# Emission of radiation by a system of identical excited three-level emitters

V. V. Lomonosov and M. Yu. Talantov

*Russian Scientific Center 'Kurchatov Institute,' 123182 Moscow, Russia*
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A quantum mechanical analysis of the time development of radiation from a system of $N$ identical three-level emitters initially in excited states is given for the two-stage de-excitation scheme. For $N=1$ and $N=2$ (the exact resonance case), the corresponding transition probabilities are evaluated and the frequency and angular dependence of the radiation is discussed.

In Ref. 1, using the time-dependent quantum mechanical theory due to Weisskopf, Wigner, and Heitler,[2,3] the time development of formation of coherent spontaneous photon radiation (or the Dicke effect[4]) has been obtained for a system of $N$ identical two-level atoms initially in their excited state. In particular, for a two-atom system ($N=2$), the time-dependent probability is predicted for the process of the induced emission of a photon by an excited atom acted upon by a spontaneous photon from another atom. Reference 5, based on the results of Ref. 1, considers the space-time interference picture due to the detection of one or two photons from a system of two simultaneously excited and spontaneously emitting atoms; it is shown that in the general case, when the atoms interact strongly via the radiation field, the interference phenomenon is rather complex in character and is determined by the combination of the Brown–Twiss effect[6] and the "classical" interference from two coherent emitters. It should be noted that the methods employed in these studies enable one to obtain the wave function of a system of identical atoms for any time, taking all the relevant level widths and phase relations into account.

In this work, the same quantum mechanical formalism is used to consider effects caused by the radiation of photons from a system of identical three-level emitters which are initially in their uppermost excited state. As far as the experimental situation in laser and Mössbauer spectroscopy is concerned, the detected radiation often comes from emitters for which a three-level description is adequate. In this case it is evident that—unlike two-level systems—not only coherent spontaneous radiation involving two (atomic or nuclear) transitions, but also the mutual influence of the transitions should be observable.

Most generally, the time evolution of a system of $N$ three-level emitters (hereafter referred to as atoms) located at fixed points $\mathbf{R}_1$, $\mathbf{R}_2$,...$\mathbf{R}_N$ and having nonoverlapping wave functions $u_n(\mathbf{r}-\mathbf{R}_i)$ ($n=0$ is the ground state of the atom; $n=1$, 2 are the first and second excited states, respectively; $i=1$–$N$) is described by the equation

$$i\frac{d}{dt}\langle a|U(t)|i\rangle=\sum_{|z\rangle}\langle a|H(t)|z\rangle\langle z|U(t)|i\rangle, \qquad (1)$$

where the operator $U(t)$ determines the state of the system for any time $t>0$, it being assumed that at $t=0$ all the atoms are in the upper ($n=2$) excited state; $|i\rangle$, $|a\rangle$, and $|z\rangle$ are the initial, final, and intermediate states, respectively; and $H(t)$ is the standard Hamiltonian[2] for the interaction of an atom with a radiation field. Only transitions between the levels 0 and 1 and 1 and 2 are allowed, the transitions 0–2 are forbidden, and also the usual requirements of the Weisskopf–Wigner approximation are imposed, namely, the summation over the intermediate states in (1) includes only one-photon transitions, and the emission (absorption) of each photon leads to the de-excitation (excitation) of the atom. To the intermediate states summed over in (1) one can associate an invariant number $\Lambda(|z\rangle)$ which satisfies the following rules:

$$\Lambda(|z\rangle)=\Lambda(|i\rangle),\quad \Lambda(|i\rangle)=2N, \qquad (2)$$

$$\Lambda(|z\rangle)=2N_2^{\mathrm{ex}}+N_1^{\mathrm{ex}}+N_k^{\mathrm{phot}}+N_q^{\mathrm{phot}}\ \text{ in }\ |z\rangle.$$

Here $N_2^{\mathrm{ex}}$ ($N_1^{\mathrm{ex}}$) is the number of atoms in the upper (lower) excited state, and $N_k^{\mathrm{phot}}$ and $N_q^{\mathrm{phot}}$ are the numbers of photons emitted in 2–1 and 1–0 transitions (see Fig. 1). Let $j_i$ be the label of an atom in the first excited state and $s_i$ the number of an atom in the ground state. Then any intermediate system state is conveniently indexed as follows:

$$|z\rangle=\left|\begin{matrix} j_1,\ j_2\cdots j_M \\ s_1,\ s_2\cdots s_L \\ \mathbf{k}_1,\ \mathbf{k}_2\cdots\mathbf{k}_M \\ \mathbf{q}_1,\ \mathbf{q}_2\cdots\mathbf{q}_L \end{matrix}\right\rangle,\quad M,L=0,1,2...N, \qquad (3)$$

where $\mathbf{k}_i$ and $\mathbf{q}_i$ are photon wave vectors. In the following, writing zeros in place of indices indicates the absence of corresponding atoms or photons in a given state. For the matrix elements in (1) we introduce the notation

$$\left\langle\begin{matrix} j_1,\ j_2\cdots j_M \\ s_1,\ s_2\cdots s_L \\ \mathbf{k}_1,\ \mathbf{k}_2\cdots\mathbf{k}_M \\ \mathbf{q}_1,\ \mathbf{q}_2\cdots\mathbf{q}_L \end{matrix}\right|U(t)\left|\begin{matrix}0\\0\\0\\0\end{matrix}\right\rangle=\alpha\begin{matrix} j_1,\ j_2\cdots j_M \\ s_1,\ s_2\cdots s_L \\ \mathbf{k}_1,\ \mathbf{k}_2\cdots\mathbf{k}_M \\ \mathbf{q}_1,\ \mathbf{q}_2\cdots\mathbf{q}_L \end{matrix}(t), \qquad (4)$$

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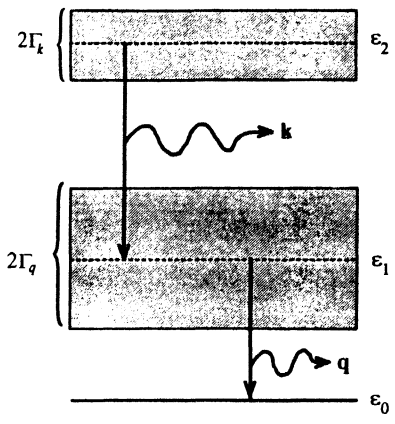


FIG. 1. Schematic energy level diagram for an isolated three-level atom.

$$\left\langle \begin{matrix} j_1,\ j_2\cdots j_M \\ s_1,\ s_2\cdots s_L \\ \mathbf{k}_1,\ \mathbf{k}_2\cdots\mathbf{k}_M \\ \mathbf{q}_1,\ \mathbf{q}_2\cdots\mathbf{q}_L \end{matrix} \middle| H^{(j)}(t) \middle| \begin{matrix} j_1',\ j_2'\cdots j_{M-1}' \\ s_1',\ s_2'\cdots s_L' \\ k_1',\ k_2'\cdots k_{M-1}' \\ q_1',\ q_2'\cdots q_L' \end{matrix} \right\rangle = C_k^{(j)} \exp[-i(\Omega_{21}-k)t], \tag{5}$$

$$C_k^{(j)} = (2Vk)^{-1/2}\left[-ie\int d^3r\bar{u}_2(\mathbf{r}-\mathbf{R}_j)\gamma_\mu e_\mu u_1(\mathbf{r}-\mathbf{R}_j) \times\exp(-i\mathbf{kr})\right], \tag{6}$$

$$\left\langle \begin{matrix} j_1,\ j_2\cdots j_{M-1} \\ s_1,\ s_2\cdots s_L \\ \mathbf{k}_1,\ \mathbf{k}_2\cdots\mathbf{k}_M \\ \mathbf{q}_1,\ \mathbf{q}_2\cdots\mathbf{q}_L \end{matrix} \middle| H^{(s)}(t) \middle| \begin{matrix} j_1',\ j_2'\cdots j_M' \\ s_1',\ s_2'\cdots s_{L-1}' \\ k_1',\ k_2'\cdots k_M' \\ q_1',\ q_2'\cdots q_{L-1}' \end{matrix} \right\rangle = C_q^{(s)}\exp[-i(\Omega_{10}-q)t], \tag{7}$$

$$C_q^{(s)} = (2Vq)^{-1/2}\left[-ie\int d^3r\bar{u}_1(\mathbf{r}-\mathbf{R}_s)\gamma_\mu e_\mu u_0(\mathbf{r}-\mathbf{R}_s) \times\exp(-i\mathbf{qr})\right]. \tag{8}$$

Here $e$ represents the electromagnetic coupling constant, $\gamma_\mu$ is the Dirac vector, and $e_\mu$ the polarization vector. In the following, the factorization $C_{\mathbf{p}}^{(d)} = \exp(-i\mathbf{R}_d\mathbf{p})\mathbf{C}_{\mathbf{p}}$ ($d = j, s$; $p = k,q$) is used. $\Omega_{ij} = \varepsilon_i - \varepsilon_j$ are the energy differences between the atomic levels (Fig. 1); $V$ is the normalization volume of the radiation field; $k = |\mathbf{k}|$ and $q = |\mathbf{q}|$ are the corresponding photon energies.

On substituting (3)–(8) into (1), carrying out the partial summation and changing to the Fourier components of the amplitudes with respect to frequencies we find

$$(\Omega - E\begin{matrix} j_1,\ j_2\cdots j_M \\ s_1,\ s_2\cdots s_L \\ \mathbf{k}_1,\ \mathbf{k}_2\cdots\mathbf{k}_M \\ \mathbf{q}_1,\ \mathbf{q}_2\cdots\mathbf{q}_L \end{matrix} + i\varepsilon)\alpha\begin{matrix} j_1,\ j_2\cdots j_M \\ s_1,\ s_2\cdots s_L \\ \mathbf{k}_1,\ \mathbf{k}_2\cdots\mathbf{k}_M \\ \mathbf{q}_1,\ \mathbf{q}_2\cdots\mathbf{q}_L \end{matrix}(\Omega)$$

$$= \delta_{M,0}\delta_{L,0} + T^{-1}\begin{bmatrix}\mathbf{k}_1,\mathbf{k}_2\ldots\mathbf{k}_M, \\ \mathbf{q}_1,\mathbf{q}_2\ldots\mathbf{q}_L\end{bmatrix}\left\{\sum_{\nu=1}^{N-M-L}\sum_{\mathbf{k}} c_{\mathbf{k}}^{(\tilde{j}_\nu)*} T \times\begin{bmatrix}\mathbf{k}_1,\mathbf{k}_2\ldots\mathbf{k}_M,\mathbf{k}, \\ \mathbf{q}_1,\mathbf{q}_2\ldots\mathbf{q}_L\end{bmatrix}\alpha\begin{matrix} j_1,\ j_2\cdots j_M,\tilde{j}_\nu \\ s_1,\ s_2\cdots s_L \\ \mathbf{k}_1,\ \mathbf{k}_2\ldots\mathbf{k}_M,\mathbf{k} \\ \mathbf{q}_1,\ \mathbf{q}_2\cdots\mathbf{q}_L \end{matrix}(\Omega)\right.$$

$$+\sum_{\nu=1}^{M}\sum_{w} c_{\mathbf{k}_w}^{(j_\nu)} T\begin{bmatrix}\mathbf{k}_1\ldots\mathbf{k}_{w-1},\mathbf{k}_{w+1}\ldots\mathbf{k}_M, \\ \mathbf{q}_1,\mathbf{q}_2\ldots\mathbf{q}_L\end{bmatrix} \times\alpha\begin{matrix} j_1\cdots j_{\nu-1},\ j_{\nu+1}\cdots j_M \\ s_1,\ s_2\ldots s_L \\ \mathbf{k}_1\ldots\mathbf{k}_{w-1},\ \mathbf{k}_{w+1}\ldots\mathbf{k}_M \\ \mathbf{q}_1,\ \mathbf{q}_2\ldots\mathbf{q}_L \end{matrix}(\Omega) + \sum_{\nu=1}^{M}\sum_{\mathbf{q}} c_{\mathbf{q}}^{(\tilde{s}_\nu)*} T$$

$$\times\begin{bmatrix}\mathbf{k}_1,\mathbf{k}_2\ldots\mathbf{k}_M, \\ \mathbf{q}_1,\mathbf{q}_2\ldots\mathbf{q}_L,\mathbf{q}\end{bmatrix}\alpha\begin{matrix} j_1\cdots j_{\mu-1},\ j_{\nu+1}\cdots j_M \\ s_1,\ s_2\ldots s_L,s_\nu \\ \mathbf{k}_1,\ \mathbf{k}_2\ldots\mathbf{k}_M \\ \mathbf{q}_1,\ \mathbf{q}_2\cdots\mathbf{q}_L,\mathbf{q} \end{matrix}(\Omega)$$

$$+\sum_{\nu=1}^{L}\sum_{w}^{L} c_{\mathbf{q}_w}^{(s_\nu)} T\begin{bmatrix}\mathbf{k}_1,\mathbf{k}_2\ldots\mathbf{k}_M, \\ \mathbf{q}_1\ldots\mathbf{q}_{w-1},\mathbf{q}_{w+1}\ldots\mathbf{q}_L\end{bmatrix}$$

$$\left\{\times\alpha\begin{matrix} j_1,\ j_2\cdots j_M,\ j_\nu \\ s_1\ldots s_{\nu-1},\ s_{\nu+1}\ldots s_L \\ \mathbf{k}_1,\ \mathbf{k}_2\ldots\mathbf{k}_M \\ \mathbf{q}_1\ldots\mathbf{q}_{w-1},\ \mathbf{q}_{w+1}\ldots\mathbf{q}_L \end{matrix}(\Omega).\right. \tag{9}$$

Here $E$ is the energy of the system as a whole; $\varepsilon$ is the positive infinitesimal specifying pole avoiding rules; $\delta_{L,M}$ the Kronecker symbol; and $T[x]$ the Einstein function

$$T\begin{bmatrix}\mathbf{a}_1,\ \mathbf{a}_2\cdots\mathbf{a}_M, \\ \mathbf{a}_{M+1}\cdots\mathbf{a}_L\end{bmatrix} = \left[\prod_i (n_i!)\right]^{1/2}, \tag{10}$$

where $n_i$ is the number of times the quantity $\mathbf{a}_i$ occurs among $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_L$.

The system of equations (9) enables one to obtain in a formal manner an exact solution (exact meaning within the above approximations) for the complete set of state amplitudes of a system of an arbitrary number of three-level emitters. It is not hard to show that the number of equations in the system is determined by the number of the atoms considered and is equal to

$$\sum_{M=0}^{N}\sum_{L=0}^{N-M} C_N^M C_{N-M}^L = 3^N. \tag{11}$$

## THE CASE OF AN ISOLATED ATOM ($N=1$)

For $N=1$, from (9) we have

$$(\Omega-\varepsilon_2-i\varepsilon)\alpha_0^0=1+\sum_{\mathbf{k}} C_{\mathbf{k}}\alpha_{\mathbf{k}}, \tag{12}$$

$$(\Omega-\varepsilon_1-k-i\varepsilon)\alpha_{\mathbf{k}}=C_{\mathbf{k}}^*\alpha_0^0+\sum_{\mathbf{q}} C_{\mathbf{q}}\alpha_{\mathbf{kq}}, \tag{13}$$

$$(\Omega-k-q-i\varepsilon)\alpha_{\mathbf{kq}}=C_{\mathbf{q}}^*\alpha_{\mathbf{k}}. \tag{14}$$

Solving the system (12)–(14) and changing to the time representation we obtain the following expressions for the amplitudes:

$$\alpha_0^0(t)=\exp(-\Gamma_k t), \tag{15}$$

$$\alpha_{\mathbf{k}}(t)=C_{\mathbf{k}}^*\frac{\exp[-i(\varepsilon_2-\varepsilon_1-k)t]\exp(-\Gamma_k t)-\exp(-\Gamma_q t)}{\varepsilon_2-\varepsilon_1-k-i(\Gamma_k-\Gamma_q)}, \tag{16}$$

$$\alpha_{\mathbf{kq}}(t)=C_{\mathbf{k}}^*C_{\mathbf{q}}^*\left\{\frac{1}{(\varepsilon_1-q-i\Gamma_q)(\varepsilon_2-k-q-i\Gamma_k)}-\frac{\exp[-i(\varepsilon_1-q)t]\exp(-i\Gamma_q t)}{(\varepsilon_1-q-i\Gamma_k)[\varepsilon_2-\varepsilon_1-k-i(\Gamma_k-\Gamma_q)]}+\frac{\exp[-i(\varepsilon_2-k-q)t]\exp(-i\Gamma_k t)}{(\varepsilon_2-k-q-i\Gamma_k)[\varepsilon_2-\varepsilon_1-k-i(\Gamma_k-\Gamma_q)]}\right\}, \tag{17}$$

where $\Gamma_k$ and $\Gamma_q$ are the standardly defined halfwidths of the corresponding levels (see Refs. 1 and 7 and Fig. 1).

Since in real experiments one is usually interested in 2–1 transitions, it is appropriate to consider the differential probability for the emission of a $k$-photon by an individual atom per unit time,

$$I_1=\frac{dW(k,t)}{dk}=|\alpha_{\mathbf{k}}(t)|^2+\sum_{\mathbf{q}}|\alpha_{\mathbf{kq}}(t)|^2. \tag{18}$$

Substituting (15)–(17) into (18) and summing over the energies and angles of the $q$-photons and the angles of the $k$-photons gives

$$I_1=F(\Delta_+,\omega,t)-F(\Delta_-,\omega,t), \tag{19}$$

$$F(x,\omega,t)=[x/\pi(\omega^2+x^2)]\{\exp[-(\Gamma_k+\Gamma_q-x)t]-\exp[-(\Gamma_k+\Gamma_q)t]\cos(\omega t)+(\omega/x)\times\exp[-(\Gamma_k+\Gamma_q)t]\sin(\omega t)\}, \tag{20}$$

$$\Delta_\pm=\Gamma_q\pm\Gamma_k,\quad \omega=\Omega_{21}-k. \tag{21}$$

Expression (19) makes it possible to trace the time dynamics of the formation of a wave packet in a resonant medium. Evaluating the derivative $dI_1/dt$, we obtain after some manipulation the condition on the extremum points [cf Eq. (19)]:

$$\cos(\omega t)-(\omega/\Delta_-)\sin(\omega t)=\exp(\Delta_- t), \tag{22}$$

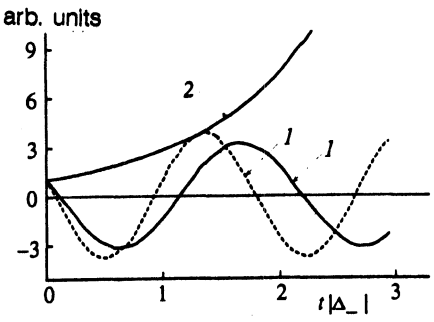


FIG. 2. Dependence of the left-hand (curves *1*) and right-hand (curves *2*) sides of Eq. (22) on time (the value of $\Delta_-$ is fixed, $\Delta_->0$). Solid lines: $\omega=3\Delta_-$; dashed lines: $\omega=3.64\Delta_-$. Curves *2* are superimposed on each other.

$\Delta_-\neq 0$. As $\Gamma_q\to 0$ ($\Delta_-\to-\Gamma_k$), we go over to the two-level case, and because the right-hand side of (19) decreases with time, for any nonresonant frequencies ($\omega\neq 0$) we have an infinite sequence of alternate maxima and minima. For a three-level atom ($\Gamma_q\neq 0$) and $\Delta_->0$, the region of the emission rate growing monotonically in time to its saturation, broadens relative to the two-level case from a single point $\omega=0$ (the center of the line) into a band $|\omega|<3.64\Delta_-$, where the numerical coefficient is obtained from the first extremum condition (see Fig. 2). As one goes further away from the line center the number of extremal points increases but remains finite (Fig. 3), each event as the rhs exponential touches the next lhs oscillation crest in Eq. (22) corresponding to an inflection point; whereas a slice through the ridge gives rise to two extrema, first a maximum and then a minimum. For $\Delta_-<0(\Gamma_q<\Gamma_k)$, the situation is qualitatively indistinguishable from that for two levels (Fig. 3). For $\Delta_-=0$, Eq. (22) is replaced by the trivial $\omega\sin(\omega t)=0$ yielding an infinite number of extrema in analogy with the case $\Delta_-<0$.

If one introduces the line halfwidth $\Gamma(t)$ at half-maximum as the frequency for which

$$I_1(\omega=\Gamma(t),t)=I_1(\omega=0,t)/2,$$

then the time dependence of this function, as calculated from (19), takes the form shown in Fig. 4. For $t\to\infty$ we obtain the familiar result $\Gamma(t=\infty)=\Gamma_k+\Gamma_q$ (Ref. 7).

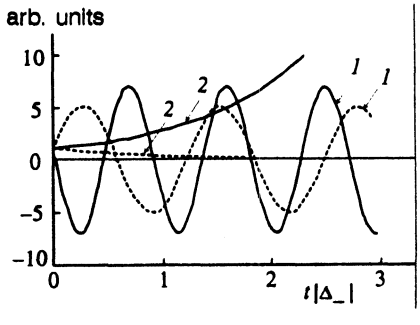


FIG. 3. The same as Fig. 2 for $\Delta_->0$, $\omega=7\Delta_-$ (solid lines) and $\Delta_-<0$, $\omega=5|\Delta_-|$ (dashed lines).

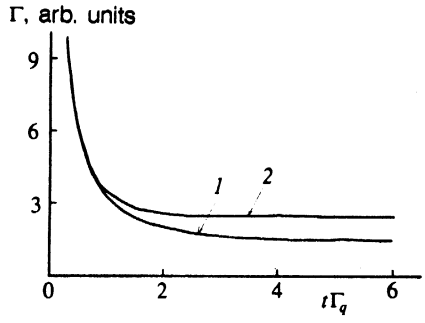


FIG. 4. Dependence of the half linewidth of the radiation of an isolated atom on time for $\Gamma_q$ fixed. Curve *1*: $\Gamma_k=0.5\Gamma_q$; curve *2*: $\Gamma_k=1.5\Gamma_q$.

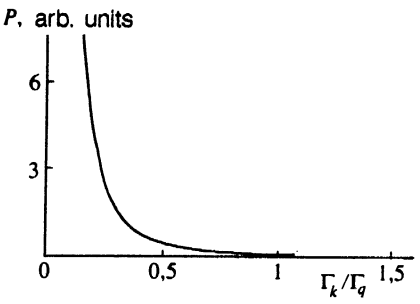


FIG. 5. Dependence of the function $P(\Gamma_k,\Gamma_q)$ on $\Gamma_k$ for a fixed value of $\Gamma_q$.

**TWO-ATOM CASE ($N=2$)**

The time development of a two-atom system is described by the system of equations given explicitly in the Appendix. The exact solution of the system (A1)–(A9) turns out to be extremely cumbersome. We present here the result for the case $t=\infty$, when the differential probability for the total de-excitation of a two-atom system is determined by the state amplitude given in (A9). We have

$$\frac{dW(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2,t=\infty)}{d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{q}_1 d\mathbf{q}_2}=|\overset{0}{\alpha}{}^{12}_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1\mathbf{q}_2}|^2. \tag{23}$$

As shown in Ref. 1, if the interatomic distance is large in comparison to the half wavelengths of interest, that is, for $|\mathbf{R}_1-\mathbf{R}_2|\gg\lambda_{21}/2$, $\lambda_{20}/2$, $\lambda_{ij}=2\pi/\Omega_{ij}$ (which is usually the case experimentally), we may neglect the nondiagonal elements of the linewidth matrix. In this approximation we obtain

$$\begin{aligned}\overset{0}{\alpha}{}^{12}_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1\mathbf{q}_2}=T^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2)[&(Y_1+Y_2)S_1(q_1,k_1)\\&+(Y_1+Y_2)S_1(q_1,k_2)+Y_1S_2(q_1,k_1)\\&+Y_2S_2(q_1,k_2)+(Y_1+Y_2)S_1(q_2,k_1)+(Y_1\\&+Y_2)S_1(q_2,k_2)+Y_1S_2(q_2,k_1)\\&+Y_2S_2(q_2,k_2)],\end{aligned} \tag{24}$$

$$\begin{aligned}S_1(A,B)=\{&(A-\varepsilon_1+i\Gamma_q)[B+q_1+q_2-\varepsilon_1-\varepsilon_2+i(\Gamma_k\\&+\Gamma_q)](k_1+k_2+q_1+q_2-2\varepsilon_1+2i\Gamma_k)(q_1+q_2\\&-2\varepsilon_1+2i\Gamma_q)\}^{-1},\end{aligned} \tag{25}$$

$$\begin{aligned}S_2(P,D)=\{&(P-\varepsilon_1+i\Gamma_q)[D+q_1+q_2-\varepsilon_1-\varepsilon_2+i(\Gamma_k\\&+\Gamma_q)](k_1+k_2+q_1+q_2-2\varepsilon_1+2i\Gamma_k)(P+D\\&-\varepsilon_2+i\Gamma_k)\}^{-1},\end{aligned} \tag{26}$$

$$\begin{aligned}Y_1=Y_0\{&\exp[i((\mathbf{q}_1+\mathbf{k}_1)R_1+(\mathbf{q}_2+\mathbf{k}_2)\mathbf{R}_2)]\\&+\exp[i((\mathbf{q}_2+\mathbf{k}_1)\mathbf{R}_1+(\mathbf{q}_1+\mathbf{k}_2)\mathbf{R}_2)]\},\end{aligned} \tag{27}$$

$$\begin{aligned}Y_2=Y_0\{&\exp[i((\mathbf{q}_1+\mathbf{k}_2)\mathbf{R}_1+(\mathbf{q}_2+\mathbf{k}_1)\mathbf{R}_2)]\\&+\exp[i((\mathbf{q}_2+\mathbf{k}_2)\mathbf{R}_1+(\mathbf{q}_1+\mathbf{k}_1)\mathbf{R}_2)]\},\end{aligned} \tag{28}$$

$$Y_0=C^*_{k_1}C^*_{k_2}C^*_{q_1}C^*_{q_2}. \tag{29}$$

For the exact resonance case ($k=\Omega_{21}$, $q=\Omega_{10}$), expression (24) after integration over the $q$-energies becomes

$$\begin{aligned}&\frac{dW(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2,t=\infty)}{d\mathbf{k}_1 d\mathbf{k}_2 d\Omega_{\mathbf{q}_1} d\Omega_{\mathbf{q}_2}}\\&=16\pi^2P(\Gamma_k,\Gamma_q)T^{-2}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2)\\&\times\{1+\cos[(\mathbf{q}_1-\mathbf{q}_2)(\mathbf{R}_1-\mathbf{R}_2)]\}\\&\times\{1+\cos[(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{R}_1-\mathbf{R}_2)]\},\end{aligned} \tag{30}$$

$$\begin{aligned}P(\Gamma_k,\Gamma_q)=&-8g_1+16g_2-8g_3-8g_4+8g_5\\&-12g_6+8g_7-2g_8,\end{aligned} \tag{31}$$

$$g_1=[4\Gamma_q(\Gamma_k+\Gamma_q)(\Gamma_k-\Gamma_q)^2(\Gamma_k+3\Gamma_q)(3\Gamma_k+\Gamma_q)]^{-1}, \tag{32}$$

$$g_2=[-32\Gamma_k\Gamma_q(\Gamma_k+\Gamma_q)(\Gamma_k-\Gamma_q)^2(3\Gamma_k+\Gamma_q)]^{-1}, \tag{33}$$

$$g_3=[-32\Gamma_q^2(\Gamma_k+\Gamma_q)(\Gamma_k-\Gamma_q)^2(\Gamma_k+3\Gamma_q)]^{-1}, \tag{34}$$

$$g_4=[-4\Gamma_q(\Gamma_k+\Gamma_q)^2(\Gamma_k-\Gamma_q)(\Gamma_k+3\Gamma_q)(3\Gamma_k+\Gamma_q)]^{-1}, \tag{35}$$

$$g_5=[-32\Gamma_k^2(\Gamma_k+\Gamma_q)(\Gamma_k-\Gamma_q)^2(3\Gamma_k+\Gamma_q)]^{-1}, \tag{36}$$

$$g_6=[-8\Gamma_k(\Gamma_k+\Gamma_q)^2(\Gamma_k-\Gamma_q)^2(3\Gamma_k+\Gamma_q)]^{-1}, \tag{37}$$

$$g_7=[-16\Gamma_k^2(\Gamma_k+\Gamma_q)^2(\Gamma_k-\Gamma_q)(3\Gamma_k+\Gamma_q)]^{-1}, \tag{38}$$

$$g_8=[-4\Gamma_q(\Gamma_k+\Gamma_q)^2(\Gamma_k-\Gamma_q)^2(3\Gamma_k+\Gamma_q)]^{-1}, \tag{39}$$

where $\Omega_{\mathbf{q}}$ is the solid angle into which the $q$-photons are emitted. The divergences of the terms (32)–(39) for $\Gamma_k=\Gamma_q$ cancel out and the function $P(\Gamma_k,\Gamma_q)$ remains continuous; the graph of the function is shown in Fig. 5. From (30), the contributions to the angular dependence of the emission rate from the $q$- and $k$-photons are seen to factorize.

In conclusion, a brief discussion of the above results is in order. As follows from the expression (19) ($N=1$) for the transition probability from the upper excited state, there may occur new effects as compared to two-level systems. First, we have a broadening of the frequency region over which the emission rate grows monotonically with the

time, and second, there is a rather peculiar dynamics of the wings. This circumstance can have an essential role to play in the dynamics of wave packet formation in the propagation of relatively short pulses in such media.

For a two-atom system and $\Gamma_q \to 0$—which is equivalent to equating to zero all the state amplitudes describing the presence of $q$-photons—we note that Eqs. (A1)–(A9) characterizes a system of two two-level atoms. Reference 1 gives a qualitative extension to the $N \gg 1$ case, which is presumably valid—at least for $\Gamma_k \gg \Gamma_q$—for the three-level situation as well.

## APPENDIX

For $N=2$, the equations for the corresponding amplitudes, according to (9), take the form

$$(\Omega-2\varepsilon_2+i\varepsilon)\alpha_0^0=1+\sum_{\mathbf{k}_1} C_{\mathbf{k}_1}^{(1)}\alpha_{\mathbf{k}_1}^0+\sum_{\mathbf{k}_2} C_{\mathbf{k}_2}^{(2)}\alpha_{\mathbf{k}_2}^2, \tag{A1}$$

$$(\Omega-\varepsilon_1-\varepsilon_2-k_1+i\varepsilon)\alpha_{\mathbf{k}_1}^1 = C_{\mathbf{k}_1}^{(1)*}\alpha_0^0+\sum_{\mathbf{q}_1} C_{\mathbf{q}_1}^{(1)}T(\mathbf{k}_1,\mathbf{q}_1)\alpha_{\mathbf{k}_1\mathbf{q}_1}^0 + \sum_{\mathbf{k}_2} C_{\mathbf{k}_2}^{(2)}T(\mathbf{k}_1,\mathbf{k}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2}^{12}, \tag{A2}$$

$$(\Omega-\varepsilon_1-\varepsilon_2-k_2+i\varepsilon)\alpha_{\mathbf{k}_2}^2 = C_{\mathbf{k}_2}^{(1)*}\alpha_0^0+\sum_{\mathbf{q}_2} C_{\mathbf{q}_2}^{(2)}T(\mathbf{k}_2,\mathbf{q}_2)\alpha_{\mathbf{k}_2\mathbf{q}_2}^0 + \sum_{\mathbf{k}_1} C_{\mathbf{k}_1}^{(1)}T(\mathbf{k}_1,\mathbf{k}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2}^{12}, \tag{A3}$$

$$(\Omega-2\varepsilon_1-k_1-k_2+i\varepsilon)\alpha_{\mathbf{k}_1\mathbf{k}_2}^{12} = T^{-1}(\mathbf{k}_1,\mathbf{k}_2)\left[C_{\mathbf{k}_1}^{(1)*}\alpha_{\mathbf{k}_2}^2+C_{\mathbf{k}_2}^{(2)*}\alpha_{\mathbf{k}_1}^1+C_{\mathbf{k}_2}^{(1)*}\alpha_{\mathbf{k}_1}^2 + C_{\mathbf{k}_1}^{(2)*}\alpha_{\mathbf{k}_2}^1+\sum_{\mathbf{q}_1} C_{\mathbf{q}_1}^{(1)}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1}^2 + \sum_{\mathbf{q}_2} C_{\mathbf{q}_2}^{(2)}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_2}^1\right], \tag{A4}$$

$$(\Omega-\varepsilon_2-k_2-q_2+i\varepsilon)\alpha_{\mathbf{k}_2\mathbf{q}_2}^0 = T^{-1}(\mathbf{k}_2,\mathbf{q}_2)\left[C_{\mathbf{q}_2}^{(2)*}\alpha_{\mathbf{k}_2}^2+\sum_{\mathbf{k}_1} C_{\mathbf{k}_1}^{(1)}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_2}^1\right], \tag{A5}$$

$$(\Omega-\varepsilon_2-k_1-q_1+i\varepsilon)\alpha_{\mathbf{k}_1\mathbf{q}_1}^1 = T^{-1}(\mathbf{k}_1,\mathbf{q}_1)\left[C_{\mathbf{q}_1}^{(1)*}\alpha_{\mathbf{k}_1}^1+\sum_{\mathbf{k}_2} C_{\mathbf{k}_2}^{(2)}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1}^2\right], \tag{A6}$$

$$(\Omega-\varepsilon_1-k_1-k_2-q_1+i\varepsilon)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1}^1 = T^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1)\left[C_{\mathbf{q}_1}^{(1)*}T(\mathbf{k}_1,\mathbf{k}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2}^{12} + C_{\mathbf{k}_2}^{(2)*}T(\mathbf{k}_1,\mathbf{q}_1)\alpha_{\mathbf{k}_1\mathbf{q}_1}^0+C_{\mathbf{k}_1}^{(2)*}T(\mathbf{k}_2,\mathbf{q}_1)\alpha_{\mathbf{k}_2\mathbf{q}_1}^0 + \sum_{\mathbf{q}_2} C_{\mathbf{q}_2}^{(2)}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1\mathbf{q}_2}^0\right], \tag{A7}$$

$$(\Omega-\varepsilon_1-\mathbf{k}_1-\mathbf{k}_2-\mathbf{q}_2+i\varepsilon)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_2}^1 = T^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_2)\left[C_{\mathbf{q}_2}^{(2)*}T(\mathbf{k}_1,\mathbf{k}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2}^{12} + C_{\mathbf{k}_1}^{(1)*}T(\mathbf{k}_2,\mathbf{q}_2)\alpha_{\mathbf{k}_2\mathbf{q}_2}^0+C_{\mathbf{k}_2}^{(1)*}T(\mathbf{k}_1,\mathbf{q}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2}^0 + \sum_{\mathbf{q}_1} C_{\mathbf{q}_1}^{(1)}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1\mathbf{q}_2}^0\right], \tag{A8}$$

$$(\Omega-k_1-k_2-q_1-q_2+i\varepsilon)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1\mathbf{q}_2}^0 = T^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2)[C_{\mathbf{q}_1}^{(1)*}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_2}^1 + C_{\mathbf{q}_2}^{(2)*}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1}^2 + C_{\mathbf{q}_2}^{(1)*}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_1)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_1}^1 + C_{\mathbf{q}_1}^{(2)*}T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}_2)\alpha_{\mathbf{k}_1\mathbf{k}_2\mathbf{q}_2}^1]. \tag{A9}$$


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