Wave functions of charged particles simultaneously in Coulomb and laser fields

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An approximate solution is derived for the time-dependent Schrödinger equation for an electron which is simultaneously in a Coulomb field and the field of an intense electromagnetic wave. The wave functions of the discrete and continuous spectra are found in an analytic form convenient for calculations on multiphonon processes without the use of time-dependent perturbation theory. Expressions are derived for the scattering cross sections of slow electrons and positrons by a Coulomb potential in the field of a light wave.

1. INTRODUCTION

The interaction of an intense electromagnetic field with atoms has been the subject of theoretical and experimental research for a fairly long time now.[1-3] The experimentalists have been using progressively more intense laser fields in recent years, while the theoreticians usually base their calculations on perturbation theory in the external field, which is capable in practice of dealing with multiphoton processes involving no more than ten photons. An exceptional case is that of a particle with a small binding energy (in a δ-function potential) in a circularly polarized field, for which calculations can be carried out at an arbitrary field intensity.[4]

In this paper we construct approximate wave functions for an electron (or positron) simultaneously in a Coulomb field and a laser field, for the discrete and continuous spectra, in a form suitable for use in calculations on various multiphoton processes (multiphoton ionization, scattering, and nonlinear susceptibilities), without the use of time-dependent perturbation theory. We restrict the discussion to electromagnetic fields whose wavelengths satisfy the condition for the applicability of the dipole approximation: $\lambda \gg a_B$. The amplitude of the electron oscillations in the electromagnetic wave, $a_0=eF/(\mu\omega^2)$ (F and ω are the intensity and frequency of the field; e and μ are the charge and mass of the particle), must satisfy the condition $a_0 < a_B$, where a_B is the Bohr radius of the atom.

In Sec. 2 we carry out unitary transformations of the Schrödinger equation for an electron in Coulomb and laser fields for the case in which the laser field is circularly polarized.

In Sec. 3 we derive equations for the spectrum (the Stark shift of the levels) and the wave functions of the discrete states of a hydrogen-like atom. In Sec. 4 we derive wave functions for electrons and positrons in the continuum. In Sec. 5 we calculate the amplitudes and the elastic scattering cross section for slow particles in a Coulomb potential in a laser field. In Sec. 6 we discuss a linearly polarized optical field.

2. UNITARY TRANSFORMATIONS FOR THE SCHRÖDINGER EQUATION IN A CIRCULARLY POLARIZED OPTICAL FIELD

Let us examine the time-dependent Schrödinger equation for an electron in a Coulomb field and a circularly polarized optical field, the latter being described by a vector potential $\mathbf{A}(t)$ which depends only on the time (this is the dipole approximation):

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=\left[-\frac{\hbar^2}{2\mu}\nabla^2-\frac{Ze^2}{r}+\frac{e}{\mu c}\hat{\mathbf{p}}\mathbf{A}(t)+\frac{e^2}{2\mu c^2}\mathbf{A}^2(t)\right]\Psi(\mathbf{r},t). \tag{1}$$

We seek a solution of (1) in the form

$$\Psi(\mathbf{r},t)=\exp\left[-\frac{ie}{\hbar\mu c}\hat{\mathbf{p}}\int^t\mathbf{A}(t')dt'-\frac{ie^2}{2\hbar\mu c^2}\int^t\mathbf{A}^2(t')dt'\right]\Phi(\mathbf{r},t). \tag{2}$$

We introduce

$$\mathbf{a}(t)=\frac{e}{\mu c}\int^t\mathbf{A}(t')dt'=\frac{a_0}{2}(\mathbf{e}_x\sin\omega t+\gamma\mathbf{e}_y\cos\omega t),$$

$$a(\theta,\varphi,t)=\frac{a_0}{2}\sin\theta\sin(\varphi+\gamma\omega t), \tag{3}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors along the corresponding axes, $a_0=eF/(\mu\omega^2)$ is the amplitude of the classical oscillations of the electron in the optical field, and $\gamma=\pm1$ specifies whether the wave has a right-hand or left-hand polarization. Substituting (2) into (1), we find an equation for $\Phi(\mathbf{r},t)$:

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t}=\left[-\frac{\hbar^2}{2\mu}\nabla^2-\frac{Ze^2}{|\mathbf{r}-\mathbf{a}(t)|}\right]\Phi(\mathbf{r},t). \tag{4}$$

We seek a solution of Eq. (4) in the form

$$\Phi(\mathbf{r},t)=\exp\left[-\frac{i\gamma\omega t}{\hbar}(\hat{L}_z-\hbar m)-\frac{iE't}{\hbar}\right]f(\mathbf{r}),\qquad m=0,1,2,..., \tag{5}$$

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where the operator $\hat{L}_z$ represents the projection of the angular momentum. Substituting (5) into (4), we find the following steady-state equation for $f(\mathbf{r})$:

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2-\gamma\omega\hat{L}_z-\frac{Ze^2}{|\mathbf{r}-\mathbf{a}_0|}\right)f(\mathbf{r})=E'f(\mathbf{r}), \qquad (6)$$

$$\mathbf{a}_0=\frac{a_0}{2}\mathbf{e}_y.$$

We expand the potential in Eq. (6) in Legendre polynomials:

$$\frac{1}{|\mathbf{r}-\mathbf{a}_0|}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\sin\theta\sin\varphi)$$

$$=\begin{cases}\dfrac{2}{a_0}+\dfrac{4r\sin\theta\sin\varphi}{a_0^2}+\ldots, & r<\dfrac{a_0}{2},\\ \dfrac{1}{r}+\dfrac{a_0}{2}\dfrac{\sin\theta\sin\varphi}{r^2}+\ldots, & r>\dfrac{a_0}{2}.\end{cases} \qquad (7)$$

The series in (7) converge for arbitrary $a_0/2$.

Equation (6) with potential (7) is exact. A solution of this equation would give us the spectrum of an atom for the case in which there are shifts of the levels in the optical field and a broadening of the levels, for an arbitrary strength of the field. In practice, however, it is essentially impossible to find such a solution, because potential (7) is not a central potential. If we restrict expansion (7) to the first centrally symmetric term, we can deal correctly with only the asymptotic forms of the potential at $r\ll a_0/2$ and $r\gg a_0/2$. If $a_0/2\ll a_B$, then we would be justified in restricting expansion (7) to the first term at both $r\ll a_0/2$ and $r\gg a_B$. An interpolation of the potential between these regions would not be accurate. The condition $a_0\ll a_B$ constrains the optical field strength F. If we assume $a_0<0.1a_B$, for example, then for a ruby laser with $\lambda=6.9\cdot10^{-5}$ cm the corresponding value of F would be less than or equal to $4\cdot10^{-4}$ of the atomic field strength, while for a KrCl laser with $\lambda=2\cdot10^{-5}$ cm the value of F would be at most $5\cdot10^{-3}$ of the atomic field strength. Note that $a_0/2\ll a_B$ is only a *sufficient* condition for the validity of the theory. A less stringent necessary condition is difficult to formulate explicitly, but in each concrete application of the theory it would be a simple matter to find a numerical estimate of the limitation. For example, data from numerical calculations of the Stark shift of various levels in a potential $(r^2+a_0^2)^{-1/2}$ [see Eq. (47) below] show that reasonable results can be found even for $a_0\lesssim 20a_B$ (Ref. 5).

To single out the centrally symmetric part of the potential from (7), we take an average of

$$V(\mathbf{r},\mathbf{a}_0)=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\sin\theta\sin\varphi) \qquad (8)$$

over solid angle and find an approximate "dressed-atom" potential:

$$V(r,a_0)\approx\begin{cases}2/a_0, & r<a_0/2,\\ 1/r, & r>a_0/2.\end{cases} \qquad (9)$$

At this point we transform to an atomic system of units:

$$e=\hbar=\mu=1,\quad a_B=\frac{\hbar^2}{\mu e^2},\quad \rho=r/a_B.$$

We seek a solution of Eq. (6) with potential (9) in the form

$$f_{E'}(\mathbf{r})=f_l(r)\,Y_l^m(\theta,\varphi), \qquad (10)$$

where Y_l^m is a spherical harmonic. The equation for $f_l(r)$ becomes

$$\left\{-\frac{1}{2}\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right)+\frac{l(l+1)}{2\rho^2}-ZV(\rho,a_0)\right\}f_l(\rho)$$

$$=Ef_l(\rho), \qquad (11)$$

where $E=E'-\gamma\hbar\omega m$, $m=0,1,2,\ldots$.

3. DISCRETE SPECTRUM

It is a simple matter to solve Eq. (11) numerically and to find a system of levels for the discrete spectrum for a given field intensity F and a given field frequency ω:

$$E_{\nu,l}(a_0),\quad l=0,1,2\ldots, \qquad (12)$$

(the levels are numbered in order of increasing energy). To find the wave function corresponding to the spectrum in analytic form we replace potential (9) by a Fuss pseudopotential:

$$V_F(\rho)=-\frac{Z}{\rho}+\frac{1}{\rho^2}\sum_{l=0}^{\infty}B_l\hat{P}_l, \qquad (13)$$

where the projection operator $\hat{P}_l$ projects onto the subspace of spherical harmonics with the given l, and the B_l are parameters. The potential V_F was first used by Simons[6] for calculations of one-photon processes in atoms and by the present author[1,2] for calculations on multiphoton processes in atoms by perturbation theory. This potential has been used to model the potential of an electron in a complex atom; in that case the parameters B_l were chosen by fitting the experimental spectrum of the atom. The potential in (13) has an obvious advantage over potential (9): it is described by a smooth function of ρ, which allows an analytic solution of Eq. (11). In the limit $\rho\to\infty$ this potential has the same asymptotic behavior as a Coulomb potential. A disadvantage of this potential is that it is not local, since B_l depends on the energy.

To solve Eq. (11) with potential (13) we introduce $\nu=Z(-1/2E)^{1/2}$ (for $E<0$), $x=2\rho\sqrt{-2E}$, and

$$\lambda=\left[\left(l+\frac{1}{2}\right)^2+2B_l\right]^{1/2}-\frac{1}{2}. \qquad (14)$$

Equation (11) now becomes

$$\frac{d^2f_{\nu,\lambda_l}(x)}{dx^2}+\frac{2}{x}\frac{df_{\nu,\lambda_l}(x)}{dx}+\left[-\frac{1}{4}+\frac{\nu}{x}-\frac{\lambda_l(\lambda_l+1)}{x^2}\right]f_{\nu,\lambda_l}=0. \qquad (15)$$

We can express the solution of Eq. (15) in terms of the confluent hypergeometric function $F(a,b,x)$:

$$f_{\nu,\lambda_l}(x)=cx^{\lambda_l}e^{-x/2}F(-\nu+\lambda_l+1,2\lambda_l+2,x). \quad (16)$$

The condition

$$\lim_{x\to\infty} f_{\nu,\lambda_l}(x)\to 0$$

leads to

$$-\nu+\lambda_l+1=-p,$$

where $p=0,1,2,\ldots$. Introducing $\nu_{p,l}=p+\lambda_l+1$, we find the energy levels in potential (13):

$$E_{p,l}=-\frac{Z^2}{2\nu_{p,l}^2}, \quad \nu_{p,l}=p+\frac{1}{2}+\left[\left(l+\frac{1}{2}\right)^2+B_l\right]^{1/2}. \quad (17)$$

Replacing $E_{p,l}$ by the levels $E_{\nu,l}(a_0)$ found above [from (12)], we find $\nu_{p,l}(a_0)$. From (17) and (14) we find the corresponding $B_l(a_0)$ and $\lambda_l(a_0)$. We can then write the solution of Eq. (11) in the form (for simplicity, we will write simply ν instead of $\nu_{p,l}$)

$$f_{\nu,l}(\rho)=c_{\nu,l}\left(\frac{2Z\rho}{\nu}\right)^{\lambda_l}\exp\left(-\frac{Z\rho}{\nu}\right)\times F\left(-\nu+\lambda_l+1,2\lambda_l+2,\frac{2Z\rho}{\nu}\right), \quad (18)$$

where $c_{\nu,l}$ is a normalization factor, given by

$$c_{\nu,l}=\left(\frac{2Z}{\nu}\right)^{3/2}\frac{1}{\Gamma(2\lambda_l+2)}\left[\frac{\Gamma(\nu+\lambda_l+2)}{2\nu p!(1-b)}\right]^{1/2}, \quad b=\frac{1}{Z}\frac{\partial\lambda_l}{\partial\nu}. \quad (19)$$

The factor $c_{\nu,l}$ differs from the ordinary normalization of the Coulomb function by the replacement $l\to\lambda_l$; the coefficient b arises because of the nonlocal nature of potential (13). To go back to our original function (2), we substitute (18) into (10), substitute the result into (5), and finally substitute into (2). Applying the shift operator $\exp(i\mathbf{a}\hat{\mathbf{p}})$, we find the function we need (aside from an inconsequential phase factor):

$$\Psi_{E,l}(r,\theta,\varphi,t)=c_{\nu,l}\left\{\frac{2Z}{\nu}[\rho-a_r(t)]\right\}^{\lambda_l}\times\exp\left\{-\frac{Z}{\nu}[\rho-a_r(t)]\right\}\times F\left(-\nu+\lambda_l+1,2\lambda_l+2,\frac{2Z}{\nu}[\rho-a_r(t)]\right)\times Y_l^m\left(\theta-\frac{a_\theta(t)}{\rho},\varphi\right), \quad (20)$$

where

$$a_r(t)=\frac{a_0}{2}\sin\theta\sin(\varphi+\gamma\omega t),$$

$$a_\theta(t)=\frac{a_0}{2}\cos\theta\sin(\varphi+\gamma\omega t) \quad (21)$$

are the projections of the vector $\mathbf{a}(t)$ in the spherical coordinate system onto the unit vectors $\mathbf{r}_0$ and θ_0. In this case we have

$$\hat{p}_r=-i\frac{\partial}{\partial\rho}, \quad \hat{p}_\theta=-i\frac{1}{\rho}\frac{\partial}{\partial\theta}.$$

Wave function (20) can be used for calculations on transitions between discrete levels of the atom, with allowance for reradiation of the intense laser field.

4. CONTINUOUS SPECTRUM

In the continuous spectrum we have $E>0$. To derive expressions suitable for describing both attraction and repulsion, we introduce $\xi=\pm Z_1Z_2/k$, where $k=\sqrt{2E}$. Using analytic continuation in ν, we find the following radial wave function for the continuum, in place of (18):

$$f_{k,l}(r)=c_l(2kr)^{\lambda_l}\exp(ikr)F(1+\lambda_l+i\xi,2\lambda_l+2,-2ikr), \quad (22)$$

where

$$c_l=\exp\left(-\frac{\pi}{2}\xi\right)\frac{|\Gamma(1+\lambda_l+i\xi)|}{\Gamma(\lambda_l+2)}. \quad (23)$$

For $|\xi|\gg 1$ (slow particles: $v\ll v_{at}$) we have

$$c_l\to\begin{cases}2(2\pi\xi)^{1/2}e^{-\pi\xi}, & \xi>0 \text{ (repulsion)},\\ 2(2\pi\xi)^{1/2}, & \xi<0 \text{ (attraction)}.\end{cases} \quad (24)$$

Going back to the time-dependent function $\Psi_{k,l}(r,\theta,\varphi,t)$, we find the following wave function for the continuum:

$$\Psi_{k,l}(r,\theta,\varphi,t)=c_l\{2k[r-a_r(t)]\}^{\lambda_l}\times\exp\{ik[r-a_r(t)]\}\times F(1+\lambda_l+i\xi,2\lambda_l+2,-2ik[r-a_r(t)])\times Y_l^m\left(\theta-\frac{a_\theta(t)}{r},\varphi\right). \quad (25)$$

The wave function in (25) can be applied to the scattering of charged particles (electrons and positrons) in a Coulomb field and to problems of multiphoton ionization. In the case of positrons (protons), we need to make the replacements $a_r\to -a_r$, $a_\theta\to -a_\theta$.

5. POTENTIAL SCATTERING OF SLOW PARTICLES

The problem of potential scattering in the presence of an intense electromagnetic wave arises in studies of stimulated multiphoton bremsstrahlung, the heating of the electrons of a weakly ionized plasma through absorption of laser light, and other cases.

The case of most interest is the scattering of slow charged particles, with $|\xi|\gg 1$. For fast particles, with $|\xi|\ll 1$, one can use the Born approximation. This problem

was solved in the first Born approximation in Ref. 7; it was solved in the second Born approximation and also the eikonal approximation in Ref. 8. The asymptotic behavior of the function in (22) as $r\to\infty$ is found from the asymptotic form of the function $F(a,b,x)$ (Ref. 9).

We have

$$\lim_{l\to\infty} f_{k,l}(r)\to\frac{1}{kr}\sin\left[kr-\xi\ln(2kr)+\sigma_l-\frac{l\pi}{2}+\delta_l(a_0)\right]. \tag{26}$$

Here $\sigma_l=\arg\Gamma(l+1+i\xi)$ is the ordinary Coulomb scattering phase shift, and $\delta_l(a_0)$ is the phase shift of the scattering by the part of potential (13) associated with the presence of the electromagnetic field:

$$\delta_l(a_0)=\arg\frac{\Gamma(\lambda_l+1+i\xi)}{\Gamma(l+1+i\xi)}+\frac{\pi}{2}(l-\lambda_l). \tag{27}$$

For $|\xi|\gg 1$, making use of the asymptotic behavior of the Γ function,

$$\delta_l\approx\begin{cases}\pi(l-\lambda_l), & \text{for attractive forces,}\\ 0 & \text{for repulsive forces.}\end{cases} \tag{28}$$

The method of calculating the phase shift δ_l as a function of λ_l from Eq. (27) is related to the method involving a calculation from the discrete spectrum of the atom in a field, extrapolated to positive energies, $E>0$, in the same way as the quantum defect μ_l calculated from the discrete spectrum of an atom and extrapolated to positive energies is related to the phase shift for the scattering by that atom, δ_l. (The quantum defect incorporates the deviation of the atomic potential from a Coulomb potential and is not a consequence of the external field, but this point is unimportant to the derivation.) Seaton[10] has shown that in the latter case this relationship is

$$\operatorname{ctg}\delta_l(k^2)=\operatorname{ctg}[\pi\mu_l(k^2)][1-\exp(2\pi i\nu)]. \tag{29}$$

The equation $\delta_l(k^2)=\pi\mu_l(k^2)$ at the threshold $k^2\to 0$ corresponds exactly (modulo π) to $\delta_l\approx\pi(l-\lambda_l)$ [Eq. (29)], since we have $\lambda_l=n_l-\mu_l-1$, where n_l is the principal quantum number of the lowest state with the given l. Calculations for atoms near the threshold[1,2] $k^2\to 0$ have shown that the phase shifts calculated from (28) in terms of λ_l differ only insignificantly from those calculated from (29) in terms of $\mu_l(k^2)$. The method of the quantum defect and the method of the Fuss potential (13) are thus closely related. In the former, the integer radial quantum number n_r is replaced by a noninteger; in the latter, the integer orbital number tends toward a noninteger number, $l\to\lambda_l$.

It follows from (26) that the wave function $f_k(\mathbf{r})$, expanded in l, has the asymptotic form

$$f_k(\mathbf{r})\underset{r\to\infty}{\to}\frac{1}{kr}\sum_{l=0}^{\infty}b_l(2l+1)i^l\exp(i\sigma_l)\sin[kr-\xi\ln(2kr)+\sigma_0+\delta_l]P_l(\cos\theta), \tag{30}$$

where the constant b_l must be chosen such that $f_k(r)$ as $r\to\infty$ simultaneously describes a plane wave distorted by the Coulomb field and an outgoing spherical wave. Using the asymptotic expansion of the wave function for a purely Coulomb field, we find[11]

$$f_k(\mathbf{r})|_{r\to\infty}\propto\exp\{ikr+i\xi\ln[k(r-z)]\}+[f^c(\theta)+f_F(a_0)]\frac{1}{r}\exp\{i[kr-\xi\ln(2kr)]\}, \tag{31}$$

where

$$f^c(\theta)=\frac{Z_1Z_2}{v^2}\frac{1}{\sin^2(\theta/2)}\times\exp\left(-i\xi\ln\sin^2\frac{\theta}{2}+i\pi+2i\sigma_0\right) \tag{32}$$

is the Coulomb scattering amplitude, and

$$f_F(a_0)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\exp(2i\sigma_l)\times\{\exp[2i\delta_l(a_0)]-1\} \tag{33}$$

is the scattering amplitude associated with the effect of the optical field on the charged particle. It follows from (28) that we have $f_F(a_0)=0$ for the scattering of positively charged particles. For the scattering of electrons we need retain only the phase shift δ_0, since at $l>0$ we have $\lambda_l\approx l$. The meaning of the the equality $\delta_l=0$ for positrons can also be seen easily from Eq. (24), since the probability of positrons penetrating the region $r<a_0$ is exponentially small under the condition $|\xi|\gg 1$: $\sim 4(2\pi\xi)\exp(-2\pi\xi)$. Substituting the function $f_k(\mathbf{r})$ from (31) into (2), and taking the limit $r\to\infty$, we find

$$\Psi_{k,l}(\mathbf{r},t)\propto\exp\left\{ik[r-a(t)]+i\xi\ln\left[2k\sin^2\frac{\theta}{2}[r-a_r(t)]\right]\right\}+[f^c(\theta)+f_F(a_0)]\frac{\exp\{ik[r-a_r(t)]\}}{r}\times\exp\{-i\xi\ln[2k(r-a_r(t))]\}, \tag{34}$$

where $a_r(t)$ is defined in (21). Using the expansion

$$\exp[iz\cos\omega t]=\sum_{n=-\infty}^{\infty}i^nJ_n(z)\exp(in\omega t)$$

from the theory of Bessel functions, and using the Neumann summation formula,[9]

$$J_p(u+v)=\sum_{k=-\infty}^{\infty}J_{p\pm k}(u)J_k(v),\quad p=0,1,2,\ldots,$$

we find

$$f_F^c(\theta)=J_p\left(\frac{1}{2}a_0\mathbf{k}(\mathbf{e}_x+\gamma\mathbf{e}_y)\right)[f^c(\theta)+f_F(a_0)] \tag{34a}$$

(we have omitted some inconsequential phase factors). The corresponding cross section $d\sigma$ for electrons is

$$d\sigma(\theta,\varphi)=J_p^2\left(\frac{a_0k}{2}\left[\sin\theta(\cos\varphi\pm\sin\varphi)\right]\right)|f^c(\theta)$$
$$+f_F(a_0)|^2d\Omega. \tag{35}$$

The distinction from the Born approximation here is that the expression for the cross section contains, in addition to the exact amplitude for Coulomb scattering without a field, the amplitude $f_F(a_0)$ and a term representing the interference of these two amplitudes. For the scattering of positrons we must set $f_F(a_0)=0$ in (35). It follows from this analysis that for slow positrons the region $r<a_0$ can immediately be eliminated from consideration. The variables in the time-independent Schrödinger equation (10) with $a_0=0$ then separate in parabolic coordinates. Reverting to r, we can then find the wave function in the momentum representation:[12]

$$\Psi_k(\mathbf{r})=\exp(-\pi/2k)\Gamma(1+i\xi)\exp(i\mathbf{kr})$$
$$\times F\left(-i\xi,1,2ikr\sin^2\frac{\theta}{2}\right). \tag{36}$$

Going over to the time-dependent function in (2), we find

$$\Psi_k(\mathbf{r},t)=\exp(-\pi/2k)\Gamma(1+i\xi)$$
$$\times\exp\{i\mathbf{k}[\mathbf{r}+\mathbf{a}(t)]\}F$$
$$\times\left(-i\xi,1,2ik\,|\mathbf{r}+\mathbf{a}(t)|\sin^2\frac{\theta}{2}\right). \tag{37}$$

Any polarization of the wave can chosen in (37), and the latter function can also be used for slow electrons with a large impact parameter (with the substitution $\mathbf{a}\to-\mathbf{a}$).

6. LINEARLY POLARIZED FIELD

Let us examine the approximations which must be made in order to study a linearly polarized field. We will see below that these approximations also apply to a circularly polarized field, so the method developed here can be generalized to elliptical polarization. In Eq. (3) we set

$$\mathbf{a}(t)=\mathbf{a}_L(t)=a_0\mathbf{e}_x\sin\omega t$$

for a linear polarization,

$$\mathbf{a}(t)=\mathbf{a}_c(t)=\frac{a_0}{2}(\mathbf{e}_x\sin\omega t\pm\mathbf{e}_y\cos\omega t)$$

for a circular polarization. We transform to oscillating coordinate system (2), using a different transformation for the Coulomb potential.[13] We use a Fourier representation for the Coulomb field:

$$V=\int\frac{\exp(i\mathbf{qr})}{q^2}d^3q, \tag{38}$$

or, in the oscillating coordinate system,

$$V(|\mathbf{r}-\mathbf{a}(t)|)=\int\frac{\exp\{iq[\mathbf{r}-\mathbf{a}(t)]\}}{q^2}d^3q. \tag{39}$$

We expand $\exp[iz\sin\omega t]$ and $\exp[iz\cos\omega t]$ in Fourier series in Bessel functions, and we use an integral representation of the Bessel function:

$$J_n(z)=\frac{i^{-n}}{\pi}\int_0^\pi\exp(iz\cos\alpha)\cos n\alpha d\alpha. \tag{40}$$

(For a circularly polarized field, we would also have to use the Neumann summation theorem for J_p here.) We then find

$$V_n(|\mathbf{r}-\mathbf{a}(t)|)=\sum_{n=-\infty}^{\infty}\frac{i^{-n}}{\pi}e^{in\omega t}\int_0^\pi\frac{\cos n\alpha d\alpha}{|\mathbf{r}-\mathbf{a}_0\cos\alpha|}, \tag{41}$$

where

$$\mathbf{a}_0=\begin{cases}\mathbf{a}_0^L=a_0\mathbf{e}_x,\\ \mathbf{a}_0^c=\dfrac{a_0}{2}(\mathbf{e}_x+\mathbf{e}_y).\end{cases} \tag{42}$$

The time-dependent potential in (41) describes the interaction of an electron with an oscillating Coulomb potential. For a circularly polarized field, a second unitary transformation of (5) rendered the potential time-independent. In the approximation of a centrally symmetric potential (9), we found an atom "dressed in a field." This was the model used in the subsequent calculations. We will show that potential (41), averaged over the field oscillation period, with only the centrally symmetric part of the expansion of this average potential retained, leads to a dressed atom which is very nearly the same as potential (9). The interaction which is found is suitable for both circularly and linearly polarized light.

Taking an average of (41) over the oscillation period, we find

$$\overline{V_n(|\mathbf{r}-\mathbf{a}(t)|)}\equiv V_0(|\mathbf{r}-\mathbf{a}_0|)$$
$$=\frac{1}{\pi}\int_0^\pi\frac{d\alpha}{|\mathbf{r}-\mathbf{a}_0\cos\alpha|}. \tag{43}$$

Let us examine the last expression:

$$\frac{1}{\pi}\int_0^\pi\frac{d\alpha}{|\mathbf{r}-\mathbf{a}_0\cos\alpha|}$$
$$=\frac{1}{(r^2+a_0^2)^{1/2}}\frac{1}{\pi}\int_0^\pi\left(1-\frac{2\mathbf{ra}_0\cos\alpha}{r^2+a_0^2}-\frac{a_0^2\sin^2\alpha}{r^2+a_0^2}\right)^{1/2}d\alpha. \tag{44}$$

The integral in (44) can be reduced to an elliptic integral of the first kind. It contains both centrally symmetric and asymmetric interaction components. To find at least some sort of solution for the problem, we restrict it to the centrally symmetric part, as was done in (9); i.e., we omit the term $2\mathbf{ra}_0\cos\alpha/(r^2+a_0^2)$. We then have

$$V_0(|\mathbf{r}-\mathbf{a}_0|)\approx\frac{1}{(r^2+a_0^2)^{1/2}}\frac{2}{\pi}K(m), \tag{45}$$

where $K(m)$ is the complete elliptic integral of the first kind, and

$$m=\frac{a_0^2}{r^2+a_0^2}$$

is the parameter of this integral. For all $r>0$ we can use a series expansion for $K(m)$ or approximate it by polynomials. In the case of the series expansion, we have the following result for the potential in (45):

$$V_0(|\mathbf{r}-\mathbf{a}_0|)\approx\frac{1}{(r^2+a_0^2)^{1/2}}\left[1+\left(\frac{1}{2}\right)^2 m+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2+\ldots\right]. \quad (46)$$

It is easy to see from (46) that this potential is approximately the same as the potential in (9). Using the value of (45) at $r=0$, we can make the approximation

$$V_0(|\mathbf{r}-\mathbf{a}_0|)\approx\frac{1}{(r^2+a_0^2)^{1/2}} \quad (47)$$

and use (47) for an arbitrary polarization with an appropriate choice of a_0^2 from (42). Solving the Schrödinger equation with potential (45) numerically, we find a system of levels $\bar{E}_{\nu,l}(a_0)$ different from (12). Constructing a pseudopotential (13) for this system, we find new values of λ_l^L and λ_l^c for linearly and circularly polarized waves. All the other equations remain the same as before, with the substitutions $\lambda_l\to\lambda_l^L$ and $\lambda_l\to\lambda_l^c$. In addition, for the projections of the vector $\mathbf{a}(t)$ for the linear polarization onto the unit vectors $\mathbf{r}_0$ and $\boldsymbol{\theta}_0$, the following expressions now hold instead of (21):

$$a_r^L(t)=a_0\sin\theta\cos\varphi\sin\omega t,$$

$$a_\theta^L(t)=a_0\cos\theta\cos\varphi\sin\omega t.$$

7. CONCLUSION

The wave functions constructed for the discrete and continuous spectra in this paper may also prove extremely useful for calculations of multiphoton processes under conditions consistent with the use of standard perturbation theory in the field, for the following reasons. First, the wave functions are in closed analytic form here, and they do not contain a multiple summation over the entire spectrum as in perturbation theory, in which one can find only numerical solutions for the wave functions. Second—and this is a particularly important point—within the framework of perturbation theory it is essentially impossible to find a multiphoton wave function for an electron in the continuous spectrum of a Coulomb potential and the field of a light wave. This is the wave function required for problems involving scattering and above-threshold absorption of photons in the course of multiphoton ionization. The analytic expression for this wave function [see (25)] contains both limiting cases: with $Z=0$, the equation reduces to the nonrelativistic Volkoff function for an electron in the field of a wave, and with $F=0$ it becomes the continuum wave function in a Coulomb potential. The scattering amplitude found with the help of this function [see (34a)] cannot be calculated by time-dependent perturbation theory, which implements only the Born approximation of the scattering problem. We might add that the calculations of the shift of the atomic levels in the field could be refined without difficultly by incorporating the first noncentrally symmetric terms of series (7) or the corresponding terms of series (41) by perturbation theory, since the Green's function required here has been constructed for pseudopotential (13) in some of our previous papers.[1,2] As a result, we find a refinement of λ_l in Eq. (14). The analytic form of the wave functions remains the same, of course.

The methods developed in this study for constructing electron wave functions in a hydrogen atom in a laser field can also be used to model the wave functions of complex atoms.

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