $\Delta\omega$ of change of the relative phase

$$\mathrm{Im}(f_1-f_2)=\Delta\omega\cdot\alpha+...,\quad \Delta\omega=\omega_2(0)-\omega_1(0),$$

$$\omega_{1,2}(\alpha)=-\frac{\partial\,\mathrm{Im}f_{1,2}(\alpha)}{\partial\alpha} \tag{15}$$

of these waves away from the Stokes line. It is evident that this interference occurs in the interval $\Delta\alpha$ satisfying the uncertainty relation

$$\Delta\alpha\cdot\Delta\omega\gtrsim1. \tag{16}$$

However, the recessive wave forms within such an interval $\Delta\alpha_R$ where its amplitude develops. Clearly, $\Delta\alpha_R$ cannot be less than $1/\Delta\omega$ because over an interval this small, there is not enough time for interference to occur, and so the recessive wave cannot be separated from its dominant counterpart. It may be expected, however, that apart from the natural width a larger width should exist. These two widths are encountered when using two methods, namely, the integral approach and the asymptotic expansion in powers of ν^{-1}, to represent the same quantity $F(\nu,\alpha)$ as a sum of the recessive and dominant waves.

In the integral approach the dominant wave D is defined such that the phases of all contributions into D, down to the exponentially small ones, do not differ by more than $\pi/2$, with the implication that D is made up of coherent components.

On the other hand, when expanding in an asymptotic series of the type (2), the phase of the kth term of S_m, equal to Im $f_2-kd\alpha$, varies linearly with number k, coinciding at $k=0$ with the phase of the dominant wave and at $k=m$, with that of the recessive wave, Im $f_2-md\alpha$ $=$Im f_1. Therefore even for α well outside of the interval $\Delta\alpha_R\sim1/\Delta\omega$, when the R and D waves are quite separate from the uncertainty relation point of view, i.e., $|\mathrm{Im}(f_1-f_2)|\approx md|\alpha|\gg\pi$, the sum of the last r terms of S_m may have a phase different from that of the recessive wave by no more than π if

$$rd|\alpha|=\frac{r}{m}|\mathrm{Im}(f_1-f_2)|<\pi, \tag{17}$$

i.e., this group of terms is not coherent with the dominant but rather with the recessive wave. If $1\ll r\ll m$, this sum may have the order of magnitude of the recessive wave and the interval $\delta\alpha\sim\pi/rd$, in which this coherence is conserved, will be large compared to the minimal interval $\sim1/\Delta\omega\sim1/m$ necessary to distinguish the recessive wave from the dominant. The number r of coherent terms is determined by the dependence of the absolute magnitude of the terms of the asymptotic series on their index and on the parameter ν near the minimum and therefore r (as well as m) is a function of ν. For a series whose terms behave like $\Gamma(k+b)\nu^{-k}$, the index satisfies $r\sim\sqrt{m}\sim\sqrt{\nu}$.

Formally, the kth term in the sum S_m may be considered near the Stokes line as a wave with a "frequency" $\omega_k=\omega_2(0)+kd=\omega_1(0)-(m-k)d$. The frequency interval $\delta\omega\sim rd$ of the last r terms of S_m (coherent with the recessive wave) is the uncertainty in the frequency of the recessive wave and hence of the difference $\Delta\omega$. Since this uncertainty, $\delta\omega\sim(r/m)\Delta\omega$, is small compared to $\Delta\omega$, it follows that the corresponding interval $\delta\alpha_{R_m}$ $\sim(m/r)\Delta\alpha_R\gg\delta\alpha_R$.

Thus, the slower rate of formation of the wave R_m relative to R is due to the fact that, in accordance with the representation $F=S_m+R_m$, we transfer to the dominant wave S_m a number of terms coherent to the recessive wave. Although the number r of these wave components is large, still it is small in comparison with m and so is insufficient to secure rapid formation of the recessive wave.

The important qualitative difference between the dominant and recessive waves lies in their behavior under the T transformation, which consists of the replacement $\alpha\to-\alpha$ and a complex conjugation. In this transformation,

$$f_{1,2}(\alpha)\to f^*_{1,2}(-\alpha)=f_{1,2}(\alpha),$$

$$w(\alpha)\to w^*(-\alpha)=-w(\alpha)\,. \tag{18}$$

Therefore, up to an irrelevant phase factor and an exponentially small correction δ, the dominant wave D transforms into itself, $D\to D+\delta$, and so possesses, in some sense, a positive T parity, whereas the recessive wave, up to the same factor, transforms into

$$R\sim ig(w)e^{f_1}\to -ig(-w)e^{f_1} \tag{19}$$

and does not have any definite T parity with the Stokes strip since $g(-w)=1-g(w)$ according to (10). However outside of the Stokes strip, when $|w|\gg1$,

$$ig(w)e^{f_1}\approx\begin{cases}\dfrac{ie^{f_1-w^2}}{2\sqrt{\pi}w}, & \operatorname{Re}w\gg1\,,\\[2ex] ie^{f_1}+\dfrac{ie^{f_1-w^2}}{2\sqrt{\pi}w}, & \operatorname{Re}w\ll-1\,.\end{cases} \tag{20}$$

Therefore, before the Stokes strip the R wave is $2\sqrt{\pi}|w|$ times smaller than its Stokes line value R_s, remains unchanged under the T inversion, and its phase is that of the dominant wave shifted by $\arg(iw^{-1})$ because $f_1-w^2=\operatorname{Re}f_1+i\operatorname{Im}f_2$. Behind the Stokes strip the wave $R\approx2R_s$ possesses its own phase $\operatorname{Im}f_1+\pi/2$, changes sign under the T inversion, and is accompanied by a small correction of the same type as was R before the strip.

Thus within the Stokes strip a recessive wave with a frequency $\omega_1\neq\psi_2$ and negative T parity is formed.

As to the behavior of the dominant and recessive waves under the T transformation in the representation $F=S_m+R_m$, we note that in the examples discussed in Ref. 6 the term S_m transforms into itself, within the same factor as D but without any δ correction, i.e., $S_m\to S_m$; whereas R_m behaves according to Eqs. (19) and (20) with w replaced by a real ξ, see Sec. I. Therefore the phase of the R_m wave being formed is $\operatorname{Im}f_1+\pi/2$ and its T parity changes from positive to negative while remaining undefined within the Stokes strip. Since $D=S_m+R_m-R$, it follows that the δ correction, having no definite T parity inside the wide Stokes strip, vanishes outside of the strip like $\delta\sim ie^{f_1-\xi^2}/2\sqrt{\pi}\xi$.

3. TIME-DEPENDENT SOLUTIONS OF THE WAVE EQUATION WITH A STATIC UNIFORM ELECTRIC FIELD

The solutions of the wave equation for a particle of charge e and mass m in a static uniform electric field ε are expressed in terms of parabolic cylinder functions. We will consider time-dependent solutions characterized by the eigenvalues p_1, p_2, p_3 of the momentum operator $-i\partial/\partial\mathbf{x}$. Solutions of this type occur when the field ε is described by a time-dependent potential $\mathbf{A}=-\varepsilon x^0$, $A^0=0$. In this case, for a scalar particle we have two independent solutions to the wave equation,

$${}^{+}_{-}\Phi_{\mathbf{p}}=e^{i\mathbf{px}}D_{-i\nu-1/2}(\pm e^{i\pi/4}z)\,, \tag{21}$$

$$\nu=\frac{m_\perp^2}{2|e\varepsilon|},\quad z=\sqrt{2|e\varepsilon|}\left(x^0+\frac{p_3}{e\varepsilon}\right),$$

$$m_\perp^2=m^2+p_\perp^2,\quad p_\perp^2=p_1^2+p_2^2. \tag{22}$$

Here and in what follows the axis 3 is chosen parallel to the field ε.

Because of the formation of pairs by the electric field, each one of the solutions (21) contains both positive- and negative-frequency waves. However, the solution ${}^{+}\Phi$ for $x^0\to+\infty$ contains only a positive-frequency quasiclassical wave, whereas the solution ${}_{-}\Phi$ for $x^0\to-\infty$ contains only a negative-frequency quasiclassical wave,

$${}^{+}_{-}\Phi_{\mathbf{p}}\propto e^{iS_\pm+i\mathbf{px}},\quad x^0\to\pm\infty,\quad S_\pm=\mp\nu\xi(\theta),$$

$$\xi(\theta)=\theta\sqrt{1+\theta^2}+\operatorname{arcsinh}\theta,\quad \theta=z/2\sqrt{\nu}\,. \tag{23}$$

Here $S_\pm$ are the positive-frequency and negative-frequency action functions. In this case the wave frequency

$$\pi^0+eA^0=-\frac{\partial S_\pm}{\partial x^0}=\pm\sqrt{m_\perp^2+\pi_3^2}=\pm m_\perp\sqrt{1+\theta^2},$$

$$\pi_3=p_3+e\varepsilon x^0\,, \tag{24}$$

is equal in absolute value to the kinetic energy of the charged particle.

It is known that, together with the functions $D_\lambda(\pm x)$, the functions $D_{-\lambda-1}(\pm ix)$ satisfy the same equation for the parabolic cylinder functions; see §8.2 in Ref. 7. Any pair of these functions form a system of linearly independent solutions. In our case the functions of the last pair are complex conjugate to those of the first, and if we substitute them for the D functions in (21) we obtain another system,

$${}^{-}_{+}\Phi_{\mathbf{p}}=e^{i\mathbf{px}}D_{i\nu-1/2}(\pm e^{-i\pi/4}z) \tag{25}$$

of linearly independent solutions of the wave equation. For $x^0\to\infty$, the solution ${}^{-}\Phi$ contains only a negative-frequency quasiclassical wave, while for $x^0\to-\infty$ the solution ${}_{+}\Phi$ contains only a positive-frequency quasiclassical wave:

$${}^{-}_{+}\Phi_{\mathbf{p}}\propto e^{iS_\mp+i\mathbf{px}},\quad x^0\to\pm\infty\,. \tag{26}$$

Since the Feynman propagator propagates waves with positive frequency forward in time and those with negative frequency backward, it is convenient to call (21) a Feynman system, and (25), an anti-Feynman system. Note that the interaction between the charged particle and the electromagnetic field has the consequence that in the analytic dependence of the Feynman and anti-Feynman solutions on the square of the mass, there appears in this last quantity an imaginary addition of the appropriate sign,

$$\mp i\nu-\frac{1}{2}=\mp i\frac{m_\perp^2\mp i|e\varepsilon|}{2|e\varepsilon|}. \tag{27}$$

Now let us employ the representation[8]

$$D_{-i\nu-\mu}(e^{i\pi/4}z)=A(\nu,\mu)\int_C dt t^{-\mu}e^{f(t,\nu,\theta)}, \tag{28}$$

$$f(t,\nu,\theta)=i\nu(\tfrac{1}{2}t^2+20t-\mathrm{lu}\;t+\theta^2),$$

$$A(\theta,\mu)=\left(\frac{\nu^{1-\mu-i\nu}}{2\pi i}\right)^{1/2}e^{-3\pi\nu/4+3i\pi\mu/4},$$

in which the contour of integration C starts and terminates at infinity in the third and first quadrants of the complex-t plane and goes around the branch point $t=0$ clockwise. At $\mu=1/2$ this representation describes both of the solutions (21) which differ only in the sign of the parameter θ. The spinor solutions are determined by a linear combination of the two functions (28) with $\mu=0$, 1. In this case $\mu-1/2$ is half of the eigenvalue of the T-odd Dirac matrix α_3.

The equation $f'(t)=0$ yields two real saddle points

$$t_{1,2}(\theta)=-\theta\pm\sqrt{1+\theta^2},\quad t_1>0,\quad t_2<0,\quad t_1\cdot t_2=-1. \tag{29}$$

At these points

$$\mathrm{Re}\,f(t_1)=0,\quad \mathrm{Re}\,f(t_2)=\pi\nu, \tag{30}$$

i.e., the saddle point t_2 is high, t_1 low, and their heights are independent of θ.

Examination of the LDS_1 and LDS_2 determined by the equations $\mathrm{Im}\,f(t,\nu,\theta)=\mathrm{Im}\,f_{1,2}$ shows that for $\theta>0$ the contour C can be transferred into the LDS_2, which goes through the saddle point t_2 only. Such an LDS_2 is described by the curve $\rho=\rho(\varphi)$ satisfying the equation

$$\rho\cos\varphi=-\theta\mp\left[\frac{1}{2}\rho^2+\ln\frac{\rho}{\rho_2}-\theta\sqrt{1+\theta^2}+\frac{1}{2}\right]^{1/2}, \tag{31}$$

$$t=\rho e^{i\varphi},\quad \rho=|t|,\quad \varphi=\arg t,\quad \rho_2=|t_2|.$$

The upper and lower signs refer to the parts of LDS_2 separated by the point of the minimum of $\rho(\varphi)$, where $\rho'(\varphi)=0$, and the expression in the brackets under the square root sign vanishes. For $\rho\to\infty$ the left and right ends of LDS_2 approach to the asymptote $\mathrm{Im}\,t=\mathrm{Re}\,t+2\theta$, remaining respectively at the left and the right of it. An example of such an LDS_2 is depicted in Fig. 1. For $\theta=0$, the LDS from the high saddle point $t_2=-1$ comes to the low saddle point $t_1=1$ where it undergoes a $\pi/2$ kink to the left and moves off to infinity. Such an LDS is described by the same Eq. (31), with $\theta=0$, $\rho=1$, and is also shown in Fig. 1.

Finally, for $\theta<0$ the contour C along a LDS is given by the sum $C=C_2+C_1$. The contours C_2 and C_1 follow the LDS_2 and LDS_1 respectively and have both their ends at infinity. Namely, the contour C_2 starts at infinity in the third quadrant of the t plane, passes over the saddle point t_2, bypasses clockwise the branch point $t=0$ and goes below the cut $\mathrm{Re}\,t\geqslant 0$ onto the adjacent sheet $-2\pi\leqslant\arg t<0$ of the Riemann surface. Following the LDS_2 in this sheet, the contour arrives at the saddle point $t_2=|t_2|e^{-i\pi}$, where it undergoes a $\pi/2$ kink to the left and goes to infinity in the quadrant $-\pi\leqslant\arg t<-\pi/2$ of this sheet. Note that the height of the saddle point t_2 on his sheet is $e^{-2\pi\nu}$ times its basic sheet value because

$$\mathrm{Re}\,f(|t_2|e^{\mp i\pi},\nu,\theta)=\mp\pi\nu. \tag{32}$$

[cf. Eq. (30)]. The contour C_2 is described by Eq. (31) with $\theta<0$.

The contour C_1 starts at the end point of C_2, but runs along LDS_1 to the saddle point t_1 and then moves onto the basic sheet and goes off to infinity by approaching the same asymptote from the right. The left and right parts of this contour are described by Eq. (31), under the square root sign of which one must change the sign of θ and replace ρ_2 by $\rho_1=|t_1|$. The contours C_1 and C_2 going along the LDS_1 and LDS_2 at $\theta<0$ are shown in Fig. 2. Figure 3 shows the 180° turn of the LDS_2 near the low saddle point, due to the change in sign of θ. It is clear that the point $\theta=0$ belongs to the Stokes line in the complex-θ plane, see below. Since

$$\mathrm{Im}(f_1-f_2)=2\nu\xi(\theta), \tag{33}$$

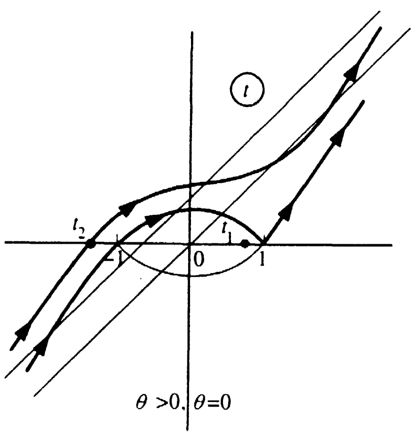

FIG. 1.

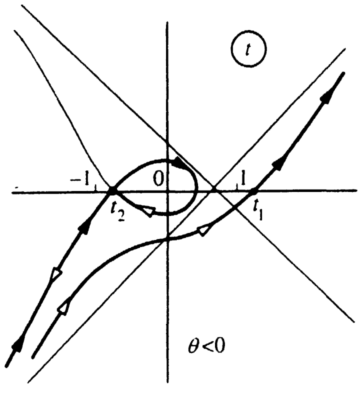

FIG. 2.

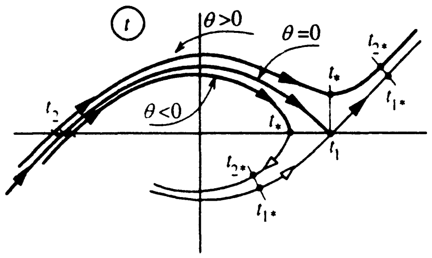

FIG. 3.

we have that, according to the condition $|w|\lesssim 1$, the natural Stokes line width $\Delta\theta\sim(4\nu)^{-1}\ll 1$.

Thus, for $\theta>0$ we can take the contour the integral (28) along the LDS_2 through the only saddle point t_2, and then evaluate the asymptotics by a standard method.[9] Then we obtain for $\nu\gg 1$, $\theta\gg(4\nu)^{-1}$

$$\begin{aligned} D_{-i\nu-\mu}(ze^{i\pi/4}) &\approx Ae^{f_2}\sqrt{-\frac{2\pi}{f_2''}}\,t_2^{-\mu} \\ &= Q(\nu,\mu)a_\mu(\theta)e^{-i\nu\xi(\theta)}, \\ Q(\nu,\mu) &= (2\nu^{\mu+i\nu})^{-1/2}e^{\pi\nu/4-i\pi\mu+i\nu/2}, \\ a_\mu(\theta) &= (\theta+\sqrt{1+\theta^2})^{1/2-\mu}(1+\theta^2)^{-1/4}. \end{aligned} \tag{34}$$

Note that the corrections to the functions $\xi(\theta)$ and $a_\mu(\theta)$ are real and of order ν^{-2} for all θ. For $\theta<0$, we can choose the contour for the integral (28) to go along LDS_2+LDS_1 and calculate the asymptotic forms of the corresponding integrals. Then for $\nu\gg 1$, $\theta\ll-(4\nu)^{-1}$ we obtain

$$\begin{aligned} D_{-i\nu-\mu}(ze^{i\pi/4}) &\approx A\left\{e^{f_2}\sqrt{-\frac{2\pi}{f_2''}}\,t_2^{-\mu}\cdot K+e^{f_1}\sqrt{-\frac{2\pi}{f_1''}}\,t_1^{-\mu}\right\} \\ &= Q\{Ka_\mu(\theta)e^{-i\nu\xi(\theta)} \\ &\quad +e^{-\pi\nu+i\pi\mu}a_\mu(-\theta)e^{i\nu\xi(\theta)}\}, \end{aligned} \tag{35}$$

where the factor $K=1-\frac{1}{2}e^{-2\pi\nu+2i\pi\mu}$ takes account of the contribution from the saddle point on the second sheet.

The asymptotic expressions (34) and (35) agree with the exact relation for the D functions,[7]

$$\begin{aligned} D_{-i\nu-\mu}(ze^{i\pi/4}) &= e^{i\pi\mu-\pi\nu}D_{-i\nu-\mu}(-ze^{i\pi/4}) \\ &\quad +\frac{\sqrt{2\pi}e^{-\pi\nu/2-i\pi(1-\mu)/2}}{\Gamma(\mu+i\nu)} \\ &\quad \times D_{i\nu-\mu'}(-ze^{-i\pi/4}), \end{aligned} \tag{36}$$

which represents $D_{-i\nu-\mu}(ze^{i\pi/4})$ in terms of a) this function itself with an opposite sign of z, and b) the complex conjugate of a) with μ replaced by $\mu'=1-\mu$, because for z real

$$D_{i\nu-\mu}(ze^{-i\pi/4})=D^*_{-i\nu-\mu}(ze^{i\pi/4}). \tag{37}$$

In fact if we take $z<0$ then, substituting into the right-hand side of (36) the asymptotic expression (34), using the Stirling formula

$$\frac{\sqrt{2\pi}}{\Gamma(\mu+i\nu)}\approx e^{\pi\nu/2-i\pi\mu/2+i\pi/4+i\nu}\nu^{1/2-\mu-i\nu}(1-\tfrac{1}{2}e^{-2\pi\nu+2i\pi\mu}),$$
$$\nu\gg 1, \tag{38}$$

and taking account of the recessive term, we obtain the asymptotic formula (35). The asymptotic expressions (34) and (35) also reproduce, with the proper accuracy $\sim\nu^{-1}\ll 1$, the dominant and the recessive terms of the exact Wronskians

$$D_{-i\nu-\mu}(e^{i\pi/4}z)\frac{\overleftrightarrow{d}}{dz}D_{-i\nu-\mu}(-e^{i\pi/4}z)=\frac{\sqrt{2\pi}e^{i\pi/4}}{\Gamma(\mu+i\nu)}, \tag{39}$$

$$\begin{aligned} &D_{-i\nu-\mu}(e^{i\pi/4}z)\frac{\overleftrightarrow{d}}{dz}D^*_{-i\nu-\mu'}(-e^{i\pi/4}z) \\ &=e^{-\pi\nu/2+i\pi(\mu-1/2)/2}, \quad \mu'=1-\mu, \end{aligned} \tag{40}$$

see again Eq. (38).

The relation (36) clearly demonstrates the behavior of the D functions under time reversal, especially when rewritten in the form

$$B(\mu,z)=e^{-\pi\nu+i\pi\mu}B(\mu,-z)+\sqrt{1-e^{-2\pi\nu+i2\pi\mu}}\,\tilde{B}(\mu,z), \tag{41}$$

where $B(\mu,z)=K^{-1/2}D_{-i\nu-\mu}(ze^{i\pi/4})$, K denoting the coefficient in front of D in the second term in (36). It is seen that under the T transformation, which is designated by the tilde sign and consists of the replacements $z\to-z$, $\mu\to 1-\mu$ and the complex conjugation operation, the B function transforms into itself up to terms $\sim e^{-\pi\nu}$ exponentially small for $\nu\gg 1$ ($K\tilde{K}=1-e^{-2\pi\nu+2i\pi\mu}$). The first term on the right-hand side of (41), due to the creation or absorption of pairs by the field with a probability amplitude $e^{-\pi\nu+i\pi\mu}$, makes the direct and time-reversed processes physically distinguishable. This distinction is confirmed by the fact that the T-invariant Wronskian (40) is different from zero.

Although it happens only when $\theta\leqslant 0$ that the low saddle point gets onto the contour C_2 traced along the LDS from the high saddle point, its influence on the integral starts to appear even at small positive values of θ, when the minimal distance from the LDS_2 to the low saddle point, which actually coincides with $|t_*-t_1|\approx 2\sqrt{|\theta|}$, turns out to be comparable with, or less than, the radius of influence of the low saddle point:

$$2\sqrt{|\theta|}\lesssim|2/f_1''|^{1/2}\approx\nu^{-1/2}. \tag{42}$$

This condition corresponds to the time interval $\Delta x^0\lesssim(2m_\perp)^{-1}$ in which it is impossible to distinguish between waves with positive and negative frequencies. In fact, according to (24), the frequency difference between such waves is $2m_\perp$, and to measure it requires a time interval Δx^0 obeying the uncertainty relation $2m_\perp\Delta x^0\gtrsim 1$. Thus, as the parameter θ is varied, we find that the exponentially small terms of the asymptotic forms of the function appear not abruptly but rather in a continuous manner, in a narrow region $\Delta\theta\sim(4\nu)^{-1}\ll 1$ (Stokes line width) near $\theta=0$.

Note that Eq. (36) relates the solution with a definite sign of frequency at one time infinity with positive-frequency and negative-frequency solutions at another time infinity. In fact, from Eqs. (21), (25), (34), and (35), it follows that

$$
\begin{aligned}
&{}^{+}\Phi_p=c_1\Phi_p+c_{2-}\Phi_p, \quad {}_{-}\Phi_p=c_1{}^{-}\Phi_p+c_2{}^{+}\Phi_p,\\
&{}^{-}\Phi_p=c_1^{*}{}_{-}\Phi_p+c_2^{*}{}_{+}\Phi_p, \quad {}_{+}\Phi_p=c_1^{*}{}^{+}\Phi_p+c_2^{*}{}^{-}\Phi_p,
\end{aligned}
\tag{43}
$$

where

$$
c_1=\sqrt{1+e^{-2\pi\nu}}e^{-i\gamma}, \quad c_2=e^{-\pi\nu+i\pi/2}, \quad |c_1|^2-|c_2|^2=1\,. \tag{44}
$$

4. STATIONARY SOLUTIONS OF THE WAVE EQUATION

The stationary solutions of the wave equation for a particle in a static uniform electric field ε arise when the potential is taken in the form $\mathbf{A}=0$, $A^0=-\varepsilon x_3$, and are characterized by the energy p^0 and components p_1, p_2 of the transverse momentum $\mathbf{p}_\perp$. Two independent scalar solutions

$$
\begin{aligned}
&{}^{-}_{+}\Phi_{p^0p_\perp}=e^{ip_\perp x_\perp-ip^0x^0}D_{i\nu-1/2}(\pm e^{i\pi/4}z),\\
&z=\sqrt{2|e\varepsilon|}\left(x_3+\frac{p^0}{e\varepsilon}\right),
\end{aligned}
\tag{45}
$$

form an anti-Feynman set, according to (27). In either solution there are waves with both a positive and a negative momentum along the field, but ${}^{-}\Phi_{p^0p_\perp}$ has the feature that, as $x_3\to+\infty$, it retains only a quasiclassical wave with a negative momentum along the field, while ${}_{+}\Phi_{p^0p_\perp}$, for $x_3\to-\infty$, retains only a quasiclassical wave with a positive momentum along the field:

$$
\begin{aligned}
&{}^{-}_{+}\Phi_{p^0p_\perp}\propto e^{iS_\mp+ip_\perp x_\perp-ip^0x^0}, \quad x_3\to\pm\infty\,,\\
&S_\mp=\mp\nu(\theta\sqrt{\theta^2-1}-\operatorname{arccosh}\theta), \quad \theta=z/2\sqrt{\nu}\,.
\end{aligned}
\tag{46}
$$

Here $S_\mp$ are the action functions of a particle having a negative or positive generalized momentum along the field,

$$
\begin{aligned}
&\pi_3+eA_3=\frac{\partial S_\mp}{\partial x_3}=\mp\sqrt{\pi_0^2-m_\perp^2}=\mp m_\perp\sqrt{\theta^2-1},\\
&\pi^0=p^0+e\varepsilon x_3,
\end{aligned}
\tag{47}
$$

which in this case coincides with the kinematic momentum.

The other, Feynman-type system of stationary solutions is obtained from (45) by complex conjugation of the D functions,

$$
{}^{+}_{-}\Phi_{p^0p_\perp}=e^{ip_\perp x_\perp-ip^0x^0}D_{-i\nu-1/2}(\pm e^{-i\pi/4}z)\,. \tag{48}
$$

A distinctive feature of this system is that ${}^{+}\Phi$, in the limit $x_3\to+\infty$, reduces to a quasiclassical wave with only a positive momentum along the field, while ${}_{-}\Phi$, as $x_3\to-\infty$, reduces to a quasiclassical wave only with a negative momentum along the field:

$$
{}^{+}_{-}\Phi_{p^0p_\perp}\propto e^{iS_\pm+ip_\perp x_\perp-ip^0x^0}, \quad x_3\to\pm\infty\,. \tag{49}
$$

For an examination of the systems (45) and (48), it suffices to consider only one of them, say the first. The corresponding D function is conveniently represented by the integral

$$
\begin{aligned}
&D_{i\nu-\mu}(2\sqrt{\nu}\,\theta\,e^{i\pi/4})=B(\nu,\mu)\int_C dt\,t^{-\mu}e^{f(t,\nu,\theta)},\\
&f(t,\nu,\theta)=i\nu(\tfrac{1}{2}t^2+2\theta t+\ln t+\theta^2),\\
&B(\nu,\mu)=\left(\frac{\nu^{1+i\nu-\mu}}{2\pi i}\right)^{1/2}e^{3\pi\nu/4+3i\pi\mu/4},
\end{aligned}
\tag{50}
$$

in which the contour C is the same as in (28) and the function f differs in the sign of $\ln t$. For the saddle points we have

$$
t_{1,2}(\theta)=-\theta\mp\sqrt{\theta^2-1}, \quad t_1\cdot t_2=1\,. \tag{51}
$$

At these points

$$
\operatorname{Re} f(t_{1,2})=-\pi\nu, \quad \theta\geqslant1\,, \tag{52}
$$

$$
\begin{aligned}
&\operatorname{Re} f(t_{1,2})=-\nu(\pi\mp\theta\sqrt{1-\theta^2}\pm\arccos\theta),\\
&\qquad -1\leqslant\theta\leqslant1\,,
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
&\operatorname{Re} f(t_2)=0, \quad \operatorname{Re} f(t_1=\rho_1e^{i0})=0,\\
&\operatorname{Re} f(t_1=\rho_1e^{2i\pi})=-2\pi\nu, \quad \theta\leqslant-1\,.
\end{aligned}
\tag{54}
$$

Thus, for $\theta\geqslant1$, the saddle points lie on the negative real half-axis $t_1<-1<t_2<0$, their heights being equal and independent of θ. For $|\theta|<1$, the saddle points are complex conjugate and lie in the t plane on a circle of unit radius, while the heights of the saddle points are different and θ-dependent. For $\theta<-1$, the saddle points are on the positive real axis, $0<t_1<1<t_2$, and the heights differ considerably and do not depend on θ.

Examination of the $\mathrm{LDS}_{1,2}$ shows that for $\theta>1$ the contour C can be traced along the LDS_1 which is described by the function $\rho=\rho(\varphi)$ satisfying the equation

$$
\rho\cos\varphi=-\theta\pm\left[\frac{1}{2}\rho^2-\ln\frac{\rho}{\rho_1}-\theta\sqrt{\theta^2-1}-\frac{1}{2}\right]^{1/2}, \tag{55}
$$

and is plotted in Fig. 4a. For $\rho\to\infty$, the upper and lower tails of the function tend to the asymptote $\operatorname{Im} t=\operatorname{Re} t+2\theta$ while remaining respectively below and above this straight line.

For $|\theta|\leqslant1$, the $\mathrm{LDS}_{1,2}$ coincide and are described by Eq. (55) in which $\rho_1=1$ and the term $\theta\sqrt{\theta^2-1}$ is absent.

For $\theta=1$ the saddle points merge into one, $t_1=t_2=-1$, and since $f''_{1,2}=0$ and $f'''_{1,2}\neq0$, the saddle point which forms has three lines of steepest descent at an angle of $2\pi/3$ to one another: straight up, down to the left, and down to the right (a "monkey saddle," see Ref. 9). Therefore a LDS suffers a kink of $\pi/3$ at this saddle point; see Fig. 4b.

For $|\theta|<1$, the saddle point t_2 becomes higher than t_1 and the LDS_2 at $t=t_1$ suffers a $\pi/2$ kink; see Figs. 4c,d. For $\theta=-1$, the saddle points again merge into one, $t_1=t_2=1$, forming a monkey saddle with descents straight down, up to the left, and up to the right. The LDS rises to this saddle

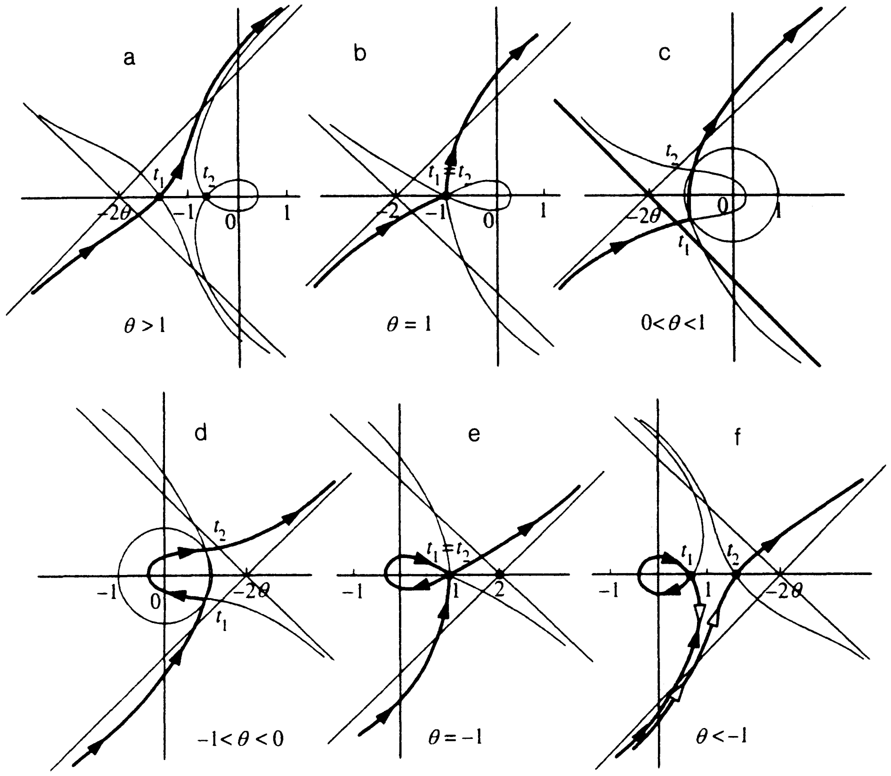

FIG. 4.

point along the lower descent with a phase $\arg t \to 2\pi$, makes a 120° turn to the left at the saddle point, and then, making a clockwise bypass around the branch point $t=0$, takes the left descent to climb the saddle point with a phase $\arg t \to 0$. Here again the LDS turns through 60° to the left and moves off to infinity while approaching from above to the asymptote $\operatorname{Im} t = \operatorname{Re} t - 2$, see Fig. 4e.

For $\theta < -1$ the contour of integration, taken along the saddle-point LDSs, divides itself into two contours terminated at infinity, $C = C_1 + C_2$. The contour C_1 goes along the LDS_1 approaching t_1 from below with a phase $\arg t \to 2\pi$; when on the saddle point, it makes a turn to the right through an angle of 90° and then goes around the branch point $t=0$ clockwise to arrive at the saddle point t_1 with zero phase. It then proceeds from the saddle point t_1 down under the cut and onto the adjacent sheet $-2\pi < \arg t < 0$ and goes to infinity with a phase $\arg t \to -3\pi/4$. This contour is described by Eq. (55).

The contour C_2 begins at the end point of C_1 and goes along the LDS_2 to the saddle point t_2; here it goes onto the basis sheet and then goes to infinity by approaching the asymptote $\operatorname{Im} t = \operatorname{Re} t + 2\theta$ from above; see Fig. 4f. This LDS is described by Eq. (55) in which, under the square root sign, one must change the sign of θ and replace ρ_1 by $\rho_2 = |t_2|$.

Thus, for $\theta/g1$ the first of the functions (45) is represented by an integral along a LDS through a single saddle point and has an asymptotic form determined by this saddle point alone:

$$D_{i\nu-\mu}(e^{i\pi/4}z) \approx \frac{\nu^{(i\nu-\mu)/2}(\theta + \sqrt{\theta^2-1})^{1/2-\mu}}{\sqrt{2}(\theta^2-1)^{1/4}}$$

$$\times \exp\left(-\frac{\pi\nu}{4} - \frac{i\pi\mu}{4} - \frac{i\nu}{2} + iS_- \right). \qquad (56)$$

For $\theta < -1$, using (36) with the sign of ν reversed, the D function may be represented in terms of two functions, D itself with $\theta > 1$ and its complex conjugate with $\mu \to 1-\mu$, which have as their respective asymptotic forms Eq. (56) and its complex conjugate with $\mu \to 1-\mu$. This means that in the region $\theta < -1$ the first and second terms in Eq. (36) with the sign of ν reversed represent waves with negative and positive momenta, respectively.

According to (47), in the region $|\theta| < 1$ the momentum is pure imaginary. Here particles and antiparticles, incident from the right and left respectively, undergo annihilation. Since in $\pi^0 < -m_\perp$ $(\theta < -1)$ the antiparticle and wave momenta are opposite to one another, the amplitude of the wave traveling to the left is larger than that of the wave incident from the left because their squares are proportional to the number of the incident and reflected particles.

We also note that close to the points $\theta = \pm 1$ the domains of influence of the saddle points t_1 and t_2 overlap, the phase difference between waves with opposite momenta becomes small,

$$S_+ - S_- \approx \frac{4}{3}\nu(\theta^2-1)^{3/2} \lesssim 1, \text{ or } \Delta x_3 \lesssim \frac{m}{2|e\varepsilon|}\nu^{-2/3}, \qquad (57)$$

and these waves are indistinguishable from one another. In fact, the momentum difference $\Delta\pi_3 = 2m_\perp \sqrt{\theta^2-1}$ is only measurable on the length Δx_3 which obeys the uncertainty relation $\Delta\pi_3 \Delta x_3 \geqslant 1$ opposite to the condition (57).

Inside the interval $-1<\theta<1$, far away from the narrow transition regions (57), the function under consideration has an asymptotic representation determined by the two saddle points t_1 and t_2:

$$D_{iv-\mu}(e^{i\pi/4}z)$$
$$\approx \frac{v^{(iv-\mu)/2}e^{-\pi v/4+i\pi(1-\mu)/4-iv/2}}{\sqrt{2}\sin\alpha}$$
$$\times\left\{e^{v(\alpha-\sin\alpha\cos\alpha)+i(\mu-1/2)\alpha}\right.$$
$$\left.-\frac{i}{2}e^{-v(\alpha-\sin\alpha\cos\alpha)-i(\mu-1/2)\alpha}\right\},\quad \alpha=\arccos\theta. \tag{58}$$

Note that the contribution from the lower saddle point t_1 [the second, recessive, term in (58)] enters with weight 1/2 because the LDS_2 goes only along one of the descents from t_1, see Figs. 4c,d. Moreover, under the reversal of time (i.e., complex conjugation plus the replacement $\mu\to 1-\mu$) this term changes sign whereas the first, dominant, one does not. It is only because the contributions from both the higher and lower saddle points are included that the asymptotic expression (58) and its T transform possess, to within the approximation considered, the same T invariant flux (Wronskian) as the corresponding exact functions,

$$D^*_{iv-\mu'}(e^{i\pi/4}z)\left(-i\frac{\overleftrightarrow{\partial}}{\partial z}\right)D_{iv-\mu}(e^{i\pi/4}z)$$
$$=-e^{-\pi[vn]/2-i\pi(\mu-1/2)/2},\quad \mu'=1-\mu. \tag{59}$$

In the vicinity of the points $\theta=+1$ and $\theta=-1$ satisfying the condition (57), both the asymptotic forms (58) and (36) [the latter using (58) on its right-hand side] become invalid. In these regions the saddle points $t_{1,2}$ approach each other to the extent that their influence may be described in terms of a monkey saddle. This means that in the expansion of $f(t)$ near the saddle points in the integral (20), not only the second but also the third derivative must be retained. The contributions from terms with higher derivatives are small,

$$\left|f_{1,2}^{(n)}\cdot(t-t_{1,2})^n{}_{\text{eff}}\right|\sim v^{1-n/3}\ll 1,\quad n\geqslant 4. \tag{59'}$$

As always in these cases, the results are expressed in terms of the Airy functions,

$$D_{iv-\mu}(e^{i\pi/4}z)\approx C(v,\mu)$$
$$\times\begin{cases}2e^{-\pi v-i\pi(\mu-1/3)}\mathrm{Ai}(e^{-2i\pi/3}y),\\ 2\,\mathrm{Ai}(y)+e^{-2\pi v-2i\pi(\mu-1/3)}\mathrm{Ai}(e^{2i\pi/3}y).\end{cases} \tag{60}$$

Here $y=v^{2/3}(1-\theta^2)$, $C(v,\mu)=B(v,\mu)v^{-1/3}e^{-iv/2}$, and the first and second line refer to the respective regions $|\theta\mp 1|\lesssim v^{-2/3}\ll 1$.

A few remarks may be made about Eq. (60) above.

1) In the $|\theta-1|\lesssim v^{-2/3}$ region there exists one recessive wave with an amplitude $\sim e^{-\pi v}$ relative to the wave amplitude for the region $|\theta+1|\lesssim v^{-2/3}$. Its asymptotic form is determined by the monkey saddle $t_{1,2}=-1$ lying to the left from the cut $0\leqslant t<\infty$, and by the integration contour which passes through this saddle point via the LDSs along the rays $\arg(t+1)=7\pi/6$, $\pi/2$.

2) In the region $|\theta+1|\lesssim v^{-2/3}$ there are two waves which have opposite momenta and whose amplitudes differ by a recessive term $\sim e^{-2\pi v}$. The asymptotic behavior of the resultant wave is determined by the saddle point $t_{1,2}=1$ on the cut. The dominant contribution to the asymptotic form is given by the monkey saddle on the upper rim of the cut ($\arg t_{1,2}=0$) and by the contour which passes via the LDSs along the rays $\arg(t-1)=5\pi/5$, $\pi/6$. In accordance with the method considered, the exponentially small contribution to the asymptotic form is determined by one half of the integral along such a LDS through the monkey saddle on the lower rim of the cut ($\arg t_1=2\pi$) which is symmetric relative the steepest ascent line from this to the main saddle point and so consists of rays $\arg(t-t_1)=3\pi/2$, $5\pi/6+2\pi$ and hence gets onto the adjacent sheet $2\pi\leqslant\arg t<4\pi$ of the Riemann surface.

3) Since the variable z (or θ) has only entered the argument of the Airy functions, the Wronskians for the D functions (See §8.2 in Ref. 7) in the region (57) are proportional to those for the Airy functions (see Eqs. 10.4.11–10.4.13 in Ref. 8). Thus, using the asymptotic formula (60) we obtain

$$D_{iv-\mu\mu}(E^{i\pi/4}z)\frac{\overleftrightarrow{d}}{dz}D_{iv-\mu}(-e^{i\pi/4}z)$$
$$\approx e^{\pi(v+i\mu)/2-iv}v^{1/2-\mu+iv}\cdot\left(1-\tfrac{1}{2}e^{-2\pi(v+i\mu)}\right). \tag{61}$$

Since the exact value of the left-hand side is $\sqrt{2\pi}i/\Gamma(\mu-iv)$, what we have on the right are the leading terms of the dominant and recessive series of the asymptotic form of this function for $v\gg 1$, $|\mu|\ll\sqrt{v}$. The leading (dominant) term is just the Stirling formula. The value of the Wronskian (59) for the other, T-conjugate, pair of independent solutions is also reproduced by the asymptotic forms (60).

The exponentially small correction $\sim e^{-2\pi v}$ in the Wronskian (61), as well as the value of the Wronskian (59) computed from the asymptotic form (60) near $\theta=-1$, are due entirely to the exponentially small contribution from the saddle point t_1 on the lower rim of the cut. Accordingly, the Wronskians (61) and (59) control the correctness not only of the dominant but also of the recessive terms in the asymptotic forms (60). In particular, the above rule for calculating the contribution from the lower saddle point which gets onto the LDS from its higher counterpart is controlled. If, say, this contribution were taken simply to equal the integral along the descent $\arg(t-t_1)=3\pi/2$, then in the second term in the lower formula of Eq. (60) we would have instead $e^{2i\pi/3}\mathrm{Ai}(e^{2i\pi/3}y)$ the function $e^{i\pi/2}\mathrm{Hi}(y)$, which would render the Wronskians (61) and (59) incorrect.

4) The Airy functions of a complex argument in Eq. (60) can be expressed in terms of real functions of real y as

$$2e^{\pm i\pi/3}\mathrm{Ai}(e^{\mp 2i\pi/3}y)=\mathrm{Ai}(y)\pm i\,\mathrm{Bi}(y). \tag{61'}$$

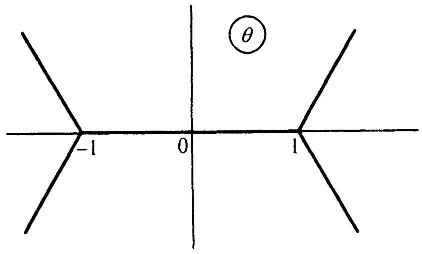

FIG. 5.

When moving θ away and *outward* from the ends of the interval $|\theta|\leqslant 1$ the functions Ai(y) and Bi(y) show oscillations, while on going *inward* Ai(y) falls off exponentially and Bi(y) exponentially grows. From this and Eqs. (58) and (60) it follows that the amplitudes of the recessive waves that come out from the slit $|\theta|\leqslant 1$ are formed inside the slit while their phases are formed in the intervals $|\theta\mp 1|\lesssim\nu^{-2/3}$.

It is known that for an Airy function Ai(z) in the complex-z plane, the Stokes lines are the rays $\arg z=0$, $2\pi/3$, $4\pi/3$. Representing y in the first and second lines of Eq. (60) by the respective forms $y=-2\nu^{2/3}(\theta-1)$ and $y\approx 2\nu^{2/3}(\theta+1)$, we find that the Stokes line for the function $D_{i\nu-\mu}(e^{i\pi/4}z)$ in the complex-θ plane emanate from the points $\theta=\pm 1$ in the form of the rays $\arg(\theta-1)=\pi/3$, π, $5\pi/3$ and $\arg(\theta+1)=0$, $2\pi/3$, $4\pi/3$. The general pattern of the Stokes lines for $D_{i\nu-\mu}(e^{i\pi/4}z)$ is shown in Fig. 5.

From the corresponding formulas of Secs. 3 and 4 it is seen that the stationary and nonstationary solutions are related by a rotation through an angle of $\pm\pi/2$ in the complex-θ plane. In fact, the Feynman stationary solutions subject to the rotation $z\to iz$ (i.e., $\theta\to i\theta$ or $x_3\to ix^0$, $p^0\to -ip_3$) go over to the Feynman nonstationary solutions

$$
{}^{+}_{-}\Phi_{p^0p_\perp}\propto D_{-i\nu-\mu}(\pm e^{-i\pi/2}z)\to D_{-i\nu-\mu}(\pm e^{i\pi/4}z)\sim{}^{+}_{-}\Phi_{p}, \qquad (62)
$$

whereas the anti-Feynman stationary solutions go over to the anti-Feynman nonstationary under the rotation in the opposite direction $z\to -iz$ (i.e., $\theta\to -i\theta$ or $x_3\to -ix^0$, $p^0\to ip_3$):

$$
{}_{+}^{-}\Phi_{p^0p_\perp}\propto D_{i\nu-\mu}(\pm e^{i\pi/4}z)\to D_{i\nu-\mu}(\pm e^{-i\pi/4}z)\sim{}_{+}^{-}\Phi_{p}. \qquad (63)
$$

The different sign of rotation for the Feynman and anti-Feynman solutions are due to the fact that they transform into one another under the T transformation, consisting of the complex conjugation in combination with the replacements $x^0\to -x^0$, $\mathbf{p}\to -\mathbf{p}$.

5. PHYSICAL INTERPRETATION OF THE INTEGRAL REPRESENTATIONS AND OF THE ASYMPTOTIC CONDITIONS

The parameter θ in (23) can be written in the form

$$
\theta=\sqrt{\frac{|e\epsilon|}{2\nu}}\left(x^0+\frac{p_3}{e\varepsilon}\right)=\frac{\pi_\parallel}{m_\perp},\quad \pi_\parallel=\operatorname{sgn}(e\varepsilon)\pi_3,
$$

$$
\pi_3=p_3+e\varepsilon x^0, \qquad (64)
$$

showing it to be identical (in units of $m_\perp$) to the component of the particle kinetic momentum the direction of the force $e\varepsilon$. There is a simple physical meaning in the integral representation (28) for the parabolic cylinder functions involved in the Feynman set (21). This representation is a superposition of the solutions

$$
{}^{+}_{-}\Phi_{p_+\mathbf{p}_\perp}=e^{i\mathbf{p}_\perp\mathbf{x}_\perp}(4|e\varepsilon|)^{-1/4}\exp\left\{-\frac{ip_+x_-}{2}+\frac{ie\varepsilon x_3^2}{2}-\frac{ie\varepsilon x_+^2}{4}-\left(i\nu+\frac{1}{2}\right)\ln\frac{\pm\pi_+}{\sqrt{2|e\varepsilon|}}\right\}, \qquad (65)
$$

of the same wave equation, which are characterized by the eigenvalues p_1, p_2, and p_+ of the operator P_1, P_2, and $P_+=P^0+P_3$, where

$$
P_{1,2}=\Pi_{1,2},\quad P^0=\Pi^0-e\varepsilon x_3,\quad P_3=\Pi_3-e\varepsilon x^0,
$$

$$
\Pi_\alpha=-i\frac{\partial}{\partial x^\alpha}-eA_\alpha,
$$

$$
x_\pm=x^0\pm x_3,\quad \pi_+=p_++e\varepsilon x_+. \qquad (66)
$$

In fact, if we make use of the representation (28) and take the integration variable t according to

$$
t=\mp\pi_+/m_\perp, \qquad (67)
$$

then the solutions (21) can be written as superpositions of the solutions (65),

$$
{}^{+}_{-}\Phi_{\mathbf{p}}(x)=N\int_{-\infty}^{\infty}dp_+\mu(p_3,p_+){}^{+}_{-}\Phi_{p_+p_\perp}(x), \qquad (68)
$$

where

$$
\mu(p_3,p_+)=(2\pi|e\varepsilon|)^{-1/2}\exp\left[i\frac{(p_+-2p_3)^2-2p_3^2}{4e\varepsilon}\right],
$$

$$
N=|e\varepsilon|^{1/4}e^{\pi\nu/4-3i\pi/8}. \qquad (69)
$$

For the parabolic cylinder functions involved in the anti-Feynman set (25), we employ the integral representation complex conjugate to (28),

$$
D_{i\nu-1/2}(e^{-i\pi/4}z)=A^*(\nu,\tfrac{1}{2})\int_{C^*}dt\,t^{-1/2}e^{-f(t,\nu,\theta)}. \qquad (70)
$$

In this integral the branch point $t=0$ is circumvented counterclockwise because the integration contour C^* is complex conjugate to the contour C in the integral (28).

Using the representation (70), the functions of the anti-Feynman set can be written[10] in the form of the superposition

$$
{}_{+}^{-}\Phi_{\mathbf{p}}(x)=N^*\int_{-\infty}^{\infty}dp_-\mu(p_3,p_-){}_{+}^{-}\Phi_{p_-p_\perp}(x), \qquad (71)
$$

$$
\mu(p_3,p_-)=(2\pi|e\varepsilon|)^{1/2}\exp\left[-i\frac{(p_-+2p_3)^2-2p_3^2}{4e\varepsilon}\right], \qquad (72)
$$

of the anti-Feynman solutions

$$ {}^{-}_{+}\Phi_{p\_p_\perp}(x)=e^{ip_\perp x_\perp}(4|e\varepsilon|)^{-1/4}\exp\left\{-\frac{ip_- x_+}{2}-\frac{ie\varepsilon x_3^2}{2}+\frac{ie\varepsilon x_-^2}{4}+\left(i\nu-\frac{1}{2}\right)\ln\frac{\mp\pi_-}{\sqrt{2|e\varepsilon|}}\right\} \tag{73} $$

of the wave equation, which are characterized by the eigenvalues p_1, p_2, p_- of the operators $P_1, P_2, P_-=P^0-P_3$; see Eq. (66). By the variable of integration in Eq. (70) is meant

$$ t=\pm\pi_-/m_\perp\ ,\quad \pi_-=p_--e\varepsilon x_-\ . \tag{74} $$

Note that under the reversal of time, which consists of complex conjugation and the replacements $x^0\to -x^0$, $\mathbf{p}\to-\mathbf{p}$, the Feynman and anti-Feynman sets (65) and (73), as well as (68) and (71), change into one another. Also note that both the sets can be expressed in terms of the eigenfunctions of one and the same operator, say P_-. This can be done by using a representation other than (28) for the D functions entering the Feynman set (21). Then we obtain

$$ \begin{aligned} {}^{+}_{-}\Phi_{\mathbf{p}}(x)&=e^{i\mathbf{px}}D_{-i\nu-1/2}(\pm e^{i\pi/4}z)\\ &=\pm c_1^*N^*\int_{e\varepsilon x_-}^{\pm\infty}dp_-\mu(p_3,p_-)\,{}^{-}_{+}\Phi_{p\_p_\perp}(x)\ ,\\ c_1^*N^*&=\sqrt{2\pi}\,|e\varepsilon|^{1/4}e^{i\pi/8-\pi\nu/4}/\Gamma(i\nu+\tfrac{1}{2})\ . \end{aligned} \tag{75} $$

Here for the upper (lower) frequency index on $\Phi_{\mathbf{p}}$ on the left the function $\Phi_{p\_\mathbf{p}_\perp}(x)$ on the right is taken with the lower (upper) frequency index.

The solutions $\Phi p-p\perp\ (x)$ are of a quasiclassical form but, unlike Eqs. (21) and (25), they have a singularity at $x_-=p\ \_/e\varepsilon$, i.e., at $\pi_-=0$ [see Eq. (74)]. This surface $x_3=x^0-p\_/e\varepsilon$, moving with light velocity, is a source of particles. Hence the functions (73) satisfy the homogeneous Klein-Gordon equation only outside this surface, and the integral of the current density divergence taken over any finite 4-volume Ω enclosing some region of the source hypersurface, is nonzero:

$$ \int_\Omega \partial_\alpha j^\alpha d^4x=\int_{\partial\Omega} j^\alpha d^3S_\alpha\neq 0\ . \tag{76} $$

This follows from the fact that the values $\Phi_{p\_p_\perp}(x)$ for $\pi_-=\pm\delta\gtrless 0$, $\delta\to 0$ are substantially different.

Let us now consider the time interval Δx^0, in which the process of creation of pairs by the static uniform electric field essentially occurs. Outside of this interval, the parabolic cylinder functions in Eqs. (21) and (25) move into their asymptotic regime. For a strong field, with $\nu\lesssim 1$, the D functions in fact depend only on z and therefore the pair formation region is determined by the condition $\Delta x\sim 1$ or $\Delta\theta\sim\nu^{-1/2}$, $\Delta x^0\sim 1/\sqrt{2|e\varepsilon|}=m_\perp^{-1}\nu^{1/2}$. It is seen that the time of formation of a pair by a strong field does not depend on the mass of the particle and is determined by the "electric" length $|e\varepsilon|^{-1/2}$, which is small compared to the Compton length if $\nu\ll 1$. The independence of the formation region of m has the consequence that the probabilities of many strong field processes do not depend on the mass and, in particular, tend to a finite limit for $m\to 0$ and the fixed value of the field.[11,12]

In the case of a weak field, when $\nu\gg 1$ holds, it is seen that as $\theta\to\infty$ the action function $\nu\xi(\theta)$ enters the asymptotic regime even for $\nu\theta^{-2}\ll 1$. This means that the region of pair formation by a weak field is determined by the condition $\Delta\theta\sim\sqrt{\nu}$, $\Delta x^0\sim m_\perp^{-1}\nu^{3/2}$.

The meaning of the above estimates for the time for formation of a pair by a static field is that the pair formation probabilities due to varying and static fields cease to differ when the characteristic field variation time $T=\varepsilon/\dot{\varepsilon}$ becomes large compared to static-field pair time formation, i.e., for

$$ T\gg\Delta x^0\sim m_\perp^{-1}\begin{cases}\nu^{1/2}, & \nu\lesssim 1, \qquad (77)\\ \nu^{3/2}, & \nu\gg 1\ . \qquad (78)\end{cases} $$

The first of these conditions has been obtained and discussed by Migdal.[13,14] One of us[15] has derived both of these conditions by considering the creation of a pair by an electric field varying like $\varepsilon(x^0)=\varepsilon\cosh^{-2}(\omega x^0)$.[16] The formation time estimates as given in Eqs. (77) and (78) are consistent with those obtained for more complex processes in a slowly varying plane wave field.[15,17,18]

According to the condition (78), for a weak field the time Δx^0 is $\sqrt{\nu}$ times larger than the barrier time $m/e\varepsilon$ which would seem to characterize the pair formation process. This difference results from the extremely sensitive exponential dependence of the pair formation probability $e^{-2\pi\nu}$ on the field strength ε. Therefore changing the field by $\delta\varepsilon$ changes the exponent $-2\pi\nu$ by the amount $-2\pi\nu\delta\varepsilon/\varepsilon$ and does not affect the probability only if the fractional change in the field is small compared to the parameter characterizing the smallness of the field, $\delta\varepsilon/\varepsilon\ll\nu^{-1}\ll 1$. For the time dependence of the field is quadratic near the maximum, $\delta\varepsilon/\varepsilon\sim(\delta x^0/T)^2$. Therefore the above condition is equivalent to $T\gg\nu^{1/2}\delta x^0$ and is identical to (78) if the time δx^0 equals the barrier time, $\delta x^0=m/e\varepsilon$ or $\Delta\theta\sim 1$. If the field varies rapidly and the condition (78) is invalid, the tunneling pair formation mechanism is overpowered by a more efficient one dependent not only on the field intensity but also on T, the characteristic field variation time.

Let us discuss now the role of the barrier time, taking the state ${}^{-}\Phi_{\mathbf{p}}$ as an example. From the integral representation (71) ${}^{-}\Phi_{\mathbf{p}}(x)$ it is seen that for $\theta\gg 1$ this function is determined almost entirely by the contributions from the saddle point $t_2\approx -2\theta\ll -1$ and from the contour region $\mathrm{Re}\,t<0$ which, by (74), corresponds to antiparticles ($\pi_-<0$). The contribution from the region $\mathrm{Re}\,t>0$ which corresponds to particles ($\pi_->0$) is exponentially small even in comparison with the recessive term for $\theta=0$. Nearer the barrier time values, when $\theta\lesssim 1$ holds, this is no longer the case. The contribution to the integral (70) from the contour region $\mathrm{Re}\,t>0$ in this case is comparable with, or larger than, the probability amplitude for the creation or absorption of a real pair by the field. This contribution can be interpreted as coming into the ${}^{-}\Phi_{\mathbf{p}}$ state from the "bound" or "subbarrier" pairs.

6. STOKES LINE WIDTHS EXCEEDING THE NATURAL ONE

The farther beyond the Stokes line the parameter α, the more accurately the integrals F_2 and F_1, see Eq. (13) describe the dominant and recessive properties of the F wave. To see this, select on LDS_2 and LDS_1 some points t_{2*} and t_{1*} located on the same level line below the saddle point t_1 (see Fig. 3), and consider the integrals

$$\delta F_2=A\int_{t_{2*}}^{t_\infty} dt e^{f}, \quad \delta F_1=A\int_{t_\infty}^{t_{1*}} dt e^{f}. \tag{79}$$

The lower the line level chosen, the smaller are ΔF_2 and δF_1 in comparison with F_2 and F_1. We may assume that the integral F_k $(k=1,2)$ forms, to within δF_k, in a region near the saddle point t_k having a radius of order

$$R_{k*}=\left|\tfrac{1}{2}f_k''\right|^{-1/2}+s_{k*}, \tag{80}$$

where s_{k*} is the arc length of the LDS_k between the points t_k and t_{k*}. If in either of these regions we may approximate the function $f(t)$ by a Taylor series expansion such as

$$f(t,\nu,\alpha)\approx f_k+\tfrac{1}{2}f_k''\cdot(t-t_k)^2, \tag{81}$$

then the length s_{k*} may be estimated from the formula

$$s_{k*}=\sqrt{\frac{2(\mathrm{Re} f_k-a)}{|f_k''|}}, \quad a=\mathrm{Re}\, f_{1*}=\mathrm{Re}\, f_{2*}, \tag{82}$$

where a is a new parameter that specifies the level line of the points t_{1*} and t_{2*} and lies in the range $-\infty<a\leqslant\mathrm{Re}\, f_1$.

The above formation regions overlap if the distance between the points t_{1*} and t_{2*} is less than, or of the order of, the smallest of the radii (80),

$$|t_{1*}-t_{2*}|\lesssim\min(R_{1*},R_{2*}), \tag{83}$$

this being the radius specifying the characteristic size of the overlap region. If the level line a is not too low, $0<\mathrm{Re}\, f_1-a\lesssim\mathrm{Re}(f_2-f_1)\gg 1$, then the minimum radius is R_{1*}. Since

$$f(t_{1*})-f(t_{2*})=i\,\mathrm{Im}(f_1-f_2), \tag{84}$$

expanding $f(t_{2*})$ about t_{1*} and making use of Eqs. (80) through (82) we have instead of (83) the condition

$$\begin{aligned}|\mathrm{Im}(f_1-f_2)|&\lesssim|f_{1*}'|R_{1*}+|\tfrac{1}{2}f_{1*}''|R_{1*}^2\\&\approx|f_1''|s_{1*}R_{1*}+|\tfrac{1}{2}f_1''|R_{1*}^2\\&=1+4\sqrt{\mathrm{Re}\, f_1-a}+3(\mathrm{Re}\, f_1-a).\end{aligned} \tag{85}$$

The left-hand side of (85) depends on α and ν; and the right-hand side, also on the parameter a. The overlap condition (85) always holds if the parameter α is inside the natural Stokes line width. As long as $0\leqslant\mathrm{Re}\, f_1-a\lesssim 1$, the width of the Stokes line remains of the order of the natural one.

As α is increased beyond the natural Stokes line width, the left-hand side of (85) may become much larger than its right-hand side. However, by decreasing α the right-hand side may be made equal to the left-hand side and the corresponding value of a will be a function of α and ν. The quantities

$$\varepsilon_1=\left|\frac{\Delta F_1}{F_1}\right|, \quad \varepsilon_2=\left|\frac{\delta F_2}{F_2}\right|=\varepsilon_1\frac{r_2}{r_1}\left|\frac{F_1}{F_2}\right|, \tag{86}$$

will also become functions of these quantities alone. Here r_k is the effective radius of formation of the integral δF_k, determined by the relation $|\delta F_k|=|A|e^a r_k$, $k=1,2$. Clearly, $\varepsilon_1(\alpha)$ is the relative amplitude of the presence of the dominant wave in the recessive and $\varepsilon_2(\alpha)$ is the same for the recessive wave in the dominant. In other words, the integrals F_1 and F_2 describe the recessive and dominant properties of F to a fractional accuracy of $\varepsilon_1(\alpha)$ and $\varepsilon_2(\alpha)$ respectively.

Note that if the parameter α is not behind but before the Stokes line, then instead of the integrals F_2 and δF_2 one must consider F and δF and define δF_1 on the downward LDS_1,

$$\delta F=A\int_{t_{2*}}^{t_\infty''} dt e^{f}, \quad \delta F_1=A\int_{t_{1*}}^{t_\infty''} dt e^{f}. \tag{87}$$

With these modifications the entire argument following Eq. (79) remains valid.

As an example, consider the function $F=D_{-i\nu-1/2}(e^{i\pi/4}2\sqrt{\nu}\theta)$ defined by the integral (28). It is readily seen that, by (33), (23), and (30), the inequality (85) has the form

$$\begin{aligned}&2\nu|\theta\sqrt{1+\theta^2}+\mathrm{arcsinh}\,\theta|\lesssim 1+4r-a+3(-a),\\&a\leqslant\mathrm{Re}\, f_1=0.\end{aligned} \tag{88}$$

As long as $0\leqslant -a\lesssim 1$ holds, the width of the Stokes line is of order the natural width, $\Delta\theta\sim(4\nu)^{-1}$. In order for the recessive wave to separate itself from the dominant by the amount $\varepsilon_1\sim e^{-\pi\nu}$ with which the dominant wave differs from the recessive in the natural formation region of the latter, it is necessary to set $a=-\pi\nu$ in (88), which will give the width $\Delta\theta\sim\sqrt{3\pi/2}$, i.e., a formation time interval of the order of the barrier time, $\Delta x^0\sim\sqrt{3\pi/2}(m/|e\varepsilon|)$.

7. STOKES LINE WIDTH AND THE METHOD OF OSCULATING PARAMETERS

An instructive approach to the question of the Stokes line width is to use the method of osculating parameters,[9] according to which a partial solution $y(\theta)$ of a differential equation of the second order with a large parameter ν is sought as a superposition of quasiclassical solutions ${}^{\pm}f(\theta)$ with correcting coefficient functions $a_{\pm}(\theta)$ defined by the relations

$$y(\theta)=a_+(\theta)^+f(\theta)+a_-(\theta)^-f(\theta), \tag{89}$$

$$y'(\theta)=a_+(\theta)^+f'(\theta)+a_-(\theta)^-f'(\theta), \tag{90}$$

and the boundary condition

$$a_+(-\infty)=1, \quad a_-(-\infty)=0, \tag{91}$$

showing that we are dealing with the solution ${}_+y(\theta)$. Since the functions $a_{\pm}(\theta)$ are not differentiated in (90), the second-order differential equation for $y(\theta)$ leads to a system of two first-order equations for $a_{\pm}(\theta)$, which is sometimes convenient in the search for approximations.

There is the tendency in the physical literature to treat the two terms on the right-hand side of Eq. (89) as representing two waves with positive and negative frequencies at *arbitrary* θ rather than for $\theta\to+\infty$ only (see Ref. 20 and literature therein). The reason for this is that the quasiclassical solutions ${}^{\pm}f(\theta)$ retain the sign of the frequency for all θ and that the factors $a_{\pm}(\theta)$ are only supposed to correct these solutions. We wish to show that this approach is satisfactory only if the function ${}^{+}f(\theta)$ describing the strong wave is approximated to within terms of order $a_{-}(\infty)$, the amplitude of the weak wave $a_{-}(\theta)^{-}f(\theta)$, which are exponentially small under the conditions assumed [for example, $a_{-}(\infty)\sim e^{-\pi\nu}$].

As a preliminary, note that if the function $y(\theta)$ is known,

$$a_{\pm}(\theta)=\pm\frac{(y,{}^{\mp}f)}{({}^{+}f,{}^{-}f)},\quad \text{where}\quad (f,g)\equiv fg'-f'g. \tag{92}$$

Now suppose that $|a_{-}(\theta)|\lesssim a_{-}(\infty)\sim e^{-\pi\nu}$ for all θ whatever the approximation ${}^{+}f$. Dropping the exponentially small terms with a_{-} in Eqs. (89) and (90) we obtain

$$y(\theta)\approx{}^{+}f(\theta), \tag{93}$$

that is, the positive-frequency part of $y(\theta)$ is defined only to within terms $\sim\nu^{-n}$ if the ${}^{\pm}f$ are taken in the nth approximation, ${}^{\pm}f={}^{\pm}f_n$. Therefore, as shown below, for $\nu^{-n-1}\gg a_{-}(\infty)$ the wave $a_{-}(\theta)^{-}f(\theta)$ cannot be everywhere exponentially small. It must contain power-law-small positive-frequency terms which correct (93) to the exact function (89). Therefore the function $a_{-}(\theta)^{-}f(\theta)$ cannot be considered positive-frequency for all θ if ${}^{+}f$ is not approximated accurately enough.

We shall illustrate this using the example of the function ${}_{+}y(\theta)=CD_{i\nu-1/2}(-e^{-i\pi/4}2\sqrt{\nu}\theta)$ [see Eq. (25)]. The constant C is specified by the condition $a_{+}(-\infty)=1$. The first terms of the asymptotic expansion of the positive-frequency part of $y(\theta)$ in inverse powers of $\nu\gg1$ can be obtained from the Darwin formula.[8]

For the nth-order approximation this formula takes the form

$$\begin{gathered}{}^{+}f_n(\theta)=e^{iS+\sigma_n},\quad {}^{-}f={}^{+}f^{*},\quad S(\theta)=-\nu\xi(\theta),\\ \sigma_n=\sum_{k=0}^{n-1}(i\nu)^{-k}c_k(\theta),\quad c_0=-\tfrac14\ln(1+\theta^2),\end{gathered} \tag{94}$$

where the c_k are real functions of θ which are bounded for $k\geqslant1$ together with their derivative and have the property $c_k(-\theta)=(-1)^k c_k(\theta)$. From Eqs. (2) and (94) it follows that $a_{-}(\theta)^{-}f_n(\theta)$ consists of a positive-frequency and a negative-frequency term, which are readily written down to within corrections $\sim\nu^{-1}$ as

$$\begin{aligned}a_{-}{}^{-}f_n=&\frac{e^{iS+\sigma1}c_n'(\theta)}{4(i\nu)^{n+1}\sqrt{1+\theta^2}}(1+O(\nu^{-1}))-ie^{-\pi\nu}\\ &\times e^{-iS+\sigma^1}(1+O(\nu^{-1})).\end{aligned} \tag{95}$$

It is clear that $a_{-}{}^{-}f_n$ will be a negative-frequency wave only for

$$\frac{|c_n'(\theta)|}{4\nu^{n+1}\sqrt{1+\theta^2}}\ll e^{-\pi\nu}. \tag{96}$$

In the notation of Ref. 21, $c_n'(\theta)=-2(-i\nu)^{n+1/2}h_{3n}X^{-3n-2}$. It can be shown that for $\theta\gg1$ we have $c_n'(\theta)\approx a_n2^{-2n}\theta^{-2n-1}$ and $a_n\approx2^{2n-1}\Gamma(n+1)$ provided that $n\gg1$ holds. Then the condition (96) is equivalent to

$$\theta^2\gg\frac{1}{2\nu}\left[\frac14\Gamma(n+1)e^{\pi\nu}\right]^{1/n+1}\Bigg|_{n\sim\nu\gg1}\sim1. \tag{97}$$

It is seen that at each further step toward improving ${}^{+}f$ the width of the interval in which the positive- and negative-frequency waves in (89) are unseparated decreases rapidly, and it is only at the step $n\sim\nu$ that this width approaches a physically meaningful value, the barrier width. An inappropriately chosen convention as to the separation of the negative-frequency wave may lead to an unjustifiably large value of the Stokes line width—and hence of the formation time of this wave.

Thus, as long as ${}^{+}f$ is taken with only poor accuracy, the term $a_{-}{}^{-}f$ in (89) is not exponentially small and contains a positive-frequency term which is large compared to $e^{-\pi\nu}$. However, if in (89) we replace ${}^{+}f$ by the exact solution ${}^{+}y$ of the same wave equation while representing ${}^{-}f$ as before by an approximate expression for the other exact solution ${}^{-}y$, ${}^{\pm}y(\theta)=CD_{\mp i\nu-1/2}\times(e^{\pm i\pi/4}2\sqrt{\nu}\theta)$, then in the new expansion (89),

$$\begin{gathered}{}_{+}y=a_{+}(\theta){}^{+}y+a_{-}(\theta)f,\quad a_{+}(\theta)=\frac{({}_{+}y,{}^{-}f)}{({}^{+}y,{}^{-}f)},\\ a_{-}(\theta)=-\frac{({}_{+}y,{}^{+}y)}{({}^{+}y,{}^{-}f)},\end{gathered} \tag{98}$$

the altered function $a_{-}(\theta)$ will render the term $a_{-}{}^{-}f$ exponentially small ($\sim e^{-\pi\nu}$) as well as negative-frequency to within terms $\sim e^{-2\pi\nu}$. Nevertheless, this term will still be different from—even though representing an approximation for—the term $c_{-}{}^{-}y$ in the expansion

$$_{+}y=c_{+}{}^{+}y+c_{-}{}^{-}y,\quad c_{\pm}=\pm\frac{({}_{+}y,{}^{\mp}y)}{({}^{+}y,{}^{-}y)}, \tag{99}$$

with constant coefficients $c_{\pm}=a\pm(\infty)$ [see Eqs. (36) and (43)].

In fact, using Eqs. (98) and (99), one can represent $a_{\pm}(\theta)$ in the form

$$a_{+}(\theta)=c_{+}+c_{-}\frac{({}^{-}y,{}^{-}f)}{({}^{+}y,{}^{-}f)},\quad a_{-}(\theta)=c_{-}\frac{({}^{+}y,{}^{-}y)}{({}^{+}y,{}^{-}f)}, \tag{100}$$

and ${}^{-}y$, as an expansion in ${}^{-}f$ and ${}^{+}y$:

$$\begin{aligned}{}^{-}y&=\frac{({}^{+}y,{}^{-}y)}{({}^{+}y,{}^{-}f)}\cdot{}^{-}f+\frac{({}^{-}y,{}^{-}f)}{({}^{+}y,{}^{-}f)}\cdot{}^{+}y\\ &=\left(1+\frac{({}^{+}y,{}^{-}r)}{({}^{+}y,{}^{-}f)}\right){}^{-}f-\frac{({}^{-}y,{}^{-}r)}{({}^{+}y,{}^{-}f)}{}^{+}y.\end{aligned} \tag{101}$$

Here ${}^{-}y={}^{-}f+{}^{-}r$. Now if one takes as ${}^{-}f$ the quantity ${}^{-}f_n$, i.e., the sum of n terms of the asymptotic expansion in powers of ν^{-1}, and uses ${}^{-}r_n \sim \nu^{-n}\,{}^{-}y$, it becomes obvious that the coefficient of ${}^{-}f_n$ in Eq. (101) is unity to within $\sim \nu^{-n}$, and the coefficient of ${}^{+}y$ is small (or order ν^{-n-1}) because

$$({}^{-}y, {}^{-}r) = {}^{-}y^2({}^{-}r_n/{}^{-}y)' \sim \nu^{-n},$$

$$({}^{+}y, {}^{-}f_n) = ({}^{+}y, {}^{-}y) - ({}^{+}y, {}^{-}r_n) = 4i\nu + O(\nu^{-n+1}) \,. \tag{101'}$$

Similarly, we see that $a_{-}(\theta) = c_{-}[1 + O(\nu^{-n})]$ and $a_{+}(\theta)$ differs from c_{+} by an exponentially small correction term $\sim \nu^{-n-1} e^{-\pi\nu - 2iS}$ with a double negative frequency. Thus, $a_{-}{}^{-}f$ equals $c_{-}{-}y$ with a fractional accuracy of ν^{-n-1}. This implies that the wave $a_{-}{}^{-}f_n$ represents, in fact, the entire negative-frequency part of the function $c_{-}{-}y$ if the last terms in ${}^{-}f_n$ are of order $e^{-\pi\nu}$.

All qualitative results of this section will continue to hold if, instead of ${}^{\pm}f_n$, one takes as approximations for ${}^{\pm}y$ the integrals over mutually independent parts of the LDS_2 and LDS_1 into which the contour of integration for ${}_{+}y$ breaks down.

We are grateful to Prof. M. V. Berry whose valuable comments encouraged us to further development of the present subject. The partial financial support of the American Physical Society is also acknowledged.

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Translated by E. Streichenko

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