Detecting dislocations by measuring the energy flux of an acoustic field

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The energy flux lines near singular points of an acoustic field—dislocations and saddle points of the phase front, at which the amplitude and gradient of the field phase, respectively, vanish—are analyzed. The energy flux lines near dislocations are closed curves. There is a region near dislocations in which the energy flux (the Poynting vector) is directed opposite the predominant propagation direction. The concept of an average isolated dislocation with a characteristic wave-field distribution is proposed for a multimode waveguide. The mean distance between a field zero and a saddle point is calculated. The mean energy "trapped" by an isolated dislocation is also calculated. The ratio of the trapped energy to the total propagating energy in the waveguide is calculated. The particular features of the behavior of the Poynting vector near a dislocation can be used successfully to detect wavefront dislocations in underwater acoustics in experiments on dislocation tomography. Composite $p\mathbf{v}$ detectors can be used in such experiments.

1. INTRODUCTION

Dislocations of the phase front are a surprising feature of the interference structure of wave fields. A feature of this sort arises near zeros of the wave field, where the constant-phase surfaces are severely distorted. The picture of the phase front is reminiscent of dislocations in a solid. The term "wavefront dislocations" was first introduced by Nye and Berry[1] back in 1974. The very first publications in this field[1–9] revealed some interesting properties of dislocations. In particular, it was found that when a field zero is circumvented along a closed contour the phase increases by 2π or -2π, depending on the sign of the topological charge of the dislocation. In the present paper we go more deeply into the physics of this phenomenon. Specifically, we establish the relationship between phase-front dislocations and the energy fluxes of acoustic fields. The behavior of these energy fluxes near dislocations gives rise to a new method for detecting dislocations. It turns out that near dislocations there is always a region in which the Poynting (or "Umov–Poynting") vector is directed opposite the predominant energy propagation direction in the waveguide. A dislocation can therefore be identified by means of a "fluxmeter," i.e., a device which measures energy fluxes. In acoustics, such measurements are carried out by a $p\mathbf{v}$ detector. In principle, a corresponding method could be used to detect dislocations in the phase front of an electromagnetic field. Unfortunately, there is no satisfactory instrument available for measuring the Poynting vector in this case. An important part of dislocation tomography is identifying and observing dislocations.[10] Dislocation tomography is based on the detection of variations in the phase, since it is the phase of a field which is most sensitive to various perturbations of the waveguide specifically near wavefront dislocations. An energy method for observing dislocations is another weapon in the arsenal of dislocation tomography.

2. ENERGY FLUX LINES

Our basic purpose in the present study is to learn about the behavior of the flux density of the time-average acoustic power near a singularity in an acoustic field characterized by the Poynting vector

$$\mathbf{I}(\mathbf{r})=\tfrac{1}{2}\,\mathrm{Re}(p(\mathbf{r})\mathbf{v}^*(\mathbf{r}))=(2\rho\omega)^{-1}\mathrm{Im}(p^*(\mathbf{r})\nabla p(\mathbf{r})). \quad (1)$$

Here $p(\mathbf{r})$ is the acoustic pressure, $\mathbf{v}(\mathbf{r})=\nabla p(\mathbf{r})/i\rho\omega$ is the vibrational velocity of the particles of the medium, ρ is the density of the medium, and ω is the fixed sound frequency. In the acoustic case, a composite $p\mathbf{v}$ detector presents us with a direct and simple way to measure the vector quantity $\mathbf{I}(\mathbf{r})$. Such a detector detects separately the pressure p and the vibrational velocity $\mathbf{v}$ in the sound wave. We are thinking of the practical problem of detecting dislocations in hydroacoustic waveguides. For simplicity we assume that the waveguide is two-dimensional; i.e., we write $\mathbf{r}=(x,y)$, where x and y are, respectively, the horizontal and vertical coordinates (Fig. 1).

In an isotropic and nonabsorbing medium, the vector $\mathbf{I}(\mathbf{r})$ satisfies the following equation away from the source:

$$\mathrm{div}\mathbf{I}(\mathbf{r})=0. \quad (2)$$

A graphic picture of the spatial distribution of the power flux $\mathbf{I}(\mathbf{r})$ is constructed by the set of energy flux lines, i.e., by the set of lines along which the acoustic energy of the field is propagating. Energy flux lines can be determined well nearly everywhere in space. They are integral curves of the following first-order differential equation:

$$dy/dx=I_y(x,y)/I_x(x,y), \quad \mathbf{I}(\mathbf{r})=(I_x(\mathbf{r}),I_y(\mathbf{r})). \quad (3)$$

In the nondegenerate case, a single integral curve of differential equation (3) passes through each ordinary point in the $\mathbf{r}=(x,y)$ plane. By virtue of Eq. (2), the energy propagating between two neighboring flux lines is

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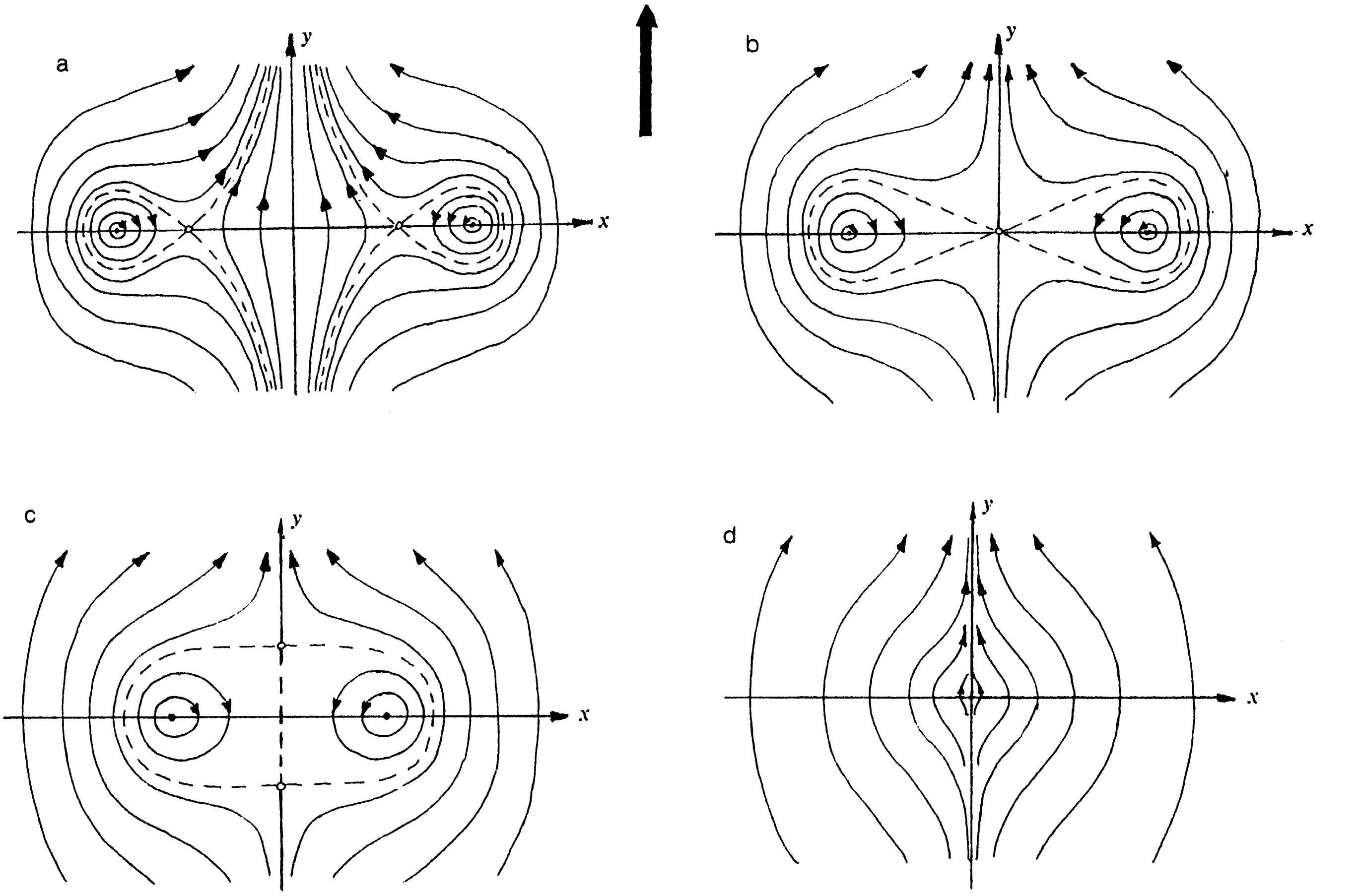

FIG. 1. Energy flux lines near zeros (dots) and saddle points (circles) in the symmetric case. The dashed lines are separatrices. The general energy propagation direction is along the y axis. a—$\varepsilon=1.5$; b—1; c—0.5; d—0.

conserved along these lines: The integral curves in the (x,y) plane can be treated as impenetrable boundaries for the energy.

An important point is that the energy flux lines coincide with phase trajectories in the (x,y) plane, i.e., with lines of the phase gradient. To demonstrate this point, we denote by $D(\mathbf{r})$ the amplitude and by $\Phi(\mathbf{r})$ the phase of the field:

$$p(\mathbf{r})=D(\mathbf{r})\exp(i\Phi(\mathbf{r})),$$
$$D(\mathbf{r})=|p(\mathbf{r})|,\quad \Phi(\mathbf{r})=\arg p(\mathbf{r}). \tag{4}$$

We denote by s a natural parameter on an energy flux line; this parameter increases along the direction $\mathbf{I}/|\mathbf{I}|$. Substituting (4) into definition (1), we find

$$\mathbf{I}(\mathbf{r})=D^2(\mathbf{r})\nabla\Phi(\mathbf{r})/2\rho\omega,\quad \nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right). \tag{5}$$

In other words, the direction of the Poynting vector is along the phase gradient $\nabla\Phi$, and all the phase trajectories are energy flux lines.

The phase of the wave field and the natural parameter are thus related by the following differential equation on an energy flux line:

$$d\Phi=|\nabla\Phi|ds. \tag{6}$$

The phase of the field increases monotonically along an energy flux line; i.e., this phase introduces a natural parametrization on a flux line.

In the simplest case of a plane wave, $\exp(i\mathbf{kr})$, the energy flux lines are straight lines parallel to the wave vector $\mathbf{k}$. In a lossless, horizontally uniform waveguide the picture of flux lines for one separate mode is also extremely simple: It is a set of straight lines running parallel to the boundaries of the waveguide. A critical restructuring of the picture of energy flux lines occurs when dislocations—singular points of the field with a singular behavior of the phase—arise. Let us recall the conditions for the occurrence of field singularities.

3. FIELD SINGULARITIES: DISLOCATIONS AND SADDLE POINTS OF THE PHASE FRONT

The acoustic field satisfies the wave equation

$$\Delta p(\mathbf{r})+k^2p(\mathbf{r})=0;\quad p(\mathbf{r})=u(\mathbf{r})+iv(\mathbf{r});\quad \Delta=\nabla^2, \tag{7}$$

where $k=\omega/c$ is the wave number, c is the sound velocity, and u and v are the real and imaginary parts of the field: $u=\mathrm{Re}\,p$, $v=\mathrm{Im}\,p$.

We assume that the field $p(\mathbf{r})$ is formed by a large number of waves which arrive from various directions; e.g., we might be dealing with a sum of modes in a vertical cross section of a waveguide. In the (x,y) plane there are thus always singular points corresponding to dislocations (x_d,y_d) and saddle points (x_s,y_s) in the distribution of the field phase:

$$D^2(\mathbf{r})=u^2(\mathbf{r})+v^2(\mathbf{r})=0 \quad \text{(dislocations, } \mathbf{r}=\mathbf{r}_d\text{)}, \tag{8}$$

$$\nabla\Phi(\mathbf{r})=D^{-2}(u\nabla v-v\nabla u)=0,$$

$$D\neq 0 \quad \text{(saddle points, } \mathbf{r}=\mathbf{r}_s\text{)}. \tag{9}$$

Both types of field singularities—the dislocations in (8) and the saddle points in (9) of the phase surface—are structurally stable formations, and they have conserved topological characteristics.[1–3] As the field parameters are varied, the dislocations move in space, while the saddle points "dance around" near field zeros.[3]

The behavior of the constant-phase lines near field zeros is completely different from that near saddle points. The constant-phase lines emerge radially from a field zero (the field phase is not defined at the zero itself). As the zero is circumvented along a small closed contour, the phase acquires an increment of $\pm 2\pi$, the sign being determined by the first topological characteristic: the charge of the dislocation. Near a saddle point, the gradient of the field phase changes the sign of the projection onto the radius vector emerging from the saddle point an even number of times (at the saddle point itself, the field gradient vanishes). There is also a second topological characteristic of the singular points: the Poincaré index, which takes on the value -1 for a saddle point and the value $+1$ for a zero of the field.[3,4]

4. GENERAL PROPERTIES OF THE BEHAVIOR OF ENERGY FLUX LINES NEAR PHASE SINGULARITIES

Points at which the flux density of the acoustic power, $\mathbf{I}(\mathbf{r})$ [see (5)], vanishes are singular points of differential equation (3). They are thus simultaneously zeros of the field [see (8); dislocations] and saddle points of the phase surface [see (9)]. Equation (3) for the energy flux lines is completely equivalent to the equation for the phase trajectories of the autonomous system.[11] The behavior of the integral curves of Eq. (3) near singular points (8) and (9) is thus determined by the values of the roots (q_1 and q_2) of the characteristic quadratic equation,[11] which in the case at hand takes the form

$$q^2-q\,\operatorname{div}\mathbf{I}(\mathbf{r})+\frac{\partial(I_x,I_y)}{\partial(x,y)}=0, \tag{10}$$

where all quantities are taken at the value $\mathbf{I}(\mathbf{r})=0$.

Using (2) and (7)–(9), we find from (10) that near a saddle point ($\mathbf{r}=\mathbf{r}_s$) the roots of the characteristic equation are real and differ in sign:

$$q_{1,2}=\pm[(uv_{xx}-vu_{xx})^2+(uv_{xy}-vu_{xy})^2]^{1/2}/2\rho\omega. \tag{11}$$

In the case of a field zero ($\mathbf{r}=\mathbf{r}_d$), in contrast, the roots q_1 and q_2 are purely imaginary:

$$q_{1,2}=\pm i|v_xu_y-u_xv_y|/2\rho\omega. \tag{12}$$

In this case the field zero is a center, and the energy flux lines are closed curves circumventing this center.

The closed nature of the energy flux lines of course does not violate energy conservation: Energy "flows into" a vortex region during the transient process by which the dislocations form (see Ref. 12 for an example of a pattern of circulating energy fluxes).

5. EXAMPLE OF AN EXACT SOLUTION

Let us examine the behavior of the energy flux lines near dislocations for the particular case of the following example[3] of an exact solution of wave equation (7):

$$p(x,y)=\left[\varepsilon-(kx)^2-\frac{k_x^2}{k_y^2}(ky)^2+\frac{2k_x}{k_y}(kx)(ky)-i\frac{k^3}{k_y^3}(ky)\right]\exp\{i(k_xx+k_yy)\}, \tag{13}$$

where $\mathbf{k}=(k_x,k_y)$ is the wave vector of the carrier wave, and ε is an adjustable parameter.

For $\varepsilon>0$, solution (13) describes two oppositely charged dislocations. The zeros of the field are on the x axis, at the points $\pm\sqrt{\varepsilon}$. These two dislocations have two conjugate saddle points. In the limit $\varepsilon\to 0$ all four field singularities move close together, and at $\varepsilon=0$ they merge at the origin of coordinates, forming an isolated field zero. Finally, at $\varepsilon<0$ the field has no singularities. In the particular case in which the carrier wave $\exp(iky)$ is propagating along the y axis ($k_x=0$) the field pattern becomes symmetric.

The change of variables $kx\to x$, $ky\to y$ puts Eq. (3) for the flux lines in the following form:

$$2xy\frac{dy}{dx}+y^2=(\varepsilon-x^2)^2-(\varepsilon-x^2)\equiv g(x). \tag{14}$$

An integral curve of differential equation[14] which passes through a nonsingular point $\mathbf{r}=(\xi,\eta)$ is determined by the real values of the function

$$y(x,\xi,\eta)=\pm[(\xi\eta^2+F(\xi)-F(x))/x]^{1/2}, \tag{15}$$

where

$$F(x)\equiv\int g(x)dx=x\left[\frac{x^4}{5}+(1-2\varepsilon)\frac{x^2}{3}+\varepsilon(\varepsilon-1)\right]. \tag{16}$$

Singular points of the family of energy flux lines in (15) are the following zeros and saddle points:

$$x_d=\pm\sqrt{\varepsilon},\quad y_d=0;\quad 0\leqslant\varepsilon; \tag{17}$$

$$\left.\begin{array}{lll} x_s=\pm\sqrt{\varepsilon-1}, & y_s=0, & 1\leqslant\varepsilon; \\ x_s=0, & y_s=\sqrt{\varepsilon(1-\varepsilon)}, & 0\leqslant\varepsilon\leqslant 1 \end{array}\right\}. \tag{18}$$

Each dislocation is surrounded by a region in which the energy flux circulates along closed trajectories (Fig. 1). The circulation directions are different for dislocations of unlike charge. In part of the volume, the direction of the energy flux is opposite to the propagation direction of the fundamental wave (as shown by the broad arrow in Fig. 1).

The separatrix, i.e., the extreme closed integral curve which always passes through a saddle point in (18) (it is shown by the dashed line in Fig. 1), separates the circulating energy flux from the energy flux which goes off to infinity. At $\varepsilon > 1$ the saddle points are separated by a distance $2\sqrt{\varepsilon-1}$, and there is an energy flux between dislocations along the propagation direction of the carrier wave (Fig. 1a). At $\varepsilon=1$, this flux comes to a halt (Fig. 1b). When the dislocations disappear at $\varepsilon<0$, the regions in which the Poynting vector is directed opposite to the wave vector of the carrier wave also disappear.

In the steady state, the separatrix separates an energy flux which is circulating in a bounded region near the field zero, and it "disconnects" this energy flux from the energy flux carried by the carrier wave. The energy flux "trapped" by each of the dislocations is

$$I_{\mathrm{tr}} = \int_{\mathbf{r}_d}^{\mathbf{r}_s} |\mathbf{I}(x,y)\times d\mathbf{r}| = \left| \int_{x_d}^{x_s} I_y(x,0)dx \right| = |F(x_d)-F(x_s)|/2\rho\omega, \quad (19)$$

where we have used $I_x(x,0)=0$. This value of the first integral is obviously independent of the end point of the integration, provided that it lies on the separatrix. The equation for the separatrix in this case is

$$y_s(x)=y(x,x_s,y_s)=\left[\frac{F(x_s)-F(x)}{x}\right]^{1/2}. \quad (20)$$

In the more general case of an arbitrary propagation direction for the carrier wave, $\exp[i(k_x x+k_y y)]$, the pattern of energy flux lines loses it symmetry with respect to y reversal, but in all cases in which there are dislocations we find that some regions appear near these dislocations in which the Poynting vector is directed opposite the propagation direction of the carrier wave, $\mathbf{k}$. This property can be utilized for an experimental observation of dislocations.

6. DISLOCATIONS AND SADDLE POINTS IN A HYDROACOUSTIC WAVEGUIDE

We consider a two-dimensional model of a uniform hydroacoustic waveguide of constant depth H with an absolutely rigid bottom. We direct the x axis horizontally along the waveguide, and the z axis vertically downward (Fig. 2). We write the field in the waveguide as the sum of N modes:

$$p(z,x)= \sum_{n=1}^{N} a_n \psi_n(z)\exp(ih_n x). \quad (21)$$

Here $h_n = \sqrt{k^2-\alpha_n^2}$ and $\alpha_n=\pi(2n-1)/2H$ are, respectively, the horizontal and vertical wave numbers of the nth mode, a_n are real amplitudes, and $\psi_n(z)=\sqrt{2}\sin\alpha_n z$ are "vertical" wave functions, normalized by the condition

$$\int_0^H \psi_n^2(z)dz=H. \quad (22)$$

We are interested in the case in which a statistical approach can be taken, and in which we can speak in terms of a "typical" or "average" dislocation with a characteristic field distribution near the amplitude zero. For this case we consider a fairly large part of the waveguide, $H\times L$, $L \gg x_{\max}$, where

$$x_{\max}=2\pi/\min_{n\neq m}|h_n-h_m|$$

is the greatest spatial period of intermode beats. We make the further assumption that the number of modes is large: $N\gg 1$.

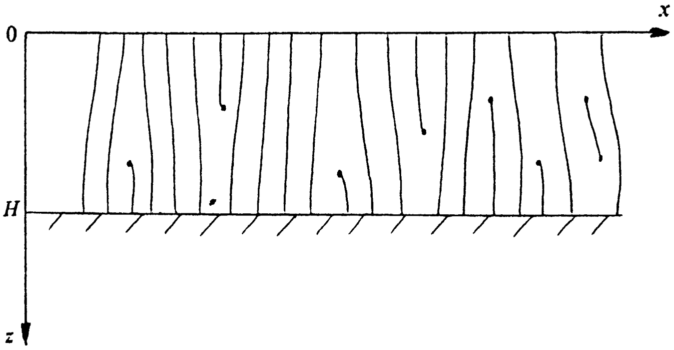

FIG. 2. Typical pattern of constant-phase lines in a cross section of an acoustic waveguide. The lines are spaced at intervals of 2π.

Numerical calculations[13] show that the field $p(z,x)$ in the region $H\times L$ obeys Gaussian statistics with a zero mean, with a uniform phase distribution over the interval $(0,2\pi)$, and with a Rayleigh amplitude distribution at values as low as $N>5$–6. The normalization corresponds to the case in which the spatial frequencies are incommensurate, and the phase of the modes at large distances undergoes numerous subdivisions by an interval $(0,2\pi)$.

The probability distribution is of course quite different from Gaussian near caustics or near sharp amplitude spikes which occur by chance as a result of constructive interference of some of the modes with large amplitudes. Falling in the category of these "bad" regions are the relatively small surface zones in which the average sound amplitude falls off linearly, reaching zero at $z=0$. We will ignore all these effects here.

For our purposes, a linear expansion of the field near an amplitude zero at (z_d,x_d), i.e.,

$$p(z_d+\zeta,x_d+\xi)=p(z_d,x_d)+\zeta\left.\frac{\partial p}{\partial z}\right|_{z_d,x_d}+\xi\left.\frac{\partial p}{\partial x}\right|_{z_d,x_d};$$
$$p(z_d,x_d)=0, \quad (23)$$

is not a good choice as a characteristic field of an average dislocation. Expansion (23) has a very small range of applicability along the x direction (this range is much shorter than the sound wavelength $\lambda=2\pi/k$), and it does not describe the field near the saddle point which always accompanies an amplitude zero. We will make use of energy considerations, which allow us to overcome both of these obstacles, while keeping the picture of the field of the average dislocation in the multimode waveguide extremely simple.

We write the field in the waveguide (21), as a specific product of two waves $p_c(x)$ and $p_d(z,x)$ (we are introducing a constant factor p_0^{-1} for convenience):

$$p(z,x) = p_c(x) p_d(z,x) / p_o. \tag{24}$$

The first of the waves, $p_c(x)$, which we call the "carrier," is a plane wave:

$$p_c(x) = p_o \exp(ih_o x), \quad h_o = \text{const}. \tag{25}$$

It does not contain dislocations. The second wave $p_d(z,x)$, in contrast, contains all the dislocations of the original field $p(z,x)$ in (21):

$$p_d(z,x) = \sum_{n=1}^{N} a_n \psi_n(z) \exp(ih_{no} x), \quad h_{no} \equiv h_n - h_o. \tag{26}$$

The zeros of this function coincide with the coordinates of the zeros of the field $p(z,x)$.

We introduce the operation of taking a spatial average over a rectangle of length L and height H, and we denote this operation by a superior bar:

$$\overline{f(z,x)} \equiv \frac{1}{HL} \int_0^H dz' \int_x^{x+L} dx' f(z',x'). \tag{27}$$

We denote by p_0^2 the mean square magnitude of wave field (21):

$$p_0^2 \equiv \overline{|p(z,x)|^2} = \frac{1}{HL} \int_0^H dz' \int_x^{x+L} dx' |p(z',x')|^2. \tag{28}$$

We require of the fields p_c and p_d that their mean square magnitudes be equal to each other and to p_0^2:

$$\overline{|p_d(z,x)|^2} = \overline{|p_c(x)|^2} = p_0^2. \tag{29}$$

It is reasonable to suggest that a dislocation wave $p_d(z,x)$ which describes vortex motion of energy in the waveguide has a zero average energy flux:

$$\overline{\mathbf{I}^d} \equiv \frac{1}{2\rho\omega} \operatorname{Im} \overline{p_d^* \nabla p_d} \equiv (\overline{I_z^d}, \overline{I_x^d}) = 0. \tag{30}$$

This vector relation can also be written as the two equations

$$\overline{I_x^d} = \frac{p_o^2}{2\rho\omega} (\{h_n\} - h_0) = 0, \quad \overline{I_z^d} = 0. \tag{31}$$

Here and below, the curly brackets mean an average over the waveguide modes, i.e., an average with the square of the mode amplitude as a weight function:

$$\{f_n\} \equiv \frac{\Sigma a_n^2 f_n}{\Sigma a_n^2} = \frac{1}{p_0^2} \Sigma a_n^2 f_n. \tag{32}$$

Since the horizontal energy flux is zero, we conclude that the magnitude of the "wave vector" of the carrier wave, h_0, should be taken to be equal to $\{h_n\}$:

$$h_0 = \{h_n\} = \sum_{n=1}^{N} a_n^2 h_n \Big/ \sum_{n=1}^{N} a_n^2. \tag{33}$$

For this choice, the average energy flux of the acoustic field in the waveguide,

$$\overline{\mathbf{I}(z,x)} = (2\rho\omega)^{-1} \operatorname{Im} \overline{p^* \nabla p} \equiv (\bar{I}_z, \bar{I}_x),$$

is equal to the energy flux in the carrier wave,

$$\bar{I}_x = (2\rho\omega)^{-1} p_0^2 h_0 = I_x^c \equiv (2\rho\omega)^{-1} \operatorname{Im} \overline{p_c^* \frac{\partial p_c}{\partial x}}. \tag{34}$$

The average vertical energy flux is identically zero:

$$\bar{I}_z = I_z^c = (2\rho\omega)^{-1} \operatorname{Im} \overline{p_c^* \frac{\partial p_c}{\partial z}} = 0. \tag{35}$$

It is by no means necessary that each of the waves p_c and p_d satisfy the wave equation and the boundary conditions. Only their product $p = p_c p_d / p_o$ has these properties. The multiplicative representation used here for the field, with a zero average energy flux for the dislocation part, turns out to be useful in statistical problems. It goes a long way toward simplifying the calculation of average values in a random field: average values of the dislocation density, of the overall length of the zero lines, of the mean square velocity of the motion of these lines, etc.[14]

We expand the dislocation part of the field, $p_d(z,x)$, in a series in the small deviation (ζ,ξ) as in (23). Using $p_d(z_d,x_d) = 0$, we write the total field $p = p_c p_d / p_0$ near the zero as

$$p(z_d + \zeta, x_d + \xi) = (a\zeta + b\xi) \exp[ih_0(x_d + \xi)]. \tag{36}$$

The derivatives of the dislocation parts which appear here,

$$a \equiv \frac{\partial p_d}{\partial z} = \sum_{n=1}^{N} a_n \psi_n'(z) \exp(ih_{no} x),$$
$$b \equiv \frac{\partial p_d}{\partial x} = i \sum_{n=1}^{N} a_n \psi_n(z) h_{no} \exp(ih_{no} x), \tag{37}$$

are calculated at the point (z_d, x_d). Expressions (36) and (37) give a characteristic field distribution for an average dislocation. Estimates found below for these derivatives show that the value of b in the paraxial approximation is fairly small in comparison with the wave number k: $|b| \ll k$. In this case, Eqs. (36) and (37) can describe the field over a distance of many wavelengths along the x direction.

The coordinates of a saddle point (z_s, x_s) are found by equating the phase gradient to zero:

$$z_s = z_d + \Delta z_s, \quad x_s = x_d, \tag{38}$$

where

$$\Delta z_s = \operatorname{Im}(ab^*) / h_0 |a|^2. \tag{39}$$

We denote by $S = |\Delta z_s|$ the vertical line segment between the field zero and the saddle point (the corresponding horizontal coordinates are the same). We can then write

$$\Delta z_s = -QS, \tag{40}$$

where

$$Q = -\operatorname{sign}(\operatorname{Im}(ab^*)) \tag{41}$$

is the first topological characteristic, i.e., the charge of the dislocation, which we have already mentioned. It is assumed that the phase of the field acquires an increment of $2\pi Q$ when a field zero is circumvented in the positive direction. In the case $Q=+1$, the saddle point is closer to the surface, and in the case $Q=-1$ it is closer to the bottom of the waveguide, than the field zero (z_d,x_d).

The horizontal component of the energy flux on segment S is

$$I_x=\frac{1}{2\rho\omega}h_o|a|^2\zeta(\zeta-\Delta z_s),$$

so the energy flux trapped inside the separatrix is

$$I_{\rm tr}=\left|\int_{\Delta z_s}^{0}I_x d\zeta\right|=\tfrac{1}{6}h_o|a|^2S^3/2\rho\omega=\frac{|\mathrm{Im}(ab^*)|^3}{12\rho\omega h_o^2|a|^4}. \qquad (42)$$

The squared modulus of the field at the saddle point is

$$|p(z_s,x_s)|^2=|\mathrm{Im}(ab^*)|^2/h_o^2|a|^2. \qquad (43)$$

7. AVERAGE CHARACTERISTICS OF DISLOCATIONS AND SADDLE POINTS IN A WAVEGUIDE

The simplicity of the equations derived here, and also certain aspects of the behavior which seem surprising at first glance (e.g., the result that the zero and the saddle point lie in the same vertical line), stem from the statistical description of the dislocations.

Let us examine the statistical characteristics of dislocations and saddle points in more detail.

The values of the derivatives a and b in (37) depend on the coordinates of the field zero (z_d,x_d). For Gaussian random fields, the mean values of quantities which are closely related to the field zeros can be calculated on the basis of the concept of "averages on a zero carrier".[14] In the particular case of a one-dimensional zero carrier [lines intersecting the plane at the points (z_d,x_d)], the mean value of the quantity $\mathscr{F}$ is given by

$$\langle\mathscr{F}(u,v,q)\rangle=n_d^{-1}\int\mathscr{F}(0,0,q)W(0,q)\left|\frac{\partial(u,v)}{\partial(z,x)}\right|dq. \qquad (44)$$

Here q represents the set of field derivatives $q=(\partial p_d/\partial z,\partial p_d/\partial x)$, and $W(p_d,q)$ is the normal probability density of the joint distribution of the dislocation part of the field, p_d, and of its derivatives. The quantity n_d is the dislocation density (the density of field zeros):

$$n_d=\int W(0,q)\left|\frac{\partial(u,v)}{\partial(z,x)}\right|dq. \qquad (45)$$

If there are a large number of propagating modes, $N\gg1$, the correlation coefficient between the field p_d and its derivative $\partial p_d/\partial z$ becomes a small quantity. As a result, the probability density breaks up into a product of six unknown Gaussian distributions with respect to each argument, with zero mean values and variances. They can be calculated under the assumption of a spatial ergodicity.[15] This assumption means that an average over an ensemble is equivalent to an average over space:

$$\langle|u_d|^2\rangle=\langle|v_d|^2\rangle=p_0^2/2,$$

$$\langle|\partial u_d/\partial z|^2\rangle=\langle|\partial v_d/\partial z|^2\rangle=p_0^2\{\alpha_n^2\}/2, \qquad (46)$$

$$\langle|\partial u_d/\partial x|^2\rangle=\langle|\partial v_d/\partial x|^2\rangle=p_0^2\{h_{no}^2\}/2.$$

Here the averaging of α_n^2 and h_n^2 over modes is carried out in accordance with (32). From (45) and (46) we find that the average density of dislocations in a multimode waveguide is

$$n_d=(2\pi)^{-1}(\{\alpha_n^2\}\{h_{no}^2\})^{1/2}. \qquad (47)$$

Expression (47) is a waveguide analog of the expression for the dislocation density in a cross section of a paraxial laser beam, which was derived in Ref. 6.

Now replacing $\mathscr{F}$ again in (44) by expressions for quantities calculated earlier, (39), (43), and (42), we find the mean distance between a zero and a saddle point,

$$\langle S\rangle=\langle|\Delta z_s|\rangle=\frac{2}{h_0}\sqrt{\frac{\{h_{no}^2\}}{\{\alpha_n^2\}}}; \qquad (48)$$

the mean value of the square magnitude at a saddle point,

$$\langle|p(z_s,x_s)|^2\rangle=p_0^2\{h_{no}^2\}/h_0^2 \qquad (49)$$

and the mean energy flux trapped by one dislocation,

$$\langle I_{\rm tr}\rangle=\frac{p_0^2}{2\rho\omega}\frac{1}{4h_0^2}\sqrt{\frac{\{h_{no}^2\}^3}{\{\alpha_n^2\}}}. \qquad (50)$$

The total energy (kinetic plus potential) E_p of the acoustic field in the area $H\times L$ is

$$E_p=(\overline{\rho|\mathbf{v}|^2/2+|p|^2/2\rho c^2})HL=p_0^2HL/\rho c^2. \qquad (51)$$

The energy trapped by all the dislocations in the same area, $E_{\rm tr}$, can be found from

$$E_{\rm tr}=n_dHL\langle I_{\rm tr}\rangle T, \qquad (52)$$

where $T=2\pi/\omega$ is the vibration period. Dividing the trapped energy $E_{\rm tr}$ by the total energy E_p from (51), we find

$$\gamma\equiv\frac{E_{\rm tr}}{E_p}=\frac{\{h_{no}^2\}^2}{8k^2h_0^2}. \qquad (53)$$

The ratio γ is the ratio of the energy which is drawn into rotation around the dislocations to the energy moving in the horizontal direction.

8. ESTIMATES OF MEAN VALUES

We assume that the total number of modes which can potentially propagate in the waveguide is large: $N_0=kH/\pi\gg1$. We consider two particular cases.

1) Relatively few of the modes actually propagate: $N\ll N_0$ but $N\gg1$. This case corresponds to the paraxial approximation, in which Brillouin rays of the modes propagate within a narrow angular sector $\alpha\sim\arcsin\delta\sim\delta$, where $\delta\equiv N/N_0\ll1$. From (33) we find, in order of magnitude,

$$h_0\sim k,\quad \{\alpha_n^2\}\sim k^2\delta^2/3,\quad \{h_{n0}^2\}\sim k^2\delta^4/45. \qquad (54)$$

The mean values of the quantities in (47)–(50) are then

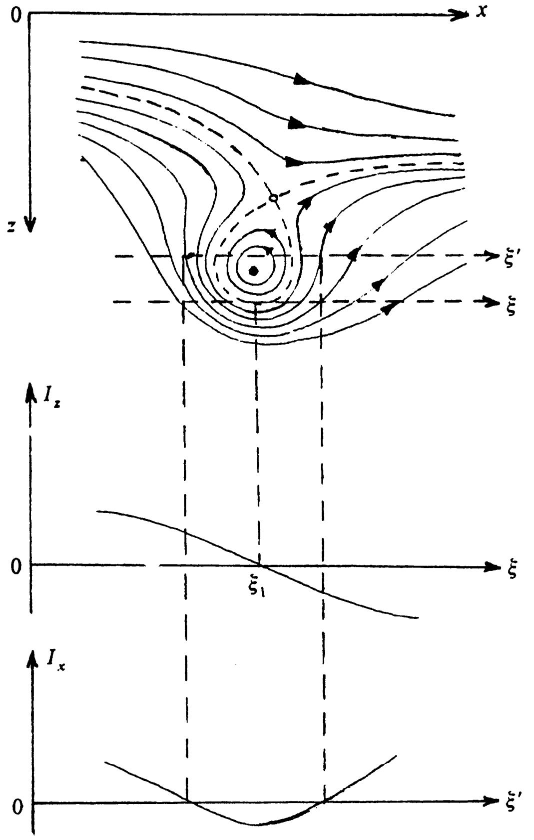

FIG. 3. Behavior of the x and z components of the Poynting vector along the horizontals ξ and ξ' at a certain distance from a field zero.

$$n_d \simeq k^2\left(\frac{N}{N_0}\right)^3 \Big/ 6\pi\sqrt{15} \sim \left(\frac{N}{N_0}\right)^3 \Big/ 2\lambda^2, \qquad (55)$$

$$\langle S\rangle \simeq \frac{N}{N_0}\lambda/4\pi, \quad \gamma \simeq 6\cdot 10^{-5}(N/N_0)^8.$$

Note that the mean dislocation density n_d is smaller by a factor of $3\sqrt{15}/\pi \sim 4$ than an estimate of the maximum dislocation density ($N_{\max}$) found in Ref. 9 on the basis of elementary considerations.

Interestingly, according to (55) the dislocation density in an acoustic waveguide is proportional to the cube of the angular spectrum, $n_d \sim \delta^3$, while for a paraxial laser beam the dislocation density is proportional to the square of the angular broadening of the beam,[6] $n_d \sim \delta^2$. There is no contradiction here; the difference in the geometry of the problem is coming into play.

2) The maximum number of propagating modes: $N = N_0$. In this case we have

$$h_0 \sim k\pi/4, \quad \{\alpha_n^2\} \sim k^2/3, \quad \{h_{no}^2\} \sim k^2/20.$$

The mean dislocation density is $n_d \simeq k^2/2\pi\sqrt{60} \sim 1/\lambda^2$. In other words, there is one dislocation per area of $\lambda\times\lambda = \lambda^2$. The mean distance from the zero to the saddle point of an isolated dislocation is comparable to λ: $\langle S\rangle \simeq \lambda/2\pi$. The fraction of the energy which is trapped is $\gamma \sim 5\cdot 10^{-4}$.

Caution must be exercised in estimating S and γ in this case, since at a high dislocation density it is difficult to accept the proposition that the dislocations can be considered to be isolated from each other and that it is sufficient to retain only the linear terms in expansion (36).

9. ACOUSTIC MEASUREMENTS

The vector energy flux density changes direction near dislocations. This circumstance can be utilized to detect dislocations.

Figure 3 shows the behavior of the z component of the Poynting vector on the horizontal ξ near a dislocation. The value of I_z vanishes and changes sign at a certain value $\xi = \xi_1$. The change in the sign of I_z is characteristic of proximity to a dislocation in the horizontal plane. In a similar way, the vanishing of the horizontal component I_x may be of assistance in finding a dislocation along the vertical direction.

Devices for measuring the Poynting vector are well known. They are "composite" or "$p\mathbf{v}$" detectors.[16,17]

From the practical standpoint, reliable detection of a reversal of the energy flux direction would require that the level of background acoustic noise, p_N^2, near the detector below in comparison with the characteristic value of the field $\langle |p(z_s,x_s)|^2\rangle$ from (49).

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