# Localized magnetoplasma and spin excitations in a 2D electron system in a strong magnetic field

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(Submitted 26 January 1993)
Zh. Eksp. Teor. Fiz. **104**, 3416–3433 (October 1993)

An interacting $2D$ system of electrons in a strong magnetic field with filled Landau levels and in a centrally symmetric external impurity potential (or in a potential of some other type) is analyzed. Localized collective neutral magnetoplasma and spin excitations are analyzed. For IR-active magnetoplasmons in the presence of an isolated Coulomb impurity, for Landau-level filling factors $\nu=1$ and 2, there are two localized states, in different parts of the spectrum. One of these states is split off downward and lies below the lower edge of the continuous spectrum (the latter corresponds to a delocalized mode with an energy $\hbar\omega_c$, which contributes the main peak in the absorption at cyclotron resonance). The second localized state is pushed upward. Depending on the sign of the impurity charge and on the filling factors $\nu$, this second state either (1) lies in the continuum, near its upper boundary (and is a resonant state), or (2) lies above (or well above) the upper boundary. At larger values of $\nu$, a large number of IR-active localized modes can appear in the spectrum. In a system with a finite density of impurities, these modes correspond to peaks in the IR absorption. The results derived here agree qualitatively with experimental data.

## 1. INTRODUCTION

Some distinctive features in the position and width of a cyclotron-resonance (CR) line of $2D$ carriers in a strong magnetic field have attracted much interest (see Refs. 1–5, for example, and the papers cited there). Experiments by Richter *et al.*[3] on GaS heterostructures selectively $\delta$-doped with either donors or acceptors revealed that the number of lines in the CR spectra and the behavior of these lines varied as functions of the Landau-level filling factor $\nu=nhc/eH<2$. A CR structure consisting of several (at least three) lines has also been observed for $2D$ electrons in a silicon MOS structure[4] containing charged impurities of both signs. In an electron system (a single-component system) with a parabolic dispersion law, interparticle interactions do not produce a shift or broadening of the CR line, because of translational invariance.[6] In order to explain these effects, it is thus necessary to take the interparticle interactions and the impurity potential into account simultaneously.

Yang and MacDonald[7] linked the features observed in the CR spectrum in Ref. 3 with one-electron transitions in the spectrum of a positively or negatively charged impurity. These features were modified as the result of (1) the random field of other impurities and (2) the existence of a finite density ($\nu=1$) of free $2D$ electrons. Interactions of these electrons were taken into account in Ref. 7 in the self-consistent Hartree approximation. So far we do not have a clear picture of the CR spectrum in the presence of an isolated impurity (or in a localizing potential of some other nature), uncluttered by the existence of disorder. When interparticle interactions are taken into account systematically, we should speak in terms of calculating the spectrum of localized collective magnetoplasma excitations. Our purpose in this paper is to construct a picture of this sort for the entire spectrum of localized excitations, including spin excitations, with various values of the angular-momentum projection $M_z$ (not exclusively for IR-active magnetoplasmons with $M_z=1$).

We are interested in a situation which is amenable to an essentially exact analysis: that in which there are filled Landau levels in the $2D$ system. A theory of magnetoplasma and spin collective modes in a $2D$ electron system with filled levels was derived in Refs. 8–10 (see also Ref. 11, which was a study of a two-component electron-hole system). Excitations of this sort are associated with transitions of a particle from high-lying Landau levels to the nearest vacant levels, and they can be thought of as neutral electron-hole ($e$-$h$) excitations, i.e., $2D$ magnetoexcitons. This situation differs from the problem of a $2D$ Wannier-Mott magnetoexciton in that exchange interactions must be taken into account.[8] The effect of the field of impurities on collective modes was studied in Refs. 1 and 2 in the self-consistent Born approximation; the results were found to be insensitive to the sign of the impurity charge. To take account of the effect of an external potential on collective modes in the present paper, we take the approach which was taken in Refs. 13–15 in the problem of the localization of $2D$ magnetoexcitons and multiparticle $e$-$h$ complexes at an impurity in a strong magnetic field.[1)]

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## 2. LOCALIZED MAGNETOPLASMA AND SPIN EXCITATIONS

### 2.1. Statement of the problem; classification of states

Let us consider an interacting $2D$ electron system with an interparticle interaction potential $U(|\mathbf{r}_1-\mathbf{r}_2|)$ in a strong magnetic field $H$ with heavily filled Landau levels against a uniform, positive neutralizing background. There is a centrally symmetric localizing potential $V(|\mathbf{r}|)$ in the system. The general approach of this paper and the qualitative results are valid for a fairly general potential $V(|\mathbf{r}|)$. In the specific calculations (Sec. 3) we consider the Coulomb interactions

$$V(r)=\mp\frac{e^2}{\varepsilon r},\quad U(r)=\frac{e^2}{\varepsilon r}. \tag{1}$$

The upper and lower signs in the expression for $V(r)$ correspond to an ionized shallow donor $D^+$ and a charged acceptor $A^-$ (we assume that the impurity is directly in the $2D$ layer). In the $A^-$ case we assume that the electron is in a deep acceptor level and that the external potential $V(r)$ corresponds to a (generally dynamic) interaction because of the small adiabatic parameter.

We assume that the magnetic field is strong:

$$\hbar\omega_c=\hbar\frac{eH}{m^*c}\gg E_0\equiv\left(\frac{\pi}{2}\right)^{1/2}\frac{e^2}{\varepsilon r_H},\quad r_H=\left(\frac{\hbar c}{eH}\right)^{1/2}, \tag{2}$$

where $E_0$ is the characteristic energy of the Coulomb interactions. We can thus ignore virtual transitions of particles between Landau levels due to the interactions, since the corresponding effects are small quantities of order[12,16] $E_0/\hbar\omega_c\ll 1$.

We first consider the general situation in which the first $n+1$ Landau levels with spin $\downarrow$ (the zeroth level up to the $n$th, inclusively) are filled, as are the first $l+1$ levels with spin $\uparrow$. The filling factors of the levels are $\nu_\downarrow=n+1$, and $\nu_\uparrow=l+1$ with $\nu\equiv\nu_\downarrow+\nu_\uparrow$. For definiteness we assume that the lower zeroth Landau level has a spin projection $\downarrow$. The low-lying excitations are associated with a transition of an electron from a filled upper level to the nearest vacant levels. In the absence of interparticle interactions, and in the absence of an interaction with the impurity, the energies of the excitations are determined by the orbital and spin splittings of the Landau levels: $\hbar\omega_c=\hbar eH/m^*c$ and $\hbar\omega_s=g^*\mu_B H$. When interparticle interactions are taken into account, dispersion arises, and zones of finite width[8] $\sim E_0$ form (see also Refs. 9–12). In this situation the states are classified on the basis of an exact quantum number, the magnetic momentum[17,18] $\mathbf{k}$, which determines the expectation value of the distance $\langle\mathbf{r}_{eh}\rangle=[\mathbf{k}\mathbf{e}_z]r_H^2$ between the particles in an $e$-$h$ pair.

When the external potential $V(|\mathbf{r}|)$ is also turned on, the states of the excitations are classified on the basis of the exact quantum number $M_z$, the projection of the angular momentum. In the strong-field approximation (the indices of the Landau levels $n_e$ and $n_h$ are conserved), we characterize the states by means of the oscillator quantum number $M=M_z-n_e+n_h$ (see also the discussion below). The moment $M$ is related to the difference between the squared radii of the $e$ and $h$ cyclotron orbits by

$$\langle\mathbf{r}_h^2-\mathbf{r}_e^2\rangle=2(M+n_h-n_e)r_H^2.$$

For each $M$ there is an infinite set of levels, and the spectrum of excitations consists of a continuum and an exciton band[2)] corresponding to sparsely distributed states of free magnetoexcitons. In addition, in the field $V(r)$ there are some split-off localized modes (cf. Refs. 12–15): discrete levels outside the continuum and quasidiscrete two-particle states (resonances) in the exciton band. In the case of an impurity potential, these resonances lie near the one-particle excited levels of an electron at an ionized donor (or of a hole at a charged acceptor).

### 2.2. Effective Hamiltonian; contribution to the energy of excitations

We consider an excitation associated with a transition of an electron from a filled Landau level $p\sigma'$ ($p\sigma'$ takes on the values $n\downarrow$ and $l\uparrow$) to one of the nearest levels with quantum numbers $k\sigma$ ($k\sigma=n+1\downarrow$ or $l+1\uparrow$). To describe one-particle states (in the $\mathbf{A}=\frac{1}{2}[\mathbf{Hr}]/2$ gauge) we use a complete set of factorized wave functions $\phi_{nm}(\mathbf{r})=\langle\mathbf{r}|nm\rangle$ of free particles in a magnetic field[19] (see also Ref. 20). The quantum number $n$ determines the index of the Landau level and the energy $\hbar\omega_c(n+1/2)$, while the oscillator quantum number $m$, with respect to which the energy is degenerate, is related to the angular-momentum projection $m_z$ by $m_z=n-m$.

It is obviously convenient to go over to the hole description of the filled levels:

$$a_{r\downarrow\, m}\to b^+_{r\downarrow\, m}\quad r=0,...,n,\quad a_{s\uparrow\, m}\to b^+_{s\uparrow\, m}\quad s=0,...,l, \tag{3}$$

where $a_{k\sigma m}$ is the Fermi operator which annihilates an electron with quantum number $m$ in Landau level $k\sigma$. We wish to stress that, for clarity, we are not changing the spin quantum number in (3). After this step, a state with filled levels becomes the vacuum state, which we denote by $|\mathrm{vac}\rangle$.

The most general type of neutral excitation with oscillator quantum number $M$ ($M=m_h-m_e$) at the given Landau levels for $M\geqslant 0$ is described by[11,13]

$$B^+_{k\sigma,p\sigma' M}=\sum_{m=0}^{\infty}A_{k\sigma,p\sigma' M}(m)a^+_{k\sigma m}b^+_{p\sigma' m+M} \tag{4}$$

(for $M<0$, it is necessary to interchange indices, $m\rightleftarrows m+M$, on the operators $a^+$ and $b^+$). The coefficients $A(m)$ and the eigenvalues must be found from the solution of the secular equation. Retaining in the complete Hamiltonian only those terms which conserve the numbers of particle in the $p\sigma'$ and $k\sigma$ levels after this switch to the hole description in (3), and measuring the energies of the excitations from

$$\hbar\omega_c(k-p)+\hbar\omega_s(\sigma_z-\sigma_z')/2,$$

where $\sigma_z=\pm 1$, we find a Hamiltonian which describes the excitation (4):

$$H_{k\sigma,p\sigma'}=\Big(\sum_{r=0}^{p}F_{rp}\Big)\hat{N}^h_{p\sigma'}-\Big(\sum_{s=0}^{k-1}F_{sk}\Big)\hat{N}^e_{k\sigma}$$
$$-\sum_{\substack{m_1,m_2\\ m_1',m_2'}}\Big\{U^{km_1',pm_2'}_{km_1,pm_2}$$
$$-\delta_{\sigma,\sigma'}U^{pm_2',km_1'}_{km_1,pm_2}\Big\}a^+_{k\sigma m_1'}b^+_{p\sigma' m_2}b_{p\sigma' m_2'}a_{k\sigma m_1}$$
$$+\sum_m V_{km}a^+_{k\sigma m}a_{k\sigma m}-\sum_m V_{pm}b^+_{p\sigma' m}b_{p\sigma' m}. \quad (5)$$

The matrix elements of the interactions from (5) are derived in the Appendix. Here the operators

$$\hat{N}^e_{k\sigma}=\sum_m a^+_{k\sigma m}a_{k\sigma m},\quad \hat{N}^h_{p\sigma'}=\sum_m b^+_{p\sigma' m}b_{p\sigma' m}$$

represent the numbers of electrons and holes in levels $k\sigma$ and $p\sigma'$, while

$$F_{rp}\equiv\sum_{m'}^{\infty}U^{pm'rm}_{rm,pm'}$$
$$=\frac{r!}{p!}\int\frac{d^2q}{(2\pi)^2}\tilde{U}(q)e^{-x}x^{p-r}[L_r^{p-r}(x)]^2 \quad (6)$$

represents the renormalization of the one-particle spectrum: the contribution to the eigenenergy of a particle in level $p$ due to exchange interactions with the filled level with index $r\leqslant p$ (see also Ref. 16). In the case $r=p$, $F_{pp}$ is the binding energy of a $2D$ magnetoexciton of zero momentum in Landau level $p$ (Ref. 12). The quantity $\tilde{U}(q)$ in (6) is the Fourier transform of the potential $U(r)$; $x\equiv q^2r_H^2/2$; and $L_n^m(x)$ are the Laguerre polynomials.

For spin-flip transitions, $\sigma\neq\sigma'$, the Hamiltonian (5) exhausts the contribution to the energy of excitations (4) in first order in $E_0$. The same is true of the $0\downarrow\rightarrow 1\downarrow$ transition without spin flip if $\nu_\uparrow=0$. In other cases we need to take account of the additional contribution of interparticle interactions, $\sim E_0$, due to "mixing" of excitations (see also Ref. 9):

$$B^+_{n+1\downarrow,n\downarrow M}|\text{vac}\rangle,\quad B^+_{l+1\uparrow,l\uparrow M}|\text{vac}\rangle. \quad (7)$$

Binary interactions of the form

$$\delta H_{nl}=\sum_{\substack{m_1,m_2\\ m_1',m_2'}}U^{n+1m_1',lm_2'}_{nm_1,l+1m_2}a^+_{n+1\downarrow m_1'}b^+_{n\downarrow m_1}b_{l\uparrow m_2'}a_{l+1\uparrow m_2}$$
$$+\text{H.a.} \quad (8)$$

make a contribution in this case. The result of the mixing is particularly simple in the case $\nu_\downarrow=\nu_\uparrow$, as in the situation without an impurity.[9] In this case, excitations

$$\frac{1}{\sqrt{2}}(B^+_{n+1\downarrow,n\downarrow M}\pm B^+_{n+1\uparrow,n\uparrow M})|\text{vac}\rangle \quad (9)$$

are correct combinations. They correspond to an singlet exciton with $S=0$ and to the spin state of a triplet exciton with $S=1$ and $S_z=0$. Corresponding to the two other spin states of the triplet exciton with $S_z=\pm 1$ are the excitations

$$B^+_{n+1\uparrow,n\downarrow M}|\text{vac}\rangle \quad \text{and} \quad B^+_{n+1\downarrow,n\uparrow M}|\text{vac}\rangle.$$

For triplet excitations, there is no binary exchange interaction, while the interaction for a singlet excitation is doubled [the Kronecker delta $\delta_{\sigma,\sigma'}$ in curly brackets in (5) is replaced by 0 and 2, respectively].

### 2.3. Magnetoplasmons and spin excitations with $M=0$: contribution to the response to a uniform alternating field

It follows from the selection rules that magnetoplasmons with $M=0$ are responsible for the CR {these are singlet magnetoplasmons in the case $\nu_\downarrow=\nu_\uparrow$ [see (9) and Eq. (12) below with $n=l$]}. It also follows from the selection rules that excitations $B^+_{n+1\uparrow,n\downarrow M=0}|\text{vac}\rangle$ (which exist if $\nu_\downarrow\neq\nu_\uparrow$) are responsible for spin resonance. Looking ahead to an application below, we will pursue this point, making use of the explicit expression for the Hamiltonian for interactions with a uniform alternating field of an electromagnetic wave of frequency $\omega$ which is propagating parallel to the magnetic field (the $z$ axis) and which has a left-hand circular polarization $\sigma^+$ (this is the wave which causes the transitions corresponding to the resonances):

$$\delta\hat{V}(t)=\frac{e}{im^*\omega}\mathscr{E}_0e^{-i\omega t}\hat{\pi}_+-\tfrac{1}{2}g^*\mu_B\mathscr{E}_0e^{-i\omega t}\hat{\sigma}_+, \quad (10)$$

$$\hat{\pi}_+=\sum_j(\pi_{jx}+i\pi_{jy}),\quad \hat{\sigma}_+=\sum_j(\sigma_{jx}+i\sigma_{jy}). \quad (11)$$

Here the operator

$$\hat{\pi}_j=-i\nabla_j+\frac{e}{c}\mathbf{A}_j$$

represents the kinematic momentum of particle $j$, and $\sigma_j$ are the Pauli matrices. For $\hat{\pi}_+$ and $\hat{\sigma}_+$ we have

$$\hat{\pi}_+|\text{vac}\rangle=\frac{1}{\sqrt{2}r_H}\sum_m\{\sqrt{n+1}\,a^+_{n+1\downarrow m}b^+_{n\downarrow m}$$
$$+\sqrt{l+1}a^+_{l+1\uparrow m}b^+_{l\uparrow m}\}|\text{vac}\rangle \quad (12)$$

and, for $\nu_\downarrow\neq\nu_\uparrow$,

$$\hat{\sigma}_+|\text{vac}\rangle=\sum_m a^+_{n\uparrow m}b^+_{n\downarrow m}|\text{vac}\rangle. \quad (13)$$

To within a constant factor, the states in (12) and (13) describe $2D$ magnetoexcitons of zero momentum,[11,12,15]

$$Q^+_{n_1n_2\mathbf{k}=0}=\frac{1}{\sqrt{N_0}}\sum_m a^+_{n_1m}b^+_{n_2m}, \quad (14)$$

where $N_0=\mathscr{S}/2\pi r_H^2$, and $\mathscr{S}$ is the area of the system.

## 3. ENERGIES OF LOCALIZED MODES

In this section of the paper we present results calculated for the binding energies of excitations localized at a donor $D^+$ and an acceptor $A^-$ for two cases: 1) filling of a spin-polarized lower Landau level $\nu_\downarrow=1$, $\nu_\uparrow=0$ ($\nu=1$); 2) filling of two levels with different spin directions, $\nu_\downarrow=\nu_\uparrow=1$ ($\nu=2$). These results were found by numerical diagonalization of the $N\times N$ matrices of Hamiltonian (5)

($N=50$–100). For the energies of localized levels of spin excitations (if there is no exchange interaction), we thereby obtain an accuracy $\sim 10^{-6}E_0$ or $\sim 10^{-3}E_0$ for magnetoplasma excitations. Delocalized states of the continuum are of course determined only to within a much larger error. For example, for a $0\downarrow \to 0\uparrow$ spin excitation the errors in the determination of the lower and upper boundaries of the continuum at $N=100$ are 0.7% and 6%, respectively[15] (see also Figs. 2 and 3).

### 3.1. Filled lower Landau level: $\nu_\downarrow = 1$, $\nu_\uparrow = 0$

*3.1.1. Spin excitations: the $0\downarrow \to 0\uparrow$ transition.* The Hamiltonian $H_{0\downarrow\, 0\uparrow}$ in (5), which describes spin-flip excitations, differs from the problem of bound states of a $2D$ magnetoexciton in that there is a shift of the origin on the energy scale. When this point is taken into account, the results of Ref. 13 can be extended to the case of spin excitations.

For definiteness we consider a positively charged donor $D^+$. By virtue of the $t \to -t$ symmetry, the results for a negatively charged center $A^-$ are found by making the replacement $M \to -M$. That this is true can also be seen from the explicit expression for Hamiltonian (5) with $k\sigma = 0\uparrow$ and $p\sigma' = 0\downarrow$.

For each $M>0$ from the continuum (whose lower and upper boundaries are $\hbar\omega_s$ and $\hbar\omega_s+E_0$, respectively,[8]), a localized discrete level is split off downward. The ground state is a localized $M=1$ mode with an energy $\hbar\omega_s - 0.1189E_0$. As $M$ increases, the position of the discrete level descends monotonically toward $\hbar\omega_s$. Furthermore, for $M \geqslant 5$, it is possible to distinguish in the spectrum some impurity resonant levels, which lie in the continuum near one-particle excited levels of the donor, $V_{0,m}$, $m=1,2,,...$ (see the Appendix). The number of such levels increases with increasing $M$ (when the *e-h* interaction is weakened), and their width decreases. For $M<0$ there are localized discrete levels in the spectrum above the upper boundary of the continuum, and there are also some resonant localized levels. The position of the upper discrete level tends toward $\hbar\omega_s+2E_0$ with increasing $|M|$.

For a state with $M=0$ which is associated with a spin resonance, there is no interaction with the spin-independent and otherwise arbitrary external potential. The frequency of the spin resonance is thus independent of whether the system contains nonmagnetic impurities (and interparticle interactions), as was to be expected.

*3.1.2. Magnetoplasma excitations: the $0\downarrow \to 1\downarrow$ transition.* Setting $k\sigma = 1\downarrow$ and $p\sigma' = 0\downarrow$, considering the first $N$ terms in expansion (4), and using the Hamiltonian $H_{1\downarrow\, 0\downarrow}$ from (5), we find, for $M \geqslant 0$, for example, a secular equation of degree $N$:

$$\mathrm{Det}_{(N)}(\langle 0\downarrow\ m'+M;1\downarrow\ m'|H_{1\downarrow\, 0\downarrow}|1\downarrow\ m;0\downarrow\ m+M\rangle - \delta_{m,m'}\varepsilon(M)) = 0, \qquad (15)$$

$$|n\sigma m;k\sigma' m'\rangle \equiv a^+_{n\sigma m} b^+_{k\sigma' m'}|\mathrm{vac}\rangle,$$

where $\varepsilon(M)$ is the energy of the excitation relative to $\hbar\omega_c$. For an interaction with a positively charged donor $D^+$, the matrix elements of the Hamiltonian are (we are using $F_{01}=E_0/2$, and we are also assuming $m' \geqslant m$)

$$\langle 0\downarrow\ m'+M;1\downarrow\ m'|H_{1\downarrow\, 0\downarrow}|1\downarrow\ m;0\downarrow\ m+M\rangle = \delta_{m,m'}\{(-V_{1,m}+V_{0,m+M})+\tfrac{1}{2}E_0\} - U^{(01)}_{m,m+M}(m'-m)+\tilde{U}^{(01)}_{m,m'}(M). \qquad (16)$$

These matrix elements are calculated with the help of (A3) and the explicit expressions[21] for the matrix elements of the direct $[U^{(01)}_{k,l}(s)]$ and exchange $\tilde{U}^{(01)}_{k,l}(s)$ interactions in the zeroth and first Landau levels [see (A8) and (A9)]. The contribution of the direct interparticle interaction to the off-diagonal elements $U^{(01)}_{m,m+M}(|m'-m|)$ falls off exponentially with increasing $|m'-m|$, while the contribution of the exchange interaction, $\tilde{U}^{(01)}_{m,m'}(|M|)$, falls off algebraically. The energies of the localized levels of a magnetoplasmon at an ionized donor [which we denote by $(D^+, MP)$] are shown in Fig. 1a. Let us explain the results.

We first note that energies 0–$0.5E_0$ correspond to the continuous spectrum.[8] Since we have

$$\langle r_h^2 - r_e^2\rangle = 2(M-1)r_H^2,$$

the states $(D^+, MP)$ with $M<0$ are excited states. Some of the $M<0$ levels are resonant levels with a finite linewidth; they lie in the continuum (near its upper boundary). The $M<0$ levels lying above the continuum are excited levels of the discrete spectrum. For $|M| \gg 1$ they correspond to a hole bound to a $D^+$ and a rather remote electron. The energy corresponding to the highest-lying levels is

$$E_0 - F_{01} + V_{0,0} = \tfrac{3}{2}E_0.$$

States with $M>0$ contain local levels in the continuum. There are also some discrete levels below the continuum boundary. For $M \geqslant 3$, there is more than one such level (for $5 \leqslant M \leqslant 8$ there are three discrete levels). The lowest states correspond to an electron which is bound to a center with quantum numbers $n=1$ and $m=1$. For $M \gg 1$ when the mixing of different $m$-channels is only a minor effect, the energy of these states is given by the diagonal matrix element

$$-V_{1,1}+V_{0,M+1}+\tfrac{1}{2}E_0 - U^{(01)}_{1,M+1}(0)+\tilde{U}^{(01)}_{1,0}(M+1). \qquad (17)$$

As $M \to \infty$ it tends toward the value $-0.25E_0$ in accordance with a power law in $M$.

A second discrete level of $(D^+, MP)$ lies near the lower boundary of the continuum. At small values of $M$ (at which the contribution of the exchange interactions is large and positive), this level falls in the continuum and is a resonant level. It corresponds to an electron with $n=1$ and $m=0$. For $M \gg 1$ the energy of the level is determined by a diagonal element, which is given by (here we are using $E_0 - F_{01} - V_{1,0} = 0$)

$$V_{0,M} - U^{(01)}_{1,M}(0) + \tilde{U}^{(01)}_{0,0}(M). \qquad (18)$$

In the limit $M \to \infty$ it vanishes algebraically.

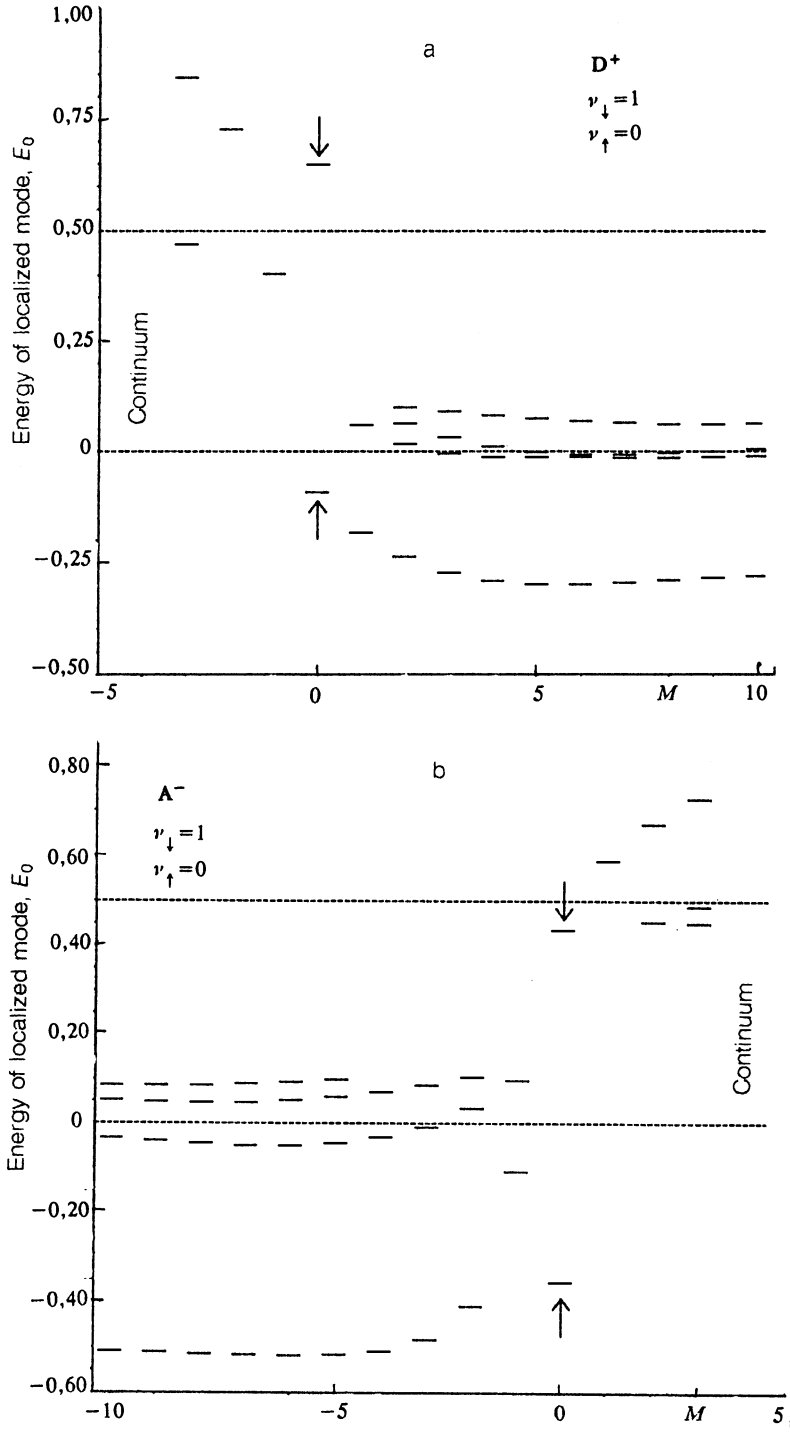


FIG. 1. Energies (in units of $E_0$, measured with respect to $\hbar\omega_c$) of localized states of magnetoplasmons with an oscillator quantum number $M$ for $\nu_\downarrow=1$ and $\nu_\uparrow=0$. For modes which are bound at an ionized donor ($D^+$, MP), Fig. 1a shows states with $-3\leqslant M\leqslant 10$; for modes bound at an ionized acceptor ($A^-$, MP), Fig. 1b shows states with $-10\leqslant M\leqslant 3$. The energies were found through a numerical diagonalization of the $50\times 50$ matrices of the Hamiltonian (5) with $k\sigma=1\downarrow$, $p\sigma'=0\downarrow$ (for $M=0$, these would be $100\times 100$ matrices). The levels in the energy interval $0$–$0.5E_0$ lie in the continuum and are resonant levels. The arrows mark IR-active modes with $M=0$.

For the states with $M=0$, which are responsible for the CR, there are two localized ($D^+$, MP) levels. One lies under the lower boundary of the continuum and corresponds to the ground state. The other lies above the upper boundary of the continuum and corresponds to an excited state. The reason for this behavior is that the difference $\delta V_m^{(10)}\equiv V_{0,m}-V_{1,m}$, which determines the energy of the interactions of a hole and an electron with a center, is large and positive in the $m=0$ case ($\delta V_0^{(10)}=0.5E_0$), while it is large and negative in the $m=1$ case ($\delta V_1^{(10)}=-0.25E_0$). These results are governed by the structure of the wave functions for Landau levels other than the ground level (see also Ref. 7). For the other $m$-channels, the values of $\delta V_m^{(10)}$ are comparatively small and negative [see (A3) and (A4), and they do not cause splitting off of localized levels.

Figure 1b shows results for the energies of magnetoplasmons localized at a negatively charged acceptor ($A^-$, MP). The low-lying states have quantum numbers $M\leqslant 0$. A hole with $n=m=0$ bound to the center corresponds to the lowest discrete level. For $M<0$ and $|M|\gg 1$, the positions of this level are determined primarily by the diagonal matrix element

$$-V_{0,0}+V_{1,|M|}+\tfrac{1}{2}E_0-U^{(01)}_{0,|M|}(0)+\tilde{U}^{(01)}_{0,0}$$
$$\times(|M|)\xrightarrow[M\to-\infty]{}-0.5E_0. \qquad (19)$$

The second low-lying discrete level of ($A^-$, MP) differs only slight from that in the ($D^+$, MP) case [because $V_{0,1}=V_{1,0}$; see (A3)–(A5)].

The positions of the two localized ($A^-$, MP) IR-active modes with $M=0$ are different from those in the ($D^+$, MP) case: The splitting off of the lower discrete level, which is determined primarily by $-\delta V_0^{(10)}$, is considerably larger ($\simeq -0.35E_0$), while the upper level falls in the continuum and becomes a resonant level.

### 3.2. Two filled Landau levels: $\nu_\downarrow=\nu_\uparrow=1$

When mixing of excitations, $0\downarrow\to 1\downarrow$ and $0\uparrow\to 1\uparrow$, is taken into account, we obtain singlet magnetoplasmons with $S=0$ and triplet spin excitations with $S=1$, $S_z=0$. Corresponding to two other spin triplet states with $S_z=\pm 1$ are the transitions $0\downarrow\to 1\uparrow$ and $0\uparrow\to 1\downarrow$. Only a single excitation with $M=0$ is electric-dipole-active (Subsection 2.3). Triplet excitations with $M=0,\pm 1$ are (exceedingly weakly) magnetic-dipole-active in the Voigt geometry, i.e., in the case in which the radiation is propagating across the magnetic field in the plane of the $2D$ system. In practice, however, triplet excitations with $M=0$ can be observed only when spin-orbit effects are taken into account. These effects cause these excitations to become weakly electric-dipole-allowed (the small factor involved here is $\sim 10^{-6}$).

We will not reproduce here the calculated energies of the localized states for this case.[22] We will simply point out some new features (not seen in the case of a filling $\nu=1$):

1) Since the positive contribution of binary exchange interactions is doubled for singlet magnetoplasmons, the local levels in both the ($A^-$, MP) and ($D^+$, MP) cases shift up the energy scale.

2) There is no positive contribution from binary exchange interactions for the triplet states. As a result, the lower edge of the continuum shifts downward and corresponds to an energy $\simeq -0.11E_0$ (a roton minimum in the $\mathbf{k}\neq 0$ case). The upper edge of the spectrum is again at $0.5E_0$ (Ref. 9). This point must be taken into account in classifying the low-lying localized levels as discrete or quasidiscrete.

### 3.3. Spectrum of an impurity cyclotron resonance

Let us examine the contribution of localized magnetoplasmons to the response of the system at the frequency $\omega$. For the absorption coefficient for left-hand-polarized radiation, $\alpha_+(\omega)$, we have [see (10) and (12)]

$$\alpha_+(\omega)\propto\sum_k|\langle M=0,k|\hat{\pi}_+|\text{vac}\rangle|^2\delta(\omega-\omega_k). \qquad (20)$$

The quantity $|M=0k\rangle$ in (20) is the set of magnetoplasma states with $M=0$ (in the case $\nu_\downarrow=\nu_\uparrow$, these are singlet states), which includes both discrete levels and the continuum. In our case we have $k=1,\ldots,N$: the states which are found through a diagonalization of the $N\times N$ matrix of the Hamiltonian. The contribution of a state to the response is thus determined by the dimensionless quantity

$$f_k\equiv 2r_H^2|\langle M=0,k|\hat{\pi}_+|\text{vac}\rangle|^2=\Big|\sum_m^N A^{(k)}_{M=0}(m)\Big|^2, \qquad (21)$$

i.e., (to within normalization) by the projection onto the state of a magnetoexciton with $\mathbf{k}=0$ [see (14)]. Here $A^{(k)}_{M=0}$ are the expansion coefficients in (4). For singlet excitations, the right side of (21) contains an additional factor of 1/2, which incorporates the filling of the two Landau levels $\downarrow$ and $\uparrow$.

Figure 2 shows the energy distribution of the values of $f_k$ (normalized to the number of states considered, $N$) for $\nu=1$; Fig. 3 shows the corresponding results for $\nu=2$. The response is dominated by the fundamental delocalized mode corresponding to the lower boundary of the continuum (with a momentum $\mathbf{k}=0$ in the absence of an impurity). For $N\gg 1$ we have $f_{\mathbf{k}=0}/N\approx 1$ for it, and its energy tends toward $\hbar\omega_c$ from above (Figs. 2 and 3 with $E=0$). The other delocalized modes fill the energy interval $0$–$0.5E_0$ densely in the limit $N\to\infty$, and for them we have $f_k\to 0$. The calculation for finite values of $N$ corresponds qualitatively to a system with finite dimensions $L\approx\sqrt{2N}r_H$.

Modes localized at an impurity, ($D^+$, MP) and ($A^-$, MP), have finite values of $f_k$. These values can be calculated accurately even at small values of $N$. In a system with a finite impurity density $n_{\text{imp}}$ ($\text{cm}^{-2}$), lines split off from $\hbar\omega_c$, with intensities $\propto f_k n_{\text{imp}}$, with a width determined by the scatter in the constants of the interaction with impurities, correspond to these impurities in the CR spectrum. As the magnetic field is strengthened, the splitting off of the impurity lines from the mode with energy $\hbar\omega_c$ increases in proportion to $H^{1/2}$. The positions of these lines depend on the sign of the impurity charge [in the ($A^-$, MP) case, they are shifted down the energy scale] and on the level filling factors $\nu$ (as we go from $\nu=1$ to $\nu=2$, the levels shift up the energy scale). The transition matrix elements $f_k$ for ($D^+$, MP) are larger than those for ($A^-$, MP), because the oscillator strength for the main transition decreases with the energy near $\hbar\omega_c$ (Figs. 2 and 3).

In a system with finite $n_{\text{imp}}$ the main absorption line also shifts and broadens. In this connection we note that for delocalized states the approximation of isolated impurities is not strictly valid, and the presence of other impurities must be taken into account in secular equation (15), (16) at

$$N\sim(2\pi\,\mathbf{r}_H^2 n_{\text{imp}})^{-1}\equiv\nu_{\text{imp}}^{-1}.$$

The effect of the random field of many impurities may cause (among other things) Anderson localization of collective excitations. However, for excitations bound at one impurity, the length scale is on the order of a few times $r_H$, and the approximation of an isolated impurity is valid under the condition $\nu_{\text{imp}}\ll 1$.

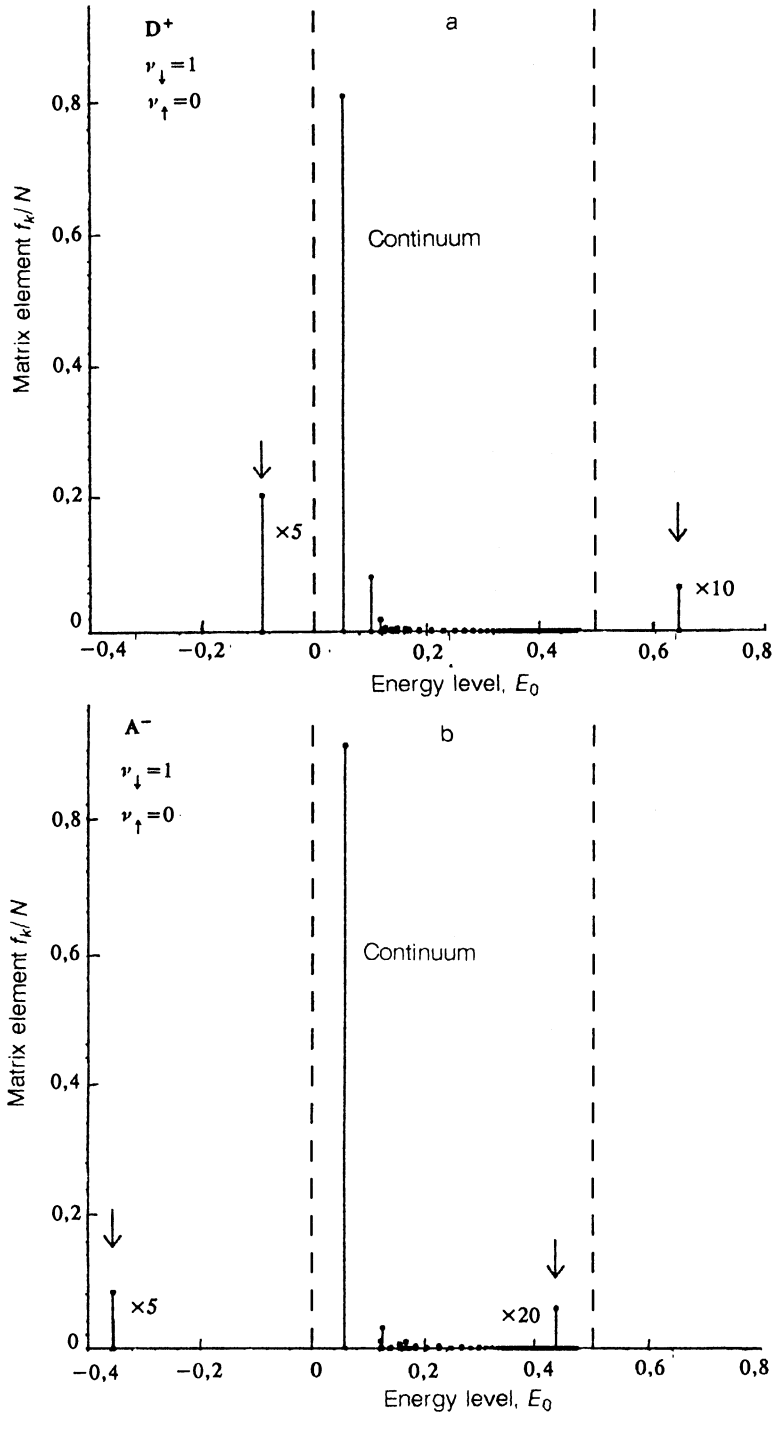


FIG. 2. Distribution with respect to the energy (measured with respect to $\hbar\omega_c$) of the transition matrix elements $f_k/N$, $f_k=|\langle M=0k|\hat{\pi}_+|\text{vac}\rangle|^2$ [see (21)], normalized to the number ($N$) of states in the basis taken into account (for $N=100$). Figure 2a shows results for magnetoplasma IR-active modes with $M=0$ in the presence of an ionized donor $D^+$; Fig. 2b shows corresponding results for an ionized acceptor $A^-$. The level filling factors are $\nu_\downarrow=1$ and $\nu_\uparrow=0$ ($\nu=1$). The arrows show ($D^+$, MP) and ($A^-$, MP) modes which are localized at an impurity. The exact boundaries of the continuum are shown by the vertical dashed lines.

## 4. CONCLUSION

For two cases of filled zeroth Landau levels, $\nu=1$ and $\nu=2$, we have derived the energies of collective magnetoplasma and spin excitations bound to a positively or negatively charged Coulomb center. In addition to the split-off discrete levels (low-lying and excited), which lie outside the continuum, we have observed a large number of resonances which fall in the continuum of delocalized excitations (near one-particle donor or acceptor levels). The wave functions of the resonances are not normalizable; for broad resonances, they contain oscillating tails of substantial amplitude. For narrow resonances, however, the behavior of the corresponding wave functions near an impurity center and at distances not too far from the center is essentially indistinguishable from the behavior of the wave functions of discrete states[15] (as for ordinary quasistationary levels[23]).

For IR-active magnetoplasmons with oscillator quantum number $M=0$ (with an angular-momentum projection $M_z=+1$), having filling factors $\nu=1$ and $\nu=2$, there are two localized levels, which differ greatly in energy. In general, the magnetoabsorption spectrum must consist of a main peak near $\hbar\omega_c$, corresponding to a delocalized cyclotron mode, and to impurity peaks, to the right and left of [3)]

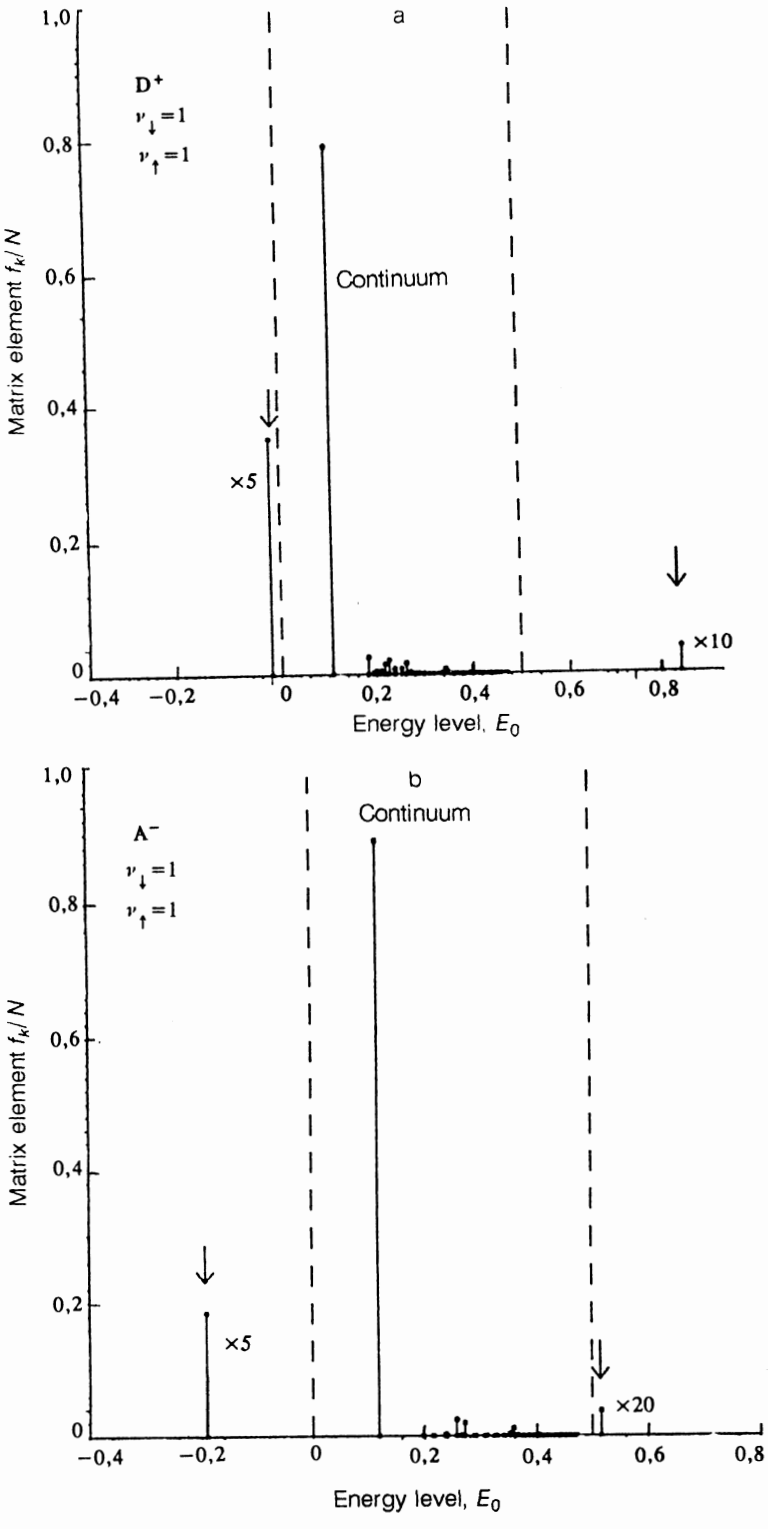


FIG. 3. The same as in Fig. 2, but for singlet magnetoplasma IR-active modes with $M=0$ for $\nu_\downarrow=1$ and $\nu_\uparrow=1$ ($\nu=2$).

$\hbar\omega_c$. A magnetoabsorption spectrum with this structure has been observed[4] for the CR of quasi-2$D$ electrons in silicon MOS structures (in which case there are charged impurities of both signs) in strong fields for filling factors $\nu=1-2$. In quasi-2$D$ GaAs heterostructures selectively doped with donors, with $\nu=1$–2, a low-energy peak has been observed[3] (as a knee on the CR), in addition to the main CR peak. That low-energy peak is, we believe, due to the excitation of a low-lying ($D^+$, MP) local magnetoplasma mode. The absence of an additional line to the right of $\hbar\omega_c$ in this case [the upper ($D^+$, MP) mode] may mean that the corresponding dipole transition matrix element is small (Figs. 2a and 3a), and the line is not resolved in the spectrum. In addition, the impurities in these structures lie at a finite distance (~25–300 Å in Ref. 3) from the layer of 2$D$ electrons. As a result, the impurity potential is weak in comparison with that in the case discussed here, in which the Coulomb center is directly in the layer of 2$D$

electrons.[4)] For GaAs heterostructures which are selectively doped with acceptors, one observes, in addition to the main CR peak, an additional peak to the right of $\hbar\omega_c$. The latter peak is associated with the excitation of an upper ($A^-$, MP) local mode. The absence of a peak on the left of $\hbar\omega_c$ in this case [corresponding to the lower-lying ($A^-$, MP) local mode] probably means that (in the one-particle picture) this excitation corresponds primarily to the $1s \to 2p^+$ transition from a $1s$ initial acceptor state which lies high above the Landau level and may not be filled.[7] In the experiments of Refs. 3 and 4, there was a substantial redistribution of oscillator strength between the magnetoabsorption peaks as the field $H$ increases further under the condition $\nu < 1$ (under conditions to which the theory of this paper does not apply). For GaAs heterostructures doped with acceptors, a single peak with a fairly narrow line remains in the spectrum.[3] These effects have yet to be explained.

There are the interesting questions of how many local impurity IR-active modes with $M=0$ there might be and what their energies might be when a larger number of Landau levels are filled. For $\nu=3$ or $\nu=4$, in which case a charged Coulomb impurity is directly in the layer of $2D$ electrons (or fairly close to it), we can expect the appearance of three localized magnetoplasma states with $M=0$. For a $D^+$ donor (or an $A^-$ acceptor), two (one) of these states lie at energies above $\hbar\omega_c$, while one (two) lies at an energy lower than $\hbar\omega_c$. This conclusion follows from the energies of the electron- and hole-impurity interactions in different $m$-channels, $\delta V_m^{(21)} \equiv V_{2,m} - V_{1,m}$, at Landau levels with indices $n=2$ and $n=1$. In the first three channels, with $m=0$, 1, and 2, the values of $\delta V_m^{(21)}$ are comparatively large and have different signs for $m=0$, 1 and $m=2$ (see the Appendix). This situation may cause splitting off of three local modes. The formation of a larger number of IR-active localized magnetoplasma modes (their broadening is also being taken into account) agrees qualitatively with the experimentally observed[3-5] broadening of the CR spectrum with increasing $\nu$.

Finally, let us examine the possibility that bound states of two collective excitations would form at a charged impurity. When the origin for the energy scale is shifted, the problem of two spin excitations, $0\downarrow \to 0\uparrow$ (a bispinon), is exactly equivalent to the problem of the impurity-bound biexciton in zeroth Landau levels.[13] The results of Ref. 13 show that in the case of identical particles with parallel spins (electrons in the case of a $D^+$ donor; holes in the case of an $A^-$ acceptor), no stable bound states arise. For excitations corresponding to transitions between the zeroth and first Landau levels, on the other hand, bound states of bimagnetoplasmons arise, apparently for both $A^-$ and $D^+$. For the $A^-$, the levels should be relatively low. The existence of such impurity-bound states is in agreement with the formation of bound states of two free $2D$ magnetoplasmons.[27]

After this paper had been submitted for publication,[5)] the paper of Cheng *et al.*[28] appeared, reporting an experimental study of the spectra of impurity transitions associated with $D^-$ centers (see also the previous footnote[3)]). That study was carried out in strong magnetic fields, $H<9$ T, on GaAs/GaAlAs quantum wells with an adjustable density of excess free electrons, $n_{ex}=2\cdot10^{10}-2.8\cdot10^{11}$ cm$^{-2}$. A significant effect of the free electrons on the spectra of impurity transitions was observed[28] only for $\nu>0.3$. The length scale of the wave functions of $D^-$ centers becomes comparable to the average distance between free electrons. In this situation, the spectra of magnetooptic impurity transitions can be described successfully in terms of collective magnetoplasma excitations (not in terms of transitions of two-electron $D^-$ centers). The results of the present study (see also Ref. 29) yield a qualitative (and, to some extent, semiquantitative) understanding of the basic spectral features observed in Ref. 28:

1) For $n_{ex}>8\cdot10^{10}$ cm$^{-2}$ ($\nu>0.37$ for $H=9$ T), a broad line (a CR knee at energies below $\hbar\omega_c$) appears in the magnetoabsorption spectra near the CR line. The intensity of this knee increases with a further increase in $n_{ex}$. We attribute this spectral feature to a low-lying localized magnetoplasma mode ($D^+$, MP) (Figs. 2a and 3a). In the limit $\nu \to 0$, this mode becomes a triplet T-transition of a $D^-$ center (see Figs. 2 and 4 of Ref. 29) and is associated in the one-particle picture with the $2p^- \to 2s$ transition of a neutral donor.

2) It was found in Ref. 28 that the line which arises from the $D^-$ singlet transition shifts toward higher energies with increasing $n_{ex}$ (at first glance, this shift would seem to contradict the picture of screened impurity centers). We attribute the positive energy shift of the impurity transition to an increase in the contribution of exchange effects with increasing $\nu$: a negative renormalization of the energies of the filled initial states and an increase in the contribution of binary exchange interactions [see (5) and (6) above]. The values which we found for the shifts of the upper ($\delta_\nu^+$) and lower ($\delta_\nu^-$) ($D^+$, MP), magnetoplasma modes (with respect to the energies of the impurity transitions from which they arise with increasing $\nu$), in units of $e^2/\varepsilon r_H$, are, $\delta_{\nu=1}^+=0.19$ and $\delta_{\nu=1}^-=0.20$, for $\nu=1$ respectively. For $\nu=2$ the increase in the contribution from exchange effects makes the shifts larger, $\delta_{\nu=1}^+=0.42$ and $\delta_{\nu=1}^-=0.29$. The values for $\delta_\nu^+$ agree satisfactorily with the experimental shift of 13 cm$^{-1}$ at $\nu\simeq1.3$ ($n_{ex}=2.8\cdot10^{11}$ cm$^{-2}$ in a field $H=9$ T), which is $\delta_{\nu=1,3}^+\simeq0.10$. The agreement between theory and experiment[28] can be improved (at least partially) by incorporating quasi-$2D$ effects, which reduce the energy scale of Coulomb interactions.

3) The picture of impurity-bound collective magnetoplasma modes also agrees with the discontinuities observed[28] in the slopes of the plots of impurity transition energy $\partial E/\partial\nu$ (as a function of $\nu$) at integer values of $\nu$. To explain this effect, we consider, for example, the situation in which only the lower spin-polarized Landau level $0\downarrow$ is filled ($\nu_\downarrow=1$, $\nu_\uparrow=0$). As the $0\uparrow$ level begins to be filled, a new branch of excitations appears. This new branch is associated with impurity transitions of electrons with spin $\uparrow$ when the index of the Landau level changes by unity. When interelectron Coulomb interactions are taken into account, this new branch of excitations mixes with excita-

tions of spin ↓ electrons, leading to the formation of two branches of collective excitations (impurity-bound magnetoplasmons). As $\nu_\uparrow$ increases, the oscillator strength is redistributed between the two branches. When two Landau levels are filled, $\nu_\downarrow = \nu_\uparrow = 1 (\nu = 2)$, singlet and triplet excitations arise. Of these, only the singlets are IR-active (see the discussion above). Similarly, as the 1↓ Landau level begins to be filled the spectrum acquires a new IR-active branch of excitations. As the filling of this level increases, the oscillator strength is redistributed between the two branches of excitations. According to this picture, at integer values of $\nu$ there should be discontinuities not only in the derivatives $\partial E/\partial \nu$ but also in the transition energies $E(\nu)$ themselves. These features would be smoothed over in real systems, however, because of thermal activation of electrons to the nearest Landau levels with the opposite spin orientation and also because of the energy overlap of these Landau levels due to their broadening caused by disorder.

## APPENDIX

The interaction matrix elements from (5), calculated between the factorized electron wave functions $\phi_{nm}(\mathbf{r})$, are defined in the standard way:

$$V_{nm}^{n'm'} \equiv \langle n'm' | V(r) | nm \rangle \tag{A1}$$

$$U_{n_1m_1,n_2m_2}^{n_1'm_1',n_2'm_2'} \equiv \langle n_2'm_2'; n_1'm_1' | U(|\mathbf{r}_1 - \mathbf{r}_2|) | n_1m_1; n_2m_2 \rangle \tag{A2}$$

(where $|n_1m_1;n_2m_2\rangle^+ \equiv \langle n_2m_2;n_1m_1|$). In these calculations we used explicit expressions for the interaction matrix elements at the zeroth and first Landau levels. For Coulomb interactions with a negatively charged center we have ($V_{nm}^{nm} \equiv V_{n,m} = V_{m,n}$)

$$V_{0,m} = \frac{(2m-1)!!}{2^m m!} E_0, \quad V_{1,m} = \frac{4m-1}{4m-2} V_{0,m}. \tag{A3}$$

For convenience, we reproduce here the first few values of $V_{0,m}$, $V_{1,m}$, and $V_{2,m}$ (in units of $E_0$):

$$V_{0,0} = 1, \quad V_{0,1} = 0.5, \quad V_{0,2} = 0.375, \quad V_{0,3} = 0.3125 \tag{A4}$$

$$V_{1,0} = 0.5, \quad V_{1,1} = 0.75, \quad V_{1,2} = 0.4375,$$

$$V_{1,3} = 0.34375 \tag{A5}$$

$$V_{2,0} = 0.375, \quad V_{2,1} = 0.4375,$$

$$V_{2,2} \simeq 0.64063, \quad V_{2,3} \simeq 0.39844. \tag{A6}$$

Note that $V_{1,m}$ and $V_{2,m}$ are not monotonic functions of $m$. The reason is the structure of the wave functions at a Landau level other than the ground level (see also Ref. 20).

The interaction matrix elements for electrons in the zeroth Landau levels and the matrix elements for the direct and exchange interactions in the zeroth and first levels [see (16)] are given by

$$U_{0m_1,0m_2}^{0m_1',0m_2'} = U_{\min(m_1,m_1'),\min(m_2,m_2')}^{(00)}(|m_1 - m_1'|), \tag{A7}$$

$$U_{1m_1,0m_2}^{1m_1',0m_2'} = U_{\min(m_1,m_1'),\min(m_2,m_2')}^{(01)}(|m_1 - m_1'|), \tag{A8}$$

$$U_{0m_1,1m_2}^{1m_2',0m_1'} = \tilde{U}_{\min(m_1,m_2'),\min(m_2,m_1')}^{(01)}(|m_1 - m_2'|), \tag{A9}$$

(where we are assuming $m_1 + m_2 = m_1' + m_2'$). Explicit expressions for (A7)–(A9) are given in Ref. 21 for a Coulomb potential $U(r)$. Those expressions are in the form of finite algebraic sums, which can be easily evaluated analytically at small values of $n$ and $m$, and which lead to accurate numerical calculations under the conditions $n,m \gg 1$. The matrix elements $U_{kl}(s)$ from (A7)–(A9) (for arbitrary upper indices), at a fixed value of $s$, fall off algebraically with increasing $k$ and $l$, while at fixed values of $k$ and $l$ they fall off exponentially with increasing $s$. For the matrix elements $U_{0M}^{(00)}(s)$ which are off-diagonal in the problem of the ground state of the 0↓ →0↑ excitation with an angular momentum $M \geqslant 0$, for example, we have exponential decay at $s \gg M$ ($s \gg 1$ at $M = 0$):

$$U_{0M}^{(00)}(s) = E_0 \frac{s^{(M-1)/2}}{\sqrt{2\pi M!}\, 2^{M+s}} \left[ 1 + \frac{2M(M+3)-1}{8s} + O\left(\frac{M^3}{s^2}\right) \right]. \tag{A10}$$

This decay is approximately a geometric progression with a denominator of 1/2.

[1)] The connection of the localized one-particle[30] collective excitations has been studied in Ref. 31.

[2)] In the case $V(\mathbf{r}) = 0$, and with the choice of a symmetric $\mathbf{A}(\mathbf{r})$ gauge, the states in the continuum can be classified on the basis of the projection $M_z$ of the total angular momentum and on the basis of a continuous quantum number, the magnitude of the magnetic momentum $k$ ($[\hat{\mathbf{k}}^2, \hat{M}_z] = 0$). In this case $k$ unambiguously determines the energy of a magnetoexciton; i.e., there is a degeneracy in terms of $M_z$ in a magnetic field, because of the electrical neutrality of the $e$-$h$ pair.

[3)] These results agree with the theoretical analysis[7] of the energies of one-electron $1s \to 2p^+$ impurity transitions (in our notation, $n=0, m=0 \to n=1, m=0$) and $2p^- \to 2s$ ($n=0, m=1 \to n=1, m=1$). Only the Hartree contribution of interelectron interactions was taken into account in the self-consistent approximation in Ref. 7, although exchange effects are important for a correct description of magnetoplasma excitations.

[4)] Another 2$D$ system for which both interelectron correlations and an impurity potential are important is a negative donor center $D^-$ (i.e., a neutral shallow donor $D^0$ which has captured an additional electron) in GaAs/GaAlAs quantum wells. The $D^-$ sign realizes the limit $\nu \to 0$, which is, in a sense, the opposite of the limit discussed in this paper. Interestingly, the structure of the IR absorption of $D^-$ in a strong magnetic field (impurity CR) consists of the following: 1) one strong line, shifted up the energy scale from $\hbar\omega_c$, for singlet $D^-$ (Ref. 24, for example), and 2) two (theoretically predicted[21,25,26]) strong lines for the $D^-$ triplet. One of the two lies near $\hbar\omega_c$, while the other is even further from $\hbar\omega_c$ than in the case of the $D^-$ singlet (it lies near the strong $D^0$ transition,[26] so it is difficult to resolve).

[5)] The following comments were added on 20 May 1993.


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Translated by D. Parsons