Dynamical effect of finite nuclear size in the nuclear excitation process during electron transitions in an atomic shell

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We study in this paper anomalies in the excitation of nuclei during atomic transitions (dynamical effect of the nuclear volume) which are similar to the anomalies in the internal electron conversion of γ rays. We obtain numerical estimates for the relative contributions of the anomalies to the magnitude of the probability for the excitation of nuclei in atomic transitions for a number of nuclei with l-forbidden $M1$ transitions, and for $E1$ transitions forbidden in the asymptotic quantum number of the Nilsson model.

1. INTRODUCTION

Since the mid-1950's, we have known of dynamical effects connected with the penetration of the electron current into the nucleus during the electron internal conversion of γ-rays,[1] and at the present time there is great interest in them for studies in nuclear physics. This problem, which is called anomalous conversion, has been the subject of a considerable number of theoretical and experimental papers both at home and abroad. The most general features of the process were explained in the first publications by Church and Weneser,[1,2] Grechukhin,[3,4] Green and Rose,[5] Voĭkhanskiĭ and Listengarten,[6] Kramer and Nilsson,[7] and a number of other authors using different nuclear models and for various nuclear transitions. The present status of anomalous conversion and the most complete lists of references can be found in the book by Band *et al.*[8] and the review by Listengarten.[9] We note merely that even now the measurement of anomalies in the electron internal conversion coefficient (ICC) is essentially the only means for obtaining unique information about, for instance, the magnitude of nuclear penetration matrix elements, the contribution of toroidal moments to the probability of electric nuclear transitions, and certain other important nuclear characteristics (for details see Refs. 8 and 9).

Meanwhile, there is yet another process in which it is possible to measure the same nuclear characteristics as in anomalous conversion. This is the nuclear excitation process during atomic transitions, otherwise known as NEAT (a term suggested by M. A. Listengarten); in the English literature on this mechanism starting in the seventies, it was referred to as Nuclear Excitation by Electron Transitions, or NEET.[10] A list of the most important publications and the theory of the process are given in Refs. 11 and 12. The crux of NEET is the possibility of exciting a nucleus close to resonance or even at resonance via the transfer of a vacancy from the lowest atomic shell upwards, when the energy and multipolarity of the nuclear transition and one of the atomic transitions are the same. This is a fairly rare process, but its probability is entirely measurable by contemporary methods (see the review of experimental work in Refs. 11 and 12).

In most theoretical models of NEET known at present, the region of the nucleus is taken into account in the simplest way: it is treated as a point. Somewhat better is the calculation in Refs. 11 and 12. We chose a nucleus with radius $R_0=1.2A^{1/3}$ fm (A is the atomic number) and uniform charge distribution. The self- consistent field of the atom and the electron wave functions (WF) were calculated in the nuclear potential

$$V(r)=\begin{cases}\dfrac{1}{2}\dfrac{e^2Z}{R_0}\left(3-\left(\dfrac{r}{R_0}\right)^2\right), & r<R_0,\\ \dfrac{e^2Z}{r}, & r>R_0.\end{cases} \tag{1}$$

We used the γ-emission nuclear matrix elements (NME). In this way we took into account the static effect of the finite size of the nucleus, but nothing more. The use of the γ-emission NME, which is completely valid for intraband $M1$ transitions, as for instance in the ^{189}Os nucleus (which is the most popular object for studying the NEET process), is not totally correct[13] for calculating the probability of NEET in the ^{197}Au and ^{193}Ir nuclei, with l-forbidden $M1$ transitions, or for ^{181}Ta and possibly ^{237}Np, with $E1$ transitions forbidden in the asymptotic quantum number of the Nilsson model. Indeed, the NEET process in ^{197}Au and ^{193}Ir occurs for $3S_{1/2}\to 1S_{1/2}$ electron transitions in their atomic shells, and in ^{181} Ta and ^{237}Np for transitions between the $S_{1/2}$ and $P_{1/2}$ shells. The WF of the $|S_{1/2}\rangle$ and $|P_{1/2}\rangle$ atomic states do not vanish in the vicinity of the nucleus. The electron current penetrates efficiently into the nucleus for transitions between the indicated shells. Thus, when the coordinates of the electron current $j_\mu(\mathbf{r})$ and the nuclear current $J_\nu(\mathbf{R})$ satisfy the condition $r<R$ (dynamical effect of the nuclear volume), the intranuclear process may also make a considerable contribution to the probability of NEET in processes where there is an anomaly in the ICC.

The aim of the present paper is to determine the role of the dynamical effect of the penetration of the electron current into the nucleus in the excitation of nuclei in atomic transitions. Taking into account the relationship between NEET and the electron internal conversion processes we shall, wherever possible, adhere to the notation normally

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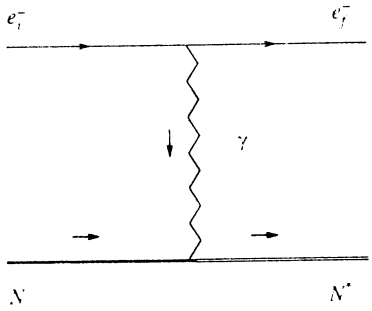

FIG. 1. NEET process diagram.

used when describing anomalous conversion. We use a system of units with $\hbar=c=1$.

2. PROBABILITY OF NEET TAKING INTO ACCOUNT THE DYNAMICAL PENETRATION EFFECT

It has been shown in Refs. 11 and 12 that the nuclear excitation process during electron transitions in an atomic shell can be described in the framework of the relativistic perturbation theory in quantum electrodynamics by a second-order Feynman diagram (Fig. 1). The use here in the electron wavefunctions of the real widths of the vacancies in the atomic shells between which the electron transition takes place enables us also to take into account the further decay of the electron hole into an upper atomic level without having to consider third- or higher-order processes.[11]

The relative probability P of the NEET process in the decay of a vacancy in one of the lower atomic shells can be evaluated using the formula[11]

$$P=\left(1+\frac{\Gamma_i}{\Gamma_f}\right)\frac{E_{\rm int}^2(L;\omega_N;i\to f,I\to F)}{(\omega_N-\omega_A)^2+\left(\frac{\Gamma_i+\Gamma_f}{2}\right)^2}, \tag{2}$$

in which $\Gamma_{i,f}$ are the widths of the vacancies in the atomic levels in the initial and final states, $|i\rangle$ and $|f\rangle$, of the electron, and ω_N and ω_A are the energies of the nuclear and atomic transitions of multipolarity L, respectively. We have denoted by $E_{\rm int}^2$ in (2) the absolute square of the interaction energy $H_{\rm int}$ of the electron current $j_\mu(\mathbf{r})$ and the nuclear current $J_\nu(\mathbf{R})$ in the second order of perturbation theory,

$$H_{\rm int}=\int_0^\infty d^3R\int_0^\infty d^3r j^\mu(\mathbf{r})\,D_{\mu\nu}(\omega_N;\mathbf{r}-\mathbf{R})J^\nu(\mathbf{R}) \tag{3}$$

averaged over the initial and summed over the final states. In (3) the photon propagator is taken in the coordinate–frequency representation[14]

$$D_{\mu\nu}(\omega_N;\mathbf{r}-\mathbf{R})=-g_{\mu\nu}\frac{\exp\,(i\omega_N|\mathbf{r}-\mathbf{R}|)}{|\mathbf{r}-\mathbf{R}|}, \tag{4}$$

$g_{\mu\nu}$ is the metric tensor, and the shell electron is the source of the field. The expressions for the currents have the form

$$J_\nu(t,\mathbf{R})=\exp(i\omega_N t)J_\nu(\mathbf{R}),\quad J_\nu(\mathbf{R})=e\Psi_F^+(\mathbf{R})\hat{J}_\nu\Psi_I(\mathbf{R}),$$

$$j_\mu(t,\mathbf{r})=\exp(-i\omega_N t)j_\mu(\mathbf{r}),\quad j_\mu(\mathbf{r})=-e\bar{\psi}_f(\mathbf{r})\gamma_\mu\psi_i(\mathbf{r}),$$

where the γ_μ are Dirac matrices, $\hat{J}_\nu$ is the operator of the nuclear electromagnetic current, $\hat{J}_\nu=(I,\hat{\mathbf{J}})$ with I the unit matrix, e is the proton electric charge, and we have denoted the WF of the nucleus and the electron by $\Psi(\mathbf{R})$ and $\psi(\mathbf{r})$, respectively. We shall use the representation

$$\psi(\mathbf{r})=\begin{pmatrix} g_{lj}(r)\Omega_{jlm}(\mathbf{n}) \\ if_{l'}j(r)\Omega_{jl'm}(\mathbf{n})\end{pmatrix}, \tag{5}$$

with the normalization condition $\int_0^\infty dr\,r^2\,[g^2(r)+f^2(r)]=1$. Here $g(r)$ is the large, and $f(r)$ the small, component of the radial WF, $\mathbf{n}=\mathbf{r}/r$, $\Omega_{jlm}(\mathbf{n})$ are the spherical spinors from Ref. 14, and l and j are the orbital and total angular momenta of the electron states.

Since the nuclear and atomic transitions have definite angular momenta and parity, one must expand the propagator (4) in the expression for the interaction energy (3)in terms of multipoles. It is convenient to take for the latter the scalar,

$$A_{Lm}(\mathbf{r};\omega)=j_L(\omega r)Y_{Lm}(\mathbf{n}),$$

$$\mathbf{A}_{Lm}^E(\mathbf{r};\omega)=\sqrt{\frac{L+1}{2L+1}}\,j_{L-1}(\omega r)\mathbf{Y}_{LL-1;m}(\mathbf{n})$$
$$-\sqrt{\frac{L}{2L+1}}\,j_{L+1}(\omega r)\mathbf{Y}_{LL+1;m}(\mathbf{n}), \tag{6}$$

$$\mathbf{A}_{Lm}^M(\mathbf{r};\omega)=j_L(\omega r)\mathbf{Y}_{LL;m}(\mathbf{n}),$$

$$\mathbf{A}_{Lm}^Y(\mathbf{r};\omega)=\sqrt{\frac{L}{2L+1}}\,j_{L-1}(\omega r)\mathbf{Y}_{LL-1;m}(\mathbf{n})$$
$$+\sqrt{\frac{L+1}{2L+1}}\,j_{L+1}(\omega r)\mathbf{Y}_{LL+1;m}(\mathbf{n})$$

electromagnetic potentials (electric components E, magnetic components M in the Coulomb gauge, and longitudinal components Y).[15] The $j_L(x)$ here are Bessel functions, the $\mathbf{Y}_{LJ;M}(\mathbf{n})$vector spherical harmonics, $\mathbf{Y}_{LJ;M}(\mathbf{n})=\Sigma_{m,\mu}(Lm1\mu|JM)Y_{Lm}(\mathbf{n})\xi_\mu$, the $(Lm1\mu|JM)$ are Clebsch–Gordan coefficients, the $Y_{Lm}(\mathbf{n})$ are spherical functions, and the $\xi_\mu(\mu=\pm1,0)$ are the components of the photon spin vector in a cyclic base.

The expansion of the convolution of the propagator with the currents in (3) is well known for the regions $r<R$ and $r>R$ and has the form[2]

$$J_0(\mathbf{R})\frac{\exp(i\omega_N|\mathbf{r}-\mathbf{R}|)}{|\mathbf{r}-\mathbf{R}|}j_0(\mathbf{r})=4\pi i\omega_N\sum_{L,M}\begin{cases}J_0(\mathbf{R})A^*_{LM}(\mathbf{R};\omega_N)B_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r}), & r>R,\\ J_0(\mathbf{R})B^*_{LM}(\mathbf{R};\omega_N)A_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r}), & r<R,\end{cases}$$

$$\mathbf{J}(\mathbf{R})\frac{\exp(i\omega_N|\mathbf{r}-\mathbf{R}|)}{|\mathbf{r}-\mathbf{R}|}j(\mathbf{r})=4\pi i\omega_N\sum_{L,M,a}\begin{cases}\mathbf{J}(\mathbf{R})\cdot\mathbf{A}^*_{LM}(\mathbf{R};\omega_N)\mathbf{B}^a_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r}), & r>R,\\ \mathbf{J}(\mathbf{R})\cdot\mathbf{B}^{a*}_{LM}(\mathbf{R};\omega_N)\mathbf{A}^a_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r}), & r<R.\end{cases} \tag{7}$$

The potentials $B_{LM}(\mathbf{r};\omega)$ and $\mathbf{B}^a_{LM}(\mathbf{r};\omega)(a=E,M,Y)$ are obtained from $A_{LM}(\mathbf{r};\omega)$ and $\mathbf{A}^a_{LM}(\mathbf{r};\omega)$ by substituting the Hankel functions of the first kind $h^{(1)}_L$ (ωr) for the Bessel functions in Eqs. (6). The asterisk in (7) indicates complex conjugation of the spin and angular functions which occur in Eqs. (6). The notation $\mathbf{J}(\mathbf{R})\cdot\mathbf{A}^a_{LM}(\mathbf{R};\omega)$ indicates a scalar product. We shall use this notation in those cases where it is necessary to make clear just which vectors are multiplied.

Using the expansions (7), we can write the interaction energy in the following form, explicitly distinguishing between the regions $r>R$ and $r<R$;

$$\begin{aligned}H_{\rm int}=4\pi i\omega_N\sum_{L,M}\Bigg\{&\int_0^\infty d^3RJ_0(\mathbf{R})\\ &\times\Bigg[A^*_{LM}(\mathbf{R};\omega_N)\int_R^\infty d^3rB_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r})\\ &+B^*_{LM}(\mathbf{R};\omega_N)\int_0^R d^3rA_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r})\Bigg]\\ &-\sum_a\int_0^\infty d^3R\mathbf{J}(\mathbf{R})\Bigg[\mathbf{A}^{a*}_{LM}(\mathbf{R};\omega_N)\int_R^\infty d^3r\\ &\times\mathbf{B}^a_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})\\ &+\mathbf{B}^{a*}_{LM}(\mathbf{R};\omega_N)\int_0^R d^3r\mathbf{A}^a_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})\Bigg]\Bigg\}.\end{aligned} \tag{8}$$

In (8) one integrates first over the electron coordinate and afterwards over the nuclear variable.

In (8), after the standard procedure of adding and subtracting the functions

$$A^*_{LM}(\mathbf{R};\omega_N)\int_0^R d^3rB_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r})$$

and

$$\mathbf{A}^{a*}_{LM}(\mathbf{R};\omega_N)\int_0^R d^3r\mathbf{B}^a_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})$$

in square brackets, we can write $H_{\rm int}$ as the sum of two terms,

$$H_{\rm int}=H^R_{\rm int}+\Delta H_{\rm int}, \tag{9}$$

the first of which is the interaction energy in the Rose model (or the model "without penetration"[8]),

$$\begin{aligned}H^R_{\rm int}=4\pi i\omega_N\sum_{L,M}\Bigg\{&\int_0^\infty d^3RJ_0(\mathbf{R})A^*_{LM}(\mathbf{R};\omega_N)\\ &\times\int_0^\infty d^3rB_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r})\\ &-\sum_a\int_0^\infty d^3R\mathbf{J}(\mathbf{R})\cdot\mathbf{A}^{a*}_{LM}(\mathbf{R};\omega_N)\\ &\times\int_0^\infty d^3r\mathbf{B}^a_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})\Bigg\},\end{aligned} \tag{10}$$

while the second is the "additional" interaction energy resulting from the dynamical effect of the penetration of the electron current into the nucleus:

$$\begin{aligned}\Delta H_{\rm int}=4\pi i\omega_N\sum_{L,M}\Bigg\{&\int_0^\infty d^3RJ_0(\mathbf{R})\\ &\times\Bigg(B^*_{LM}(\mathbf{R};\omega_N)\int_0^R d^3rA_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r})\\ &-A^*_{LM}(\mathbf{R};\omega_N)\int_0^R d^3rB_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r})\Bigg)\\ &-\sum_a\int_0^\infty d^3R\mathbf{J}(\mathbf{R})\Bigg(\mathbf{B}^{a*}_{LM}(\mathbf{R};\omega_N)\int_0^R d^3r\\ &\times\mathbf{A}^a_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})\\ &-\mathbf{A}^{a*}_{LM}(\mathbf{R};\omega_N)\int_0^R d^3r\mathbf{B}^a_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})\Bigg)\Bigg\}.\end{aligned} \tag{11}$$

The evaluation of $H^R_{\rm int}$ taking into account the finite size of the nucleus is not a problem. It was performed for a number of nuclei in Refs. 11 and 12. The problem now consists in estimating the relative contribution of the additional term $\Delta H_{\rm int}$.

3. ANOMALIES IN NEET FOR MAGNETIC DIPOLE TRANSITIONS

We consider the contribution to the NEET process for magnetic dipole transitions. To do this we split off from Eqs. (10) and (11) the parts corresponding to magnetic transitions. Since among the potentials (6) there is only one with the appropriate parity, the interaction energy for a magnetic transition of multipolarity L is given in the Rose model by

$$\begin{aligned}H^R_{\rm int}(ML)=4\pi i\omega_N\sum_{L,M}\Bigg\{&-\int_0^\infty d^3R\mathbf{J}(\mathbf{R})\\ &\times\mathbf{A}^{M*}_{LM}(\mathbf{R};\omega_N)\int_0^\infty d^3r\mathbf{B}^M_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})\Bigg\},\end{aligned} \tag{12}$$

and the additional energy is given by

$$\Delta H_{\rm int}(ML)=4\pi i\omega_N \sum_{L,M}\left\{-\int_0^\infty d^3R\mathbf{J}(\mathbf{R})\right.$$
$$\times\left(\mathbf{B}^{M*}_{LM}(\mathbf{R};\omega_N)\right.$$
$$\times\int_0^R d^3r\mathbf{A}^{M}_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})\mathbf{A}^{M*}_{LM}(\mathbf{R};\omega_N)$$
$$\left.\left.\times\int_0^R d^3r\mathbf{B}^{M}_{LM}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})\right)\right\}. \qquad (13)$$

Using the connection between the magnetic potentials $\mathbf{B}^{M*}_{LM}(\mathbf{R};\omega)=[h^{(1)}_L(\omega R)/j_L(\omega R)]\mathbf{A}^{M*}_{LM}(\mathbf{R};\omega)$, we can write $H_{\rm int}(ML)$ in the form

$$H_{\rm int}(ML)=-4\pi i\omega_N\sum_{L,M}\int_0^\infty d^3R\mathbf{J}(\mathbf{R})$$
$$\times A^{M*}_{LM}(\mathbf{r};\omega_N)\left\{\int_0^\infty d^3r\mathbf{j}(\mathbf{r})\right.$$
$$\times\mathbf{B}^{M}_{LM}(\mathbf{r};\omega_N)+\int_0^R d^3r\mathbf{j}(\mathbf{r})$$
$$\times\left(\frac{h^{(1)}_L(\omega_N R)}{j_L(\omega_N R)}\mathbf{A}^{M}_{LM}(\mathbf{r};\omega_N)\right.$$
$$\left.\left.-\mathbf{B}^{M}_{LM}(\mathbf{r};\omega_N)\right)\right\}. \qquad (14)$$

We can calculate the angular part using the well known formulas of Refs. 8 and 16:

$$\int d\Omega_{\mathbf{r}}\bar{\psi}_f(\mathbf{r})\gamma\psi_i(\mathbf{r})\mathbf{Y}_{LL;m}(\mathbf{n})$$
$$=i\frac{\Lambda(\kappa_i+\kappa_f)}{\sqrt{L(L+1)}}[g_i(r)f_f(r)+g_f(r)f_i(r)], \qquad (15)$$

where

$$\Lambda=(j_im_iLm|j_fm_f)(j_i-\tfrac{1}{2}\,L0|j_f-\tfrac{1}{2})\sqrt{\frac{2L+1}{4\pi}\frac{2j_i+1}{2j_f+1}}, \qquad (16)$$

while $\kappa=(l-j)(2j+1)$. (We note that the use of the representation (5) for the electron WF, and phases of the γ matrices corresponding to a Dirac equation of the form $E\psi=(\mathbf{p}\alpha+m\beta+V)\psi$, where $\beta=\gamma^0,\alpha=\gamma^0\gamma$, leads to a sign different from that in Ref. 8 in matrix elements containing the matrix α.)

After integrating over the angular variables, we obtain for the interaction energy

$$H_{\rm int}(ML)=-4\pi i\omega_N\sum_{L,M}\frac{-ie\Lambda}{\sqrt{L(L+1)}}\int_0^\infty d^3R\mathbf{J}(\mathbf{R})$$
$$\times\mathbf{A}^{M*}_{LM}(\mathbf{r};\omega_N)\left\{\langle\mathrm{m}^M_L(\omega_N)\rangle+(\kappa_i+\kappa_f)\right.$$
$$\times\int_0^R drr^2\left(\frac{h^{(1)}_L(\omega_N R)}{j_L(\omega_N R)}j_L(\omega_N R)\right.$$
$$\left.\left.-h^{(1)}_L(\omega_N r)\right)(g_i(r)f_f(r)+g_f(r)f_i(r))\right\},$$

where the atomic matrix element

$$\langle\mathrm{m}^M_L(\omega_N)\rangle=(\kappa_i+\kappa_f)\int_0^\infty drr^2h^{(1)}_L(\omega_N r)$$
$$\times[g_i(r)f_f(r)+g_f(r)f_i(r)] \qquad (18)$$

must be calculated numerically. The second term in (17) is an integral over the region of the nucleus where $\omega_N r\ll 1$. (Indeed, in NEET ω_N cannot exceed the binding energy of an electron in the K shell of the atom, so that we always have $\omega_N R\ll 1$ in that process.) To evaluate it, we expand the Bessel and Hankel functions in their small arguments, and we expand the electron WF in powers of r in the vicinity of the nucleus. Since one of the necessary conditions for the occurrence of an anomaly is a considerable difference in the amplitude of the electron WF in the nucleus, we need only an expansion for the wave functions of the $nS_{1/2}$ and $mP_{1/2}(n,m,=1,2,3...)$ subshells. For these WF, we can write the well known expansion (see, e.g., Ref. 8) in the following form:

$$g(r)=\frac{g(x)}{a_B^{3/2}},\quad f(r)=\frac{f(x)}{a_B^{3/2}},\quad X\equiv\frac{r}{a_B}$$
$$S_{1/2}:\ g(x)=c,\quad f(x)=cf^0x,$$
$$P_{1/2}:\ g(x)=cg^0x,\quad f(x)=c. \qquad (19)$$

We have introduced here the Bohr radius a_B to distinguish a characteristic atomic scale, while g^0,f^0 and c are constants to be calculated.

We can now evaluate the integral over the electron variable in the second term in (17) explicitly, for an $M1$ transition (we shall distinguish the latter by the Kronecker symbol δ_{M1}) it turns out to be

$$i\frac{3}{10}\left(\frac{R_0}{a_B}\right)^2\frac{c_fc_f}{(\omega_Na_B)^2}\begin{cases}(f_i^0+f_f^0)\delta_{M1}(nS_{1/2}\leftrightarrow mS_{1/2})\\(g_i^0+g_f^0)\delta_{M1}(nP_{1/2}\leftrightarrow mP_{1/2})\end{cases}. \qquad (20)$$

Finally, introducing two nuclear current magnetic-multipole transition matrix elements, a normal $\langle N^M_{LM}\rangle$ and an anomalous $\langle N^{M(an)}_{LM}\rangle$ on,

$$\langle N^M_{LM}(\omega_N)\rangle=\int_0^\infty d^3R\mathbf{J}(\mathbf{R})\cdot\mathbf{A}^{M*}_{LM}(\mathbf{R};\omega_N),$$
$$\langle N^{M(an)}_{LM}(\omega_N)\rangle=\int_0^\infty d^3R\left(\frac{R}{R_0}\right)^2\mathbf{J}(\mathbf{R})\cdot\mathbf{A}^{M*}_{LM}(\mathbf{R};\omega_N), \qquad (21)$$

and the nuclear penetration parameter of Ref. 8, $\lambda^{(0)}=\langle N^{M(an)}_{LM}\rangle/\langle N^M_{1M}\rangle$, we can write the final expression for the interaction energy for a magnetic transition in the form

$$\Delta H_{\rm int}(ML)=4\pi i\omega_N\sum_{L,M}\frac{ie\Lambda}{\sqrt{L(L+1)}}\langle N^M_{LM}(\omega_N)\rangle$$
$$\times\left\{\langle\mathrm{m}^M_{LM}(\omega_N)\rangle\right.$$

$$+i\frac{\lambda^{(0)}}{4\pi^2}\left(\frac{R_0}{a_B}\right)^2\left(\frac{\lambda_N}{a_B}\right)^2(\kappa_i\kappa_f)\frac{3c_ic_f}{10}$$
$$\times\begin{pmatrix}(f_i^0+f_f^0)\delta_{M1}(nS_{1/2}\leftrightarrow mS_{1/2})\\(g_i^0+g_f^0)\delta_{M1}(nP_{1/2}\leftrightarrow mP_{1/2})\end{pmatrix}\Bigg|, \tag{22}$$

where $\lambda_N=2\pi/\omega_N$ is the wavelength of the nuclear transition.

Since the atomic matrix element $\langle \mathrm{m}_L^M(\omega_N)\rangle$ can be calculated explicitly using Eq. (18), we can determine the relative contribution of the anomaly by comparing the terms in the braces in Eq. (22).

4. ANOMALIES IN NEET IN ELECTRIC DIPOLE TRANSITIONS

The calculation of the anomalies in electric transitions is considerably more complicated. In that case there remain in Eq. (11) for the interaction energy three terms that contain scalar, electric, and longitudinal multipoles. Working with the potentials (6) now becomes exceedingly cumbersome and inconvenient. We therefore introduce, as in Ref. 8, the pair of vector potentials

$$\mathbf{A}_{LM}^{(\pm1)}(\mathbf{r};\omega)=j_{L\pm1}(\omega r)\mathbf{Y}_{LL\pm1,M}(\mathbf{n}),\quad \mathbf{B}_{LM}^{(\pm1)}(\mathbf{r};\omega)$$
$$=h_{L\pm1}^{(1)}(\omega r)\mathbf{Y}_{LL\pm1;M}(\mathbf{n}). \tag{23}$$

The electromagnetic potentials (6) can be expressed in terms of the functions (23):

$$\mathbf{A}_{LM}^{E}(\mathbf{r};\omega)=\sum_{\nu=\pm1}(-1)^{(1+\nu)/2}$$
$$\times\sqrt{\frac{L+(1-\nu)/2}{2L+1}}\,\mathbf{A}_{LM}^{(\nu)}(\mathbf{r};\omega)$$
$$\mathbf{A}_{LM}^{Y}(\mathbf{r};\omega)=\sum_{\nu=\pm1}\sqrt{\frac{L+(1+\nu)/2}{2L+1}}\,\mathbf{A}_{LM}^{(\nu)}(\mathbf{r};\omega). \tag{24}$$

However, the main convenience of the potentials (23) consists in that, firstly, as in the case of the magnetic multipoles, the relation $\mathbf{B}_{LM}^{(\pm1)}(\mathbf{R};\omega)=(h_{L\pm1}^{(1)}(\omega R)/j_{L\pm1}(\omega R)\mathbf{A}_{LM}^{(\pm1)}(\mathbf{R};\omega)$ is satisfied and, secondly, that the easily proven equation

$$\sum_{a=E,Y}\mathbf{J}(\mathbf{R})\cdot\mathbf{A}_{LM}^{a*}(\mathbf{R};\omega_N)\mathbf{B}_{LM}^{a}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})$$
$$=\sum_{\nu=\pm1}\mathbf{J}(\mathbf{R})\cdot\mathbf{A}_{LM}^{(\nu)*}(\mathbf{R};\omega_N)\mathbf{B}_{LM}^{(\nu)}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r}) \tag{25}$$

holds for the region $r>R$, and a similar equation, with the appropriate substitution $\mathbf{A}\leftrightarrow\mathbf{B}$, holds for the region $r<R$.

If we use Eqs. (23) and (25), the interaction energy (10)–(11) for electric transitions takes the form

$$H_{\mathrm{int}}^{R}(EL)=4\pi i\omega_N\sum_{L,M}\Bigg\{\int_0^\infty d^3RJ_0(\mathbf{R})A_{LM}^{*}(\mathbf{R};\omega_N)$$
$$\times\int_0^\infty d^3rB_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r})$$
$$-\sum_{\nu=\pm1}\int_0^\infty d^3R\mathbf{J}(\mathbf{R})\cdot\mathbf{A}_{LM}^{(\nu)*}(\mathbf{R};\omega_N)$$
$$\times\int_0^\infty d^3r\mathbf{B}_{LM}^{(\nu)}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})\Bigg|, \tag{26}$$

$$\Delta H_{\mathrm{int}}(EL)=4\pi i\omega_N\sum_{L,M}\Bigg\{\int_0^\infty d^3RJ_0(\mathbf{R})$$
$$\times\Bigg(B_{LM}^{*}(\mathbf{R};\omega_N)\int_0^R d^3rA_{LM}(\mathbf{r};\omega_N)j_0(\mathbf{r})$$
$$-A_{LM}^{*}(\mathbf{R};\omega_N)\int_0^R d^3rB_{LM}(\mathbf{r};\omega_N)j_0(r)\Bigg)$$
$$-\sum_{\nu=\pm1}\int_0^\infty d^3R\mathbf{J}(\mathbf{R})\cdot\Bigg(\mathbf{B}_{LM}^{(\nu)*}(\mathbf{R};\omega_N)$$
$$\times\int_0^R d^3r\mathbf{A}_{LM}^{(\nu)}(\mathbf{r};\omega_N)\cdot\mathbf{j}(\mathbf{r})$$
$$-\mathbf{A}_{LM}^{(\nu)*}(\mathbf{r};\omega_N)\int_0^R d^3r\mathbf{B}_{LM}^{(\nu)}(\mathbf{r};\omega_N)$$
$$\times\mathbf{j}(\mathbf{r})\Bigg)\Bigg|. \tag{27}$$

Integration over the angular variables is the same in (26) and (27), and using what we have said above about the phases of the γ matrices, it gives the following results:[8]

$$\int d\Omega_{\mathbf{r}}\bar{\psi}_f(\mathbf{r})\gamma^0\psi_i(\mathbf{r})Y_{Lm}(\mathbf{n})$$
$$=\Lambda\mathscr{F}_\kappa(r)\int d\Omega_{\mathbf{r}}\bar{\psi}_f(\mathbf{r})\gamma\psi_i(\mathbf{r})\mathbf{Y}_{LL+\nu;m}(\mathbf{n})$$
$$=\frac{-i\Lambda}{\sqrt{(2L+1)[L+(\nu+1)/2]}}\mathscr{F}_\kappa^{(\nu)}(r), \tag{28}$$

where $\nu=\pm1$, while the functions $\mathscr{F}(r)$ are defined as

$$\mathscr{F}_\kappa(r)=g_i(r)g_f(r)+f_i(r)f_f(r),$$
$$\mathscr{F}_\kappa^{(\nu)}(r)=(\kappa_i-\kappa_f)(g_i(r)f_f(r)+f_i(r)g_f(r))$$
$$-(-1)^{(\nu+1)/2}\left(L+\frac{\nu+1}{2}\right)(g_i(r)f_f(r)$$
$$-f_i(r)g_f(r)). \tag{29}$$

For the interaction energy $H_{\mathrm{int}}^{R}(EL)$ in the Rose model, the integrals over the nuclear and the electron variables in Eq. (26) can be separated, which makes it possible to work separately with the nuclear and electron parts.

We transform $\int_0^\infty d^3RJ_0(R)A_{LM}^{*}(\mathbf{R};\omega_N)$ as follows. It follows from the equation of continuity for the electromagnetic current of the nucleus

$$\partial_t J_0(t,\mathbf{R})+\mathrm{div}\,\mathbf{J}(t,\mathbf{R})=0$$

that $J_0(\mathbf{R})=(i/\omega_N)\mathrm{div}\mathbf{J}(\mathbf{R})$. We substitute and integrate by parts. Since the functions $A_{LM}^{*}(\mathbf{R};\omega_N)$ have no singularities at zero and there is no current through the surface of the nucleus, one can easily show by using Gauss's theorem that this integral is equal to $-i\int_0^\infty d^3R\mathbf{J}(R)$

$\cdot \mathbf{A}^{Y*}_{LM}(\mathbf{R};\omega_N)$ [the longitudinal potential $\mathbf{A}^{Y}_{LM}(\mathbf{R};\omega_N)$ is related to $A_{LM}(\mathbf{R};\omega_N)$ by $\mathbf{A}^{Y}_{LM}(\mathbf{R};\omega_N) = (1/\omega_N)\text{grad}\, A_{LM}(\mathbf{R};\omega_N)$].

It is not possible to carry out an analogous procedure with the electron part because of the behavior of the functions $B_{LM}(\mathbf{r};\omega)$ at zero. One must perform the calculation using Eqs. (28)–(29) and the Green–Rose theorem[5] for combining Bessel functions and the wave functions (5) that are solutions of the Dirac equation:

$$\int_0^\infty dr r^2 \Bigg\{ \sum_{\nu=\pm 1} \mathscr{F}^{(\nu)}_{\kappa}(r) h^{(1)}_{L+\nu}(\omega_N r) + \frac{E_f - E_i}{|E_f - E_i|} (2L+1) \mathscr{F}_{\kappa}(r) h^{(1)}_{L}(\omega_N r) \Bigg\} - \mathscr{P}_L(r,\omega_N) h^{(1)}_L(\omega_N r)\,|_0^\infty = 0, \tag{30}$$

where the function $\mathscr{P}_L(r,\omega_N)$ is defined by

$$\mathscr{P}_L(r,\omega_N) = -\frac{2L+1}{\omega_N} r^2 [g_i(r) f_f(r) - f_i(r) g_f(r)]. \tag{31}$$

(The Green–Rose theorem in the form (30) with the crucial additional factor $(E_f - E_i)/|E_f - E_i|$ was proved in Ref. 17.) The formula for the interaction energy $H^R_{\text{int}}(EL)$now takes the form

$$H^R_{\text{int}}(EL) = 4\pi i \omega_N \sum_{L,M} \frac{ie\Lambda}{\sqrt{L(L+1)}} \Bigg\{ \langle N^E_{LM}(\omega_N)\rangle \times \langle \mathrm{m}^E_L(\omega_N)\rangle + \sqrt{\frac{L}{2L+1}} \langle N^{(+1)}_{LM}(\omega_N)\rangle \times \mathscr{P}_L(r,\omega_N) h^{(1)}_L(r\omega_N)\Big|_{r=0} \Bigg\}, \tag{32}$$

where the nuclear and the atomic matrix elements are, respectively, equal to

$$\langle N^E_{LM}(\omega_N)\rangle = \int_0^\infty d^3R \mathbf{J}(\mathbf{R}) \cdot \mathbf{A}^{E*}_{LM}(\mathbf{R};\omega_N),$$

$$\langle N^{(\nu)}_{LM}(\omega_N)\rangle = \int_0^\infty d^3R \mathbf{J}(\mathbf{R}) \cdot \mathbf{A}^{(\nu)*}_{LM}(\mathbf{R};\omega_N), \quad \nu = \pm 1, \tag{33}$$

$$\langle \mathrm{m}^E_L(\omega_N)\rangle = \int_0^\infty dr r^2 [L h^{(1)}_L(r\omega_N)(g_i(r) f_f(r) + f_i(r) g_f(r)) - h^{(1)}_{L-1}(r\omega_N)((\kappa_i - \kappa_f - L) g_i(r) f_f(r) + (\kappa_i - \kappa_f + L) f_i(r) g_f(r))]. \tag{34}$$

As is done in the theory of internal conversion, we transfer the second term in braces in (32) to the anomalous part. It is nonvanishing only for a limited number of transitions with $\varkappa_i + \varkappa_f = \pm(L+1)$ and makes an appreciable contribution to the interaction energy only when there are anomalies present. If, however, the nuclear transition does not satisfy the conditions under which anomalies are possible, the contribution of the second term in (32) to the magnitude of the interaction energy is clearly negligible. This is the reason why the second term in (32) was neglected in Refs. 11 and 12.

As to the additional interaction energy $\Delta H_{\text{int}}(EL)$, use of the potentials $\mathbf{A}^{(\nu)}_{LM}$ and $\mathbf{B}^{(\nu)}_{LM}$ makes its evaluation similar to that of $\Delta H_{\text{int}}(ML)$. After integration over the angular part of the electron WF we get

$$\Delta H_{\text{int}}(EL) = -4\pi i \omega_N \sum_{L,M} e\Lambda \int_0^\infty d^3R \times \Bigg\{ J_0(\mathbf{R}) A^*_{LM}(\mathbf{R};\omega_N) \int_0^R dr r^2 \times \Bigg(\frac{h^{(1)}_L(R\omega_N)}{j_L(R\omega_N)} j_L(r\omega_N) - h^{(1)}_L \times (r\omega_N) \Bigg) \mathscr{F}_{\kappa}(r) + i \sum_{\nu=\pm 1} \mathbf{J}(\mathbf{R}) \cdot \mathbf{A}^{(\nu)*}_{LM}(\mathbf{R};\omega_N) \int_0^R dr r^2 \times \Bigg(\frac{h^{(1)}_{L+\nu}(R\omega_N)}{j_{L+\nu}(R\omega_N)} j_{L+\nu}(r\omega_N) - h^{(1)}_{L+\nu} \times (r\omega_N) \Bigg) \frac{\mathscr{F}^{(\nu)}_{\kappa}(r)}{\sqrt{(2L+1)[L+(\nu+1)/2]}} \Bigg\}. \tag{35}$$

Since $R < R_0$, and hence the integration over the radial variable r is carried out only over the region of the nucleus, we can, firstly, expand the Bessel and Hankel functions in a small parameter and, secondly, use the representation (19) for the radial components of the electron WF in the expressions for $\mathscr{F}^{(\nu)}_{\varkappa}(r)$and $\mathscr{F}_{\varkappa}(r)$, which yields

$$\mathscr{F}_{\kappa}(r)\,|_{r\to 0} = \frac{c_i c_f (g^0_{P_{1/2}} + f^0_{S_{1/2}})}{a_B^3} \frac{r}{a_B},$$

$$\mathscr{F}^{(-1)}_{\kappa}(r=0) = \frac{3 c_i c_f \eta}{a_B^3},$$

$$\mathscr{F}^{(+1)}_{\kappa}(r=0) = 0,$$

where the factor $\eta = 1$ for $|i\rangle = |P_{1/2}\rangle$ in the $|P_{1/2}\rangle \to |S_{1/2}\rangle$ electron transition and $\eta = -1$ for $|i\rangle = |S_{1/2}\rangle$ in the $|S_{1/2}\rangle \to |P_{1/2}\rangle$ electron transition (recall that the anomalous term is considered here only for atomic $E1$ transitions between the $P_{1/2}$ and $S_{1/2}$ states).

After the appropriate substitutions one can easily get the internal integrals in (35). Expanding the potentials $\mathbf{A}^{(\nu)}_{LM}$ and A_{LM} in (35) in the small parameter $\omega_N R$, we find that

$$\Delta H_{\text{int}}(EL) = -4\pi i \omega_N \sum_{L,M} \delta_{E1}(nS_{1/2} \leftrightarrow mP_{1/2}) e\Lambda \times \int_0^\infty d^3R \Bigg\{ i \left(\frac{R_0}{a_B}\right)^3 \frac{c_i c_f (g^0_{P_{1/2}} + f^0_{S_{1/2}})}{\omega_N a_B (L+4)(3-L)}$$

$$\times\left(\frac{R}{R_0}\right)^3 J_0(\mathbf{R})\,Y^*_{LM}(\mathbf{n}_R)-\sum_{\nu=\pm1}\left(\frac{R_0}{a_B}\right)^2$$

$$\times\frac{a_B^3\mathcal{F}_\kappa^{(\nu)}(r=0)}{\omega_N a_B(L+3+\nu)(2-L-\nu)\sqrt{(2L+1)\left(L+\frac{1+\nu}{2}\right)}}$$

$$\times\left(\frac{R}{R_0}\right)^2\mathbf{J}(\mathbf{R})\cdot\mathbf{Y}^*_{LL+\nu;M}(\mathbf{n}_R), \tag{36}$$

where $\mathbf{n}_R=\mathbf{R}/R$.

We then use the fact that for the nuclear current, by virtue of continuity,

$$\int_0^\infty d^3R J_0(\mathbf{R})R^{L+2}Y^*_{LM}(\mathbf{n}_R)$$

$$=-\frac{i}{\omega}\int_0^\infty d^3R\mathbf{J}(\mathbf{R})\mathrm{grad}[R^{L+2}Y^*_{LM}(\mathbf{n}_R)]$$

$$=\frac{i}{\omega}\int_0^\infty d^3R\mathbf{J}(\mathbf{R})[(L+2)R^{L+1}\mathbf{n}_R Y^*_{LM}(\mathbf{n}_R)]$$

$$+i\sqrt{L(L+1)}R^{(L+1)}[\mathbf{n}_R\times\mathbf{Y}^*_{LL;M}(\mathbf{n}_R)], \tag{37}$$

which enables us to eliminate the operator $\mathbf{J}(\mathbf{R})\cdot\mathbf{n}_R Y^*_{LM}(\mathbf{n}_R)$ from consideration, expressing it in the appropriate way in terms of two other ones, $J_0(\mathbf{R})Y^*_{LM}(\mathbf{n}_R)$ and $\mathbf{J}(\mathbf{R})[\mathbf{n}_R\times\mathbf{Y}^*_{LL;M}(\mathbf{n}_R)]$.

Using such a substitution and also the well known expansion of the spherical vectors $\mathbf{Y}_{LL\pm1;M}(\mathbf{n}_R)$ in a longitudinal/transverse basis[18]

$$\mathbf{Y}_{LL+\nu;M}(\mathbf{n}_R)=-i\sqrt{\frac{L+(1-\nu)/2}{2L+1}}[\mathbf{n}_R\times\mathbf{Y}_{LL;M}(\mathbf{n}_R)]$$

$$+(-1)^{(1+\nu)/2}\sqrt{\frac{L+(1+\nu)/2}{2L+1}}$$

$$\times\mathbf{n}_R Y_{LM}(\mathbf{n}_R),\quad \nu=\pm1,$$

we easily obtain for the matrix elements of $\mathbf{J}(\mathbf{R})\cdot\mathbf{Y}^*_{LL\pm1;M}(\mathbf{n}_R)$

$$\int_0^\infty d^3R\left(\frac{R}{R_0}\right)^{L+1}\mathbf{J}(\mathbf{R})\cdot\mathbf{Y}^*_{LL+\nu;M}(\mathbf{n}_R)$$

$$=i\int_0^\infty d^3R\left\{(-1)^{(1+\nu)/2}\sqrt{\frac{L+(1+\nu)/2}{2L+1}}\right.$$

$$\times\frac{\omega_N R_0}{L+2}\left(\frac{R}{R_0}\right)^{L+2}J_0(\mathbf{R})Y^*_{LM}(\mathbf{n}_R)$$

$$-\left(1+(-1)^{1-\nu/2}\frac{L+\frac{1+\nu}{2}}{L+2}\right)\sqrt{\frac{L+1-\nu/2}{2L+1}}$$

$$\left.\times\left(\frac{R}{R_0}\right)^{L+1}\mathbf{J}(\mathbf{R})\cdot[\mathbf{n}_R\mathbf{Y}^*_{LL;M}(\mathbf{n}_R)]\right\}. \tag{38}$$

In fact, in these last formulas we made a choice of the nuclear penetration operators. Following the work by Voĭkhanskiĭ and Listengarten[6] we define them by the equations

$$\hat{\mathcal{M}}_1=\left(\frac{R}{R_0}\right)^2 Y^*_{LM}(\mathbf{n}_R),$$

$$\frac{\hat{\mathcal{M}}_2}{M_p R_0}=\left(\frac{R}{R_0}\right)^3\hat{\mathbf{J}}\cdot[\mathbf{n}_R\times\mathbf{Y}_{LL;M}(\mathbf{n}_R)] \tag{39}$$

(M_p is the proton mass). Note that $\hat{\mathcal{M}}_2$, which is essentially an operator of the matrix element of the transition spin currents, leads to a completely different selection rule in the asymptotic quantum number of the Nilsson model than the usual γ emission operator and charge operator $\hat{\mathcal{M}}_1$. It is therefore precisely $\hat{\mathcal{M}}_2$ which is responsible for the occurrence of most of the anomalies in the electric transitions.[6,8]

Using Eqs. (38) and the explicit form of the functions $\mathcal{F}_\kappa^{(\pm1)}(r=0)$, for ΔH_{int} (EL) we obtain

$$\Delta H_{\mathrm{int}}(EL)=-4\pi i\omega_N\sum_{L,M}\delta_{E1}(nS_{1/2}\leftrightarrow mP_{1/2})ie^2\Lambda$$

$$\times\left\{\langle\hat{\mathcal{M}}_1\rangle\left(\frac{R}{R_0}\right)^3\left(\frac{c_i c_f(g^0_{P_{1/2}}+f^0_{S_{1/2}})}{\omega_N a_B(L+4)(3-L)}\right.\right.$$

$$\left.-\frac{3c_i c_f\eta}{(2L+1)(L+2)^2(3-L)}\right)$$

$$+\langle\hat{\mathcal{M}}_2\rangle\left(\frac{R}{R_0}\right)^2\frac{1}{\omega_N a_B}\frac{1}{M_p R_0}\sqrt{\frac{L+1}{L}}$$

$$\left.\times\frac{6c_i c_f\eta}{(2L+1)(L+2)^2(3-L)}\right\}. \tag{40}$$

The matrix elements $\langle\hat{\mathcal{M}}_1\rangle$ and $\langle\hat{\mathcal{M}}_2\rangle$ in (40) are defined by the relations $\langle\hat{\mathcal{M}}_{1,2}\rangle=\int_0^\infty d^3R\Psi_F^+(\mathbf{R})\hat{\mathcal{M}}_{1,2}\Psi_I(\mathbf{R})$.

Using Eqs. (38), we can express the nuclear matrix element $\langle N^{(+1)}_{LM}(\omega_N)\rangle$ in terms of the ME $\langle\hat{\mathcal{M}}_1\rangle$ and $\langle\hat{\mathcal{M}}_2\rangle$ and rewrite the interaction energy (32) in the Rose model:

$$H^R_{\mathrm{int}}(EL)=4\pi i\omega_N\sum_{L,M}\frac{ie\Lambda}{\sqrt{L(L+1)}}\left\{\langle N^E_{LM}(\omega_N)\rangle\right.$$

$$\times\langle\mathrm{m}^E_L(\omega_N)\rangle$$

$$-\epsilon\delta_{E1}(nS_{1/2}\leftrightarrow mP_{1/2})\frac{c_i c_f\eta}{(2L+1)(L+2)}$$

$$\times\left(\frac{R_0}{a_B}\right)^{L+1}\left(\langle\hat{\mathcal{M}}_1\rangle\left(\frac{R_0}{a_B}\right)\sqrt{\frac{L+1}{L}}\frac{1}{2L+3}\right.$$

$$\left.\left.+\langle\hat{\mathcal{M}}_2\rangle\frac{1}{\omega_N a_B}\frac{1}{M_p R_0}\right)\right\}. \tag{41}$$

Using the equation of continuity for the nuclear current, the representation of the longitudinal potential in terms of the gradient of the function $A_{LM}(\mathbf{R},\omega_N)$ which we have already used above, and the relation $\mathbf{A}^E_{LM}(\mathbf{R},\omega_N)$

$\approx \sqrt{(L+1)L}\mathbf{A}^{Y}_{LM}(\mathbf{R},\omega_N)$, we can express the nuclear matrix element $\langle N^E_{LM}(\omega_N)\rangle$ (31) for an electric transition as

$$\langle N^E_L(\omega_N)\rangle = \int_0^\infty d^3R\mathbf{J}(\mathbf{R})\cdot\mathbf{A}^{E*}_{LM}(\mathbf{R};\omega_N)$$
$$\approx ie\sqrt{\frac{L+1}{L}}\frac{(\omega_N R_0)^L}{(2L+1)!!}\langle U_\gamma\rangle,$$

where the nuclear γ emission matrix element of electric multipoles is by definition equal to

$$\langle U_\gamma\rangle = \int_0^\infty d^3R J_0(\mathbf{R})\left(\frac{R}{R_0}\right)^L Y^*_{LM}(\mathbf{n}_R).$$

Using Voĭkhanskiĭ and Listengarten's nuclear penetration parameters[6] for electric multipoles $\lambda^{(1)} = \langle\hat{\mathscr{M}}_1\rangle/\langle U_\gamma\rangle$ and $\lambda^{(2)} = \langle\hat{\mathscr{M}}\,2\rangle/\langle U_\gamma\rangle$, we can write the final expression for the nucleus–electron interaction energy in NEET processed for electric transitions in the form

$$H_{\rm int}(EL) = 4\pi i\omega_N \sum_{L,M}\frac{ie\Lambda}{\sqrt{L(L+1)}}\langle N^E_{LM}(\omega_N)\rangle$$
$$\times\left\{\langle \mathrm{m}^E_L(\omega_N)\rangle + ie\delta_{E1}(nS_{1/2}\leftrightarrow mP_{1/2})\right.$$
$$\times\frac{c_f c_f}{4\pi^2}\frac{R_0}{a_B}\left(\frac{\lambda_N}{a_B}\right)^2\left(\lambda^{(2)}\frac{1}{M_p R_0}\frac{\eta}{\sqrt{2}}\right.$$
$$+\lambda^{(1)}\frac{R_0}{a_B}\left(\frac{3}{10}(g^0_{P_{1/2}}+f^0_{S_{1/2}})\right.$$
$$\left.\left.\left.-\eta\frac{\omega_N a_B}{10}\right)\right)\right\}. \qquad (42)$$

As in the case of anomalies in the ICC, the role of the term containing the nuclear charge parameter $\lambda^{(1)}$ is relatively small compared to that of the second, containing the transition spin current parameter $\lambda^{(2)}$. Here we are dealing not only with an additional small quantity introduced by an extra factor R_0/a_B, but also with the previously mentioned difference in the selection rules for the matrix elements $\langle\hat{\mathscr{M}}_2\rangle$ and $\langle U_\gamma\rangle$. As for the contribution to the anomaly from the second term in Eq. (32), its magnitude is approximately 1/3.

5. NUMERICAL ESTIMATES AND DISCUSSION OF THE RESULTS

The relative probability P for the excitation of nuclei in atomic transitions can be calculated using Eq. (2), in which the square of the interaction energy $E^2_{\rm int}$, which by definition is equal to

$$E^2_{\rm int} = \frac{1}{2J_I+1}\frac{1}{2j_i+1}\sum_{\substack{M_I,M_F\\ m_i,m_f}}|H_{\rm int}|^2,$$

can easily be evaluated using the formulas from Refs. 10 and 11:

$$E^2_{\rm int}(L;\omega_N;i\to f,I\to F)$$
$$= 4\pi e^2\omega_N^{2(L+1)}\frac{(2L+1)(j_i\tfrac{1}{2}\,L0|j_n\tfrac{1}{2})^2}{[(2L+1)!!]^2}|\langle\tilde{\mathrm{m}}^{E(M)}_L$$
$$\times(\omega_N)\rangle|^2 B(E(M)L;I\to F).$$

The reduced probability $B(E(M)L;I\to F)$ is here expressed in the standard way in terms of the matrix elements (21) and (33):

$$B(E(M)L;I\to F) = \frac{1}{2J_I+1}\sum_{M_I,M_F}|\langle N^{E(M)}_{LM}\rangle|^2,$$

and by $\langle\tilde{\mathrm{m}}^{E(M)}_L(\omega_N)\rangle$ we have denoted the total atomic matrix element including the expressions in braces in (22) and (42):

$$\langle\tilde{\mathrm{m}}^{E(M)}_L(\omega_N)\rangle\{\langle \mathrm{m}^{E(M)}_L(\omega_N)\rangle + i\delta_{E(M)1}\ldots\}.$$

We now determine the contribution of anomalies to the probability for a NEET process in the ^{193}Ir, ^{197}Au, ^{181}Ta, and ^{237}Np nuclei.

We show part of the level scheme for the iridium atom and the ^{193}Ir nucleus in Fig. 2 (the characteristics of the nuclear levels are taken from Ref. 19). For the l-forbidden magnetic dipole transition $2d_{3/2}\to 3s_{1/2}$ from the ground

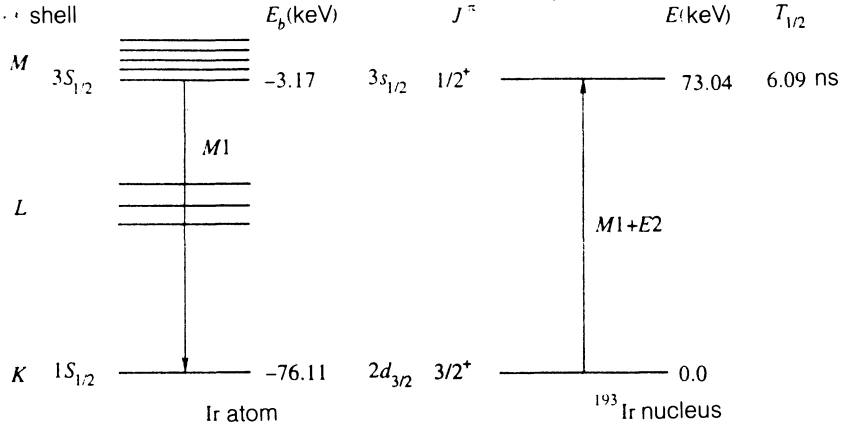

FIG. 2. Parts of the level scheme of the iridium atom and the ^{193}Ir nucleus.

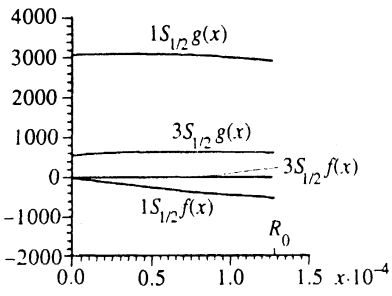

FIG. 3. Electron WF of the $1S_{1/2}$ state when there is a vacancy in the M_1 shell, and of the $3S_{1/2}$ state when there is a vacancy in the K shell for the ^{193}Ir nucleus. Abscissa: distance from the center of the nucleus in $x=r/a_B$.

state to the first excited $1/2^+$ state (73.04 keV) in ^{193}Ir, an anomaly has been predicted (see Ref. 8), but so far not observed experimentally. In the shell model, adapted suitably for a description of the properties of $M1$ in nuclei such as ^{193}Ir and ^{197}Au, there is an estimate $\lambda^{(0)}=9.7$ in Ref. 8 for the penetration parameter, and we shall use that. The relativistic electron WF were evaluated using the program of Ref. 20 in the nuclear potential (1) and the average self-consistent field of the electron shell, which was constructed using the relativistic variant of the Hartree–Fock–Slater method. The atomic matrix element (18) for an $M1$ transition between the $3S_{1/2}$ state in the Ir^+ ion with a hole in the K shell and the $1S_{1/2}$ state in the Ir^+ ion with a hole in the M shell is equal to $\langle m_1^M(\omega_N)\rangle = -2.9\cdot10^{-3}-i\cdot10.4$. The behavior of these electron WF in the region of the nucleus is shown in Fig. 3. They are approximated by Eqs. (19) with coefficients which have been calculated by the least-squares method. The result is $3S_{1/2}$: $c=580.1$, $f^0=-1605.4$; $1S_{1/2}$: $c=3121.7$, $f^0=-1598.1$. The expression in the braces in (22) takes the form $\{\langle m_1^M(\omega_N)\rangle + i\lambda^{(0)}0.16\}$. Adopting the value found for $\langle m_1^M(\omega_N)$, we find that taking the anomaly into account for the magnitude of $\lambda^{(0)}$ given in Ref. 8 reduces the probability for the NEET process by an approximate factor 1.3–1.4 as compared to the calculation in the Rose model (see Table I).

A similar situation also occurs in ^{197}Au. The magnitude of the parameter $\lambda^{(0)}$ for the l-forbidden $2d_{3/2}\to3s_{1/2}$ $M1$ transition from the ground state to the first excited $1/2^+$ state (77.35 keV) has not been established accurately. The theoretical estimate $\lambda^{(0)}=4.3$ from Ref. 8 agrees with one experimental set of data,[21] $\lambda^{(0)}=3.4\pm1.9$, cm but disagrees with another,[22] $\lambda^{(0)}=1.8\pm0.3$, cm. In the Table we have given the change in the NEET probability for both experimental values of $\lambda^{(0)}$.

Note that in both the ^{193}Ir and ^{197}Au nuclei, the cross section for the excitation of the appropriate levels in the inelastic scattering of electrons by the nuclei is less than the cross section for the NEET process. The latter is equal to the product of the cross section σ_K for the ionization of the K shell and the relative probability for NEET proper. The cross section for inelastic scattering is in both cases completely determined by the $E2$ component, and is essentially independent of the contribution from the $M1$ component (the magnitude of the latter here is no more than 10%). All this enables us to obtain important information about the properties of the $M1$ transitions considered in both nuclei in experiments with an electron beam of the kind described in Ref. 23, where the probability for NEET was measured in ^{197}Au.[1)]

An interesting situation has developed in the determination of anomalies in the ICC in the ^{237}Np nucleus. On the one hand, in the lower part of the spectrum considerable anomalies are well known in the ICC for two $E1$ transitions from the $5/2^-$ level (59.5 keV) to the $7/2^+$ (33.2 keV) and $5/2^+$ (0,0) states which lie between the $K^\pi[Nn_Z\Lambda)]=5/2^-[523]$ and $5/2^+[642]$ rotational bands. On the other hand, the presence of these anomalies agrees poorly with the known selection rules in the asymptotic quantum number of the Nilsson model.[6] Finally, for the $7/2^-$ (102.96 keV)$\to5/2^+(0,0)$ transition between the same rotational bands, no anomalies in the ICC have been observed up to the present. An experimental study of the NEET process might therefore given an independent answer to the problem of the existence of anomalies in the $E1$ transition with an energy of 103 keV in ^{237}Np (see Table I).

There is a very large anomaly in the ICC, for the $E1$ transition of energy 6.24 keV (conversion decay of the first excited level) in the ^{181}Ta nucleus. The excitation of this level in a NEET process is possible in the $3S_{1/2}\to2P_{3/2}$ and

TABLE I.

Nucleus	Transition in the nucleus	Transition in the atom	NEET probability: Neglecting the anomaly	NEET probability: Taking the anomaly into account	Penetration parameter
^{197}Au	$M1$, 77 keV	$3S_{1/2}\to1S_{1/2}$	1.4×10^{-7}	1.3×10^{-7}	$\lambda^{(0)}=1.8$
				1.2×10^{-7}	$\lambda^{(0)}=3.4$
^{193}Ir	$M1$, 73 keV	$3S_{1/2}\to1S_{1/2}$	7.4×10^{-9}	5.4×10^{-9}	$\lambda^{(0)}=9.8$
^{237}Np	$E1$, 103 keV	$2P_{1/2}\to1S_{1/2}$	2.4×10^{-13}	1.7×10^{-12}	$(\lambda^{(2)}=-165)$
		$2P_{3/2}\to1S_{1/2}$	2.9×10^{-12}	2.9×10^{-12}	
		total	3.2×10^{-12}	4.6×10^{-12}	
^{181}Ta	$E1$, 6.24 keV	$3S_{1/2}\to2P_{1/2}$	4.2×10^{-16}	6.8×10^{-14}	$\lambda^{(2)}=-620$
		$3S_{3/2}\to2P_{3/2}$	9.0×10^{-15}	9.0×10^{-15}	
		total	9.4×10^{-15}	7.7×10^{-14}	

$3S_{1/2} \to 2P_{1/2}$ electronic transitions. A strong anomaly appears in the second transition. The numerical values are given in the Table. We note merely that the partial NEET probability for the $3S_{1/2} \to 2P_{1/2}$ electronic transition increases by a factor of approximately 170 when the anomaly is taken into account. In this specific case, even the relative softness of the transition is important; as a result of it, the factor $(\lambda_N/a_B)^2$ in Eq. (41) starts to play an important role. As a result, even the total probability for the process (if we irradiate ^{181}Ta atoms with an electron beam ionizing both the $2P_{1/2}$ and the $2P_{3/2}$ shells) increases by an order of magnitude as compared to when the anomaly is not taken into account. As for absolute values of the NEET probability, in ^{181}Ta we used for their calculation data about the widths of the vacancies in atomic L and M shells from Ref. 24.

From the example of ^{181}Ta, it is well known that a measurement of the NEET probability for low-energy transitions (where the internal conversion process is possible only from the M and higher shells while NEET goes "downward," for instance, to the L shell) can give very useful information about the nuclear penetration matrix elements. The anomalies in NEET can in a number of cases manifest itself more strongly than in the ICC due to the strong localization of the electron WF in the low-lying atomic shells, and hence the large amplitude of those WF in the nucleus. For the same reason the NEET process can also turn out to be useful for a study of anomalies in transitions with a multipolarity larger than unity.

Finally, NEET is a rather rare process which requires the satisfying of rigid conditions on the energy and multipolarity of the atomic and nuclear transitions. However, the number of nuclei in which one can observe NEET can be relatively easily widened if one achieves an appropriate change in the energies of the "working" levels between which the atomic transition takes place, through multiple ionization of some of the atomic shells. Such a variation in the energy ω_A of the atomic transition does not present particular technical complications and, if necessary, can be realized for a relatively large number of atoms.

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