Behavior of relativistic particles in the field of a deep potential well

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Stationary and quasistationary solutions of spinor and scalar wave equations with a deep potential well capable of producing pairs are considered. The physical meaning of these solutions in different regions of space-time is traced, as is the manner in which these solutions reflect the nonsingle-particle nature and the spin-statistics connection. A simple method is described for constructing quasiclassical spinor solutions and the corresponding wave-transmission amplitude $c_1(E)$ and wave-reflection amplitude $c_2(E)$ as functions of the energy. Exact solutions and amplitudes $c_{1,2}(E)$ for a deep triangular potential well are found. It is shown that for such a well the continuous spectrum of stationary-state energies contains resonance values of E for which reflection is absent, i.e., $c_2(E)=0$. When the field of the well is weaker than the characteristic quantum-electrodynamic field $m^2c^3/\hbar e$ the resonance spectrum differs by an exponentially small amount from the discrete spectrum of the energies E_0 of the quasistationary states, which is determined, together with the corresponding widths Γ of the levels, by the equation $c_1(E_0+i\Gamma/2)=0$.

1. INTRODUCTION

At the present time, because of the expansion of the experimental possibilities, interest has revived in processes of pair production by strong fields, the ground states of the pairs being characterized by a so-called charged vacuum. Such states arise when one of the particles of a pair created by the field turns out to be strongly bound by the field while its antiparticle goes away to infinity.[1–6] In the one-particle interpretation of the solutions of the relativistic wave equations the production of pairs looks like the transmission of a wave through a sufficiently high potential wall (the Klein paradox[7]). According to nonrelativistic theory, a particle cannot pass through the potential wall, and therefore its motion in the well is bounded and the energy spectrum is discrete. In relativistic theory, because of the nonzero coefficient of transmission of the wave through a high wall, the motion of a particle in a sufficiently deep well takes on the appearance of unbounded motion. In this case solutions in the form of waves incident on the well from infinity are possible, i.e., the class of admissible solutions is expanded and the energy spectrum becomes continuous.

To explain the Klein paradox one must reject the one-particle interpretation of the solutions of the relativistic wave equations and assume that in certain regions of space these solutions describe a particle while in others they describe an antiparticle. Moreover, in the interpretation of the solutions we must also take into account the spin-statistics connection, i.e., distinguish the free (vacant) and occupied states.[8] In particular, decrease or increase with time of the amplitude of the quasistationary Dirac-equation solutions that correspond to the creation or absorption of a pair by the field of the well implies that they describe vacant states inside the well. The Pauli principle does not extend to scalar particles, and, therefore, solutions of the Klein–Gordon equation describe occupied states in any region of space.

One of the aims of this paper is to use the examples of a quasiclassical solution of the relativistic problem for a deep well and an exact solution for a well of special form to illustrate the non-single-particle aspect of the solutions and the manifestation of the spin-statistics connection in them. Therefore, we do not agree with the interpretation of the Klein paradox given by the authors of the well known monograph[4] who write, on page 17, that "...incident electrons can eject an electron-positron pair from the surface of the wall...", and "...pair creation is stimulated by an incident electron beam ...". Unfortunately, the problem under consideration is not yet completely solved, since it is not yet clear how to describe an electron in a level immersed in the lower continuum. Some ideas on this topic are contained in a paper by Nikishov[5] (see also Refs. 1, 4, and 9).

In Sec. 2 we consider the behavior of spinor particles in a deep potential well. We find formulas (previously known only for the scalar equation) relating the stationary quasiclassical solutions to the left and right of the turning points. These formulas are combined into a 2×2 matrix of the coefficients c_1, c_2, c_1^*, c_2^* of the expansion of the solutions with a certain sign of the momentum to the left of the well in solutions with a certain sign of the momentum to the right of it. The coefficients c_1 and c_2 contain all the information on the scattering, creation, and annihilation of particles and antiparticles in the field of the well. In the quasiclassical approximation we obtain for the width of a quasistationary level a general expression that is negative or positive for states corresponding to the creation or annihilation of a pair. This implies that spinor quasistationary states inside the well describe vacant states. The same questions for a scalar particle are considered in Sec. 3. The widths of the levels in this case are found to be opposite in sign to the spinor widths, i.e., the solutions of the Klein-Gordon equation describe everywhere occupied states.

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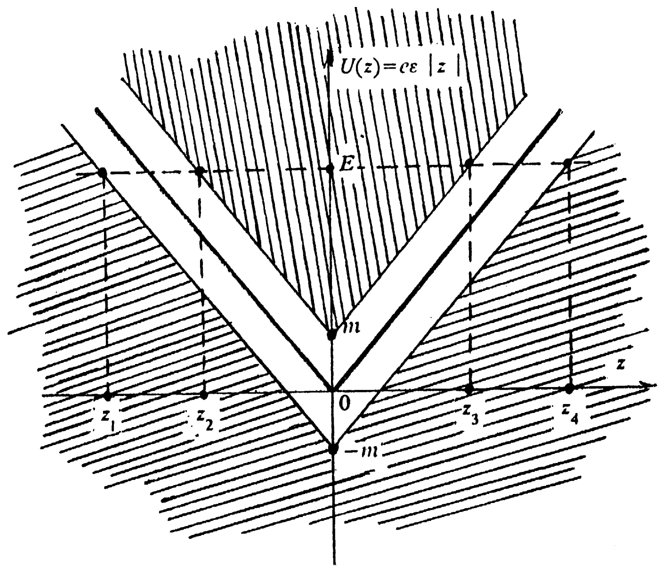

FIG. 1.

Section 4 is devoted to the exact solutions of the Dirac equation and Klein–Gordon equation for a potential well $U(z)=e\varepsilon|z|$, where $\pm\varepsilon$ is the intensity of the electric field along the z axis for $z \lessgtr 0$ (see Fig. 1). The solution of the wave equation has the form

$$\psi(\mathbf{x},t)=\text{const}\cdot\exp[i(p_1x+p_2y-Et)]\varphi(z). \tag{1}$$

In the nonrelativistic approximation there are symmetric and antisymmetric solutions:

$$\begin{aligned}\varphi^{(s)}(z)&=\mathrm{Ai}(|z|/a-k^{(s)}),\\ \varphi^{(a)}(z)&=\mathrm{sgn}\ z\cdot\mathrm{Ai}(|z|/a-k^{(a)}),\end{aligned} \tag{2}$$

$$E^{(s,a)}=p_\perp^2/2m+e\varepsilon ak^{(s,a)}, \quad p_\perp=\sqrt{p_1^2+p_2^2}. \tag{3}$$

Here, $a=(\hbar^2/2me\varepsilon)^{1/3}$ is the characteristic nonrelativistic length, and $\mathrm{Ai}(z)$ is the Airy function. The energy spectrum is determined from the equations

$$\mathrm{Ai}'(-k^{(s)})=0, \quad \mathrm{Ai}(-k^{(a)})=0. \tag{4}$$

The relativistic solutions form a stationary and a quasistationary system, satisfying different boundary conditions. Analytical expressions are found for the coefficients $c_{1,2}(E)$. The zeros of the complex function $c_1(E)$ determine the discrete spectrum $E=E_0+i\Gamma/2$ of the quasistationary states. It is shown that for an energy close to E_0 the real function $c_2(E)=0$, i.e., in the stationary solution the reflected wave vanishes and the well occupied by a particle becomes transparent to an incident antiparticle as a consequence of the coherent process of annihilation and creation of a pair.

In conclusion we note that we share the opinion, expressed in a number of papers of Nikishov,[5,6,10] that the solutions of the wave equations contain all information on the behavior of many-particle systems possessing zero or unit charge, if we disregard radiative corrections.

2. PHYSICAL MEANING OF SOLUTIONS OF THE RELATIVISTIC WAVE EQUATIONS (THE SPINOR CASE)

In this section we shall consider solutions of the Dirac equation

$$(i\hat{\Pi}+m)\psi=0, \quad \hat{\Pi}=\gamma_\mu\Pi^\mu, \quad \Pi_\mu=-i\partial_\mu-eA_\mu \tag{5}$$

with a potential $A_\mu=(\mathbf{A},A_0)$, where $\mathbf{A}=0$ and the function $U(z)=eA^0(z)$ is, as yet, arbitrary. The operators $-i\partial_0$, $-i\partial_1$, and $-i\partial_2$ commute with the operator $i\hat{\Pi}$, and, therefore, the solution can be chosen in the form (1), where $\varphi(z)$ will now be a bispinor. This solution is an eigenfunction of the polarization operator $i\gamma_5\hat{s}$, if the spacelike vector s_μ is orthogonal to the plane of the motion, i.e., to the electric field and the vector $p_{\perp\,\mu}=(p_1,p_2,0,0)$:

$$s_\mu=(p_2,-p_1,0,0)p_\perp^{-1}, \tag{6}$$

since in this case the commutator $[i\hat{\Pi},i\gamma_5\hat{s}]=2\gamma_5 s_\mu\Pi^\mu$ on the solutions (1) possesses zero eigenvalues. Following Ref. 11, we introduce four linearly independent constant bispinors $u_{\lambda\pm}$ that are eigenvectors of the commuting operators $i\gamma_5\hat{s}$ and $\gamma_0\gamma_3$:

$$i\gamma_5\hat{s}u_{\lambda\pm}=\lambda u_{\lambda\pm}, \quad \gamma_0\gamma_3u_{\lambda\pm}=\pm u_{\lambda\pm}. \tag{7}$$

It is obvious that $\lambda=\pm1$ is twice the value of the projection of the spin on the direction of s. We fix the relative phase of $u_{\lambda+}$ and $u_{\lambda-}$ by the relation

$$u_{\lambda\mp}=\gamma_3\frac{i\hat{p}_\perp+m}{m_\perp}u_{\lambda\pm}, \quad m_\perp=\sqrt{m^2+p_\perp^2}. \tag{8}$$

The two bispinors $u_{\lambda\pm}$ form an orthonormal basis:

$$u_{\lambda\pm}^+u_{\lambda\pm}=1, \quad u_{\lambda\pm}^+u_{\lambda\mp}=0, \tag{9}$$

in which the spinor function φ_λ can be expanded, with scalar functions $\varphi_{\lambda\pm}$ as coefficients:

$$\varphi_\lambda=\varphi_{\lambda+}u_{\lambda+}+\varphi_{\lambda-}u_{\lambda-}. \tag{10}$$

Henceforth, when considering solutions with a definite value of λ, we shall omit this index. Substituting (1) and (10) into Eq. (5), we obtain

$$\left[\frac{d}{dz}\mp i(E-U)\right]\varphi_\pm+m_\perp\varphi_\mp=0. \tag{11}$$

The four-current density and energy density of the spinor field are given by the formulas

$$j^0=|\varphi_+|^2+|\varphi_-|^2, \quad j_3=|\varphi_+|^2-|\varphi_-|^2, \tag{12}$$

$$T^{00}=(E-U)j^0. \tag{13}$$

The energy density T^{00} is negative in the region where $U(z)>E$ (i.e., in the Klein region). For a real particle, such a situation is unacceptable. Therefore, in the Klein region a positive value of j^0 must be interpreted as the probability density for finding not a particle but an antiparticle. In this connection we shall consider stationary states in a deep potential well.

We shall assume that the function $U(z)$ has the form of a deep well with a single minimum and without maxima. Then for the deep energy levels the equation

$(E-U)^2-m_\perp^2=0$ for the turning points has four roots $z_1<z_2<z_3<z_4$ (see Fig. 1). The functions $\varphi_\pm$ have the quasiclassical asymptotic forms

$$\varphi_\pm=\frac{e^{\mp i\pi/4}}{\sqrt[4]{\pi_3^2}}\,(A\sqrt{\pi_\pm}e^{iS}+B\sqrt{\pi_\mp}e^{-iS}), \tag{14}$$

where

$$\pi_3^2=\pi_0^2-m_\perp^2,\quad \pi_\pm=\pi^0\pm\sqrt{\pi_3^2},\quad \pi^0=E-U, \tag{15}$$

$$S(z)=\int_{z_0}^{z}dz\,\sqrt{\pi_3^2(z)}, \tag{16}$$

and z_0 is an arbitrary point, chosen by considerations of convenience. The quantities $\pi_\pm$, generally speaking, are complex, and satisfy the relations

$$\sqrt{\pi_+}\sqrt{\pi_-}=m_\perp,\quad \arg\pi_+=-\arg\pi_-, \tag{17}$$

if the cut in the complex $\pi_\pm$ plane is drawn along the negative real axis. Then the following equalities will be valid:

$$(\sqrt{\pi_\pm})^*=\sqrt{\pi_\mp},\quad |\pi^0|<m_\perp, \tag{18a}$$

$$(\sqrt{\pi_\pm})^*=\sqrt{\pi_\pm},\quad \pi^0>m_\perp, \tag{18b}$$

$$(\sqrt{\pi_\pm})^*=-\sqrt{\pi_\pm},\quad \pi^0<-m_\perp. \tag{18c}$$

Following Nikishov,[10] we introduce the spinor functions

$$^\pm\varphi={}^\pm\varphi_+u_++{}^\pm\varphi_-u_-,\quad {}_\pm\varphi={}_\pm\varphi_+u_++{}_\pm\varphi_-u_-, \tag{19}$$

which are such that ${}^+\varphi_\pm({}_+\varphi_\pm)$ have, for $z>z_4(z<z_1)$, the quasiclassical asymptotic form $\propto e^{iS}$, while ${}^-\varphi_\pm({}_-\varphi_\pm)$ have the asymptotic form $\propto e^{-iS}$, i.e., the asymptotic forms of waves with positive and negative momentum $\pi_3=\pm\partial S/\partial z$, respectively. In other words, ${}^-\varphi$ and ${}_+\varphi$ are waves converging on the well and ${}^+\varphi$ and ${}_-\varphi$ are waves diverging from the well. Of the four functions ${}^\pm\varphi$ and ${}_\pm\varphi$, only two are linearly independent, and, therefore, the following relation should be fulfilled:

$${}_+\varphi=c_1{}^+\varphi+c_2{}^-\varphi, \tag{20}$$

where c_1 and c_2 are functions of E and $p_\perp^2$. From Eqs. (11), the asymptotic forms (14), and Eq. (18c) it follows that by choosing the phases and normalizations we can ensure fulfillment of the relations

$${}^+\varphi_\pm^*=-{}^-\varphi_\mp,\quad {}_+\varphi_\pm^*=-{}_-\varphi_\mp. \tag{21}$$

Then, for the current densities corresponding to the solutions ${}^\pm\varphi$ and ${}_\pm\varphi$, the equalities ${}^-j_3=-{}^+j_3$ and ${}_-j_3=-{}_+j_3$ will be valid. We now require that the equalities ${}^\pm j_3={}_\pm j_3$ be fulfilled. Then the coefficients c_1 and c_2 will satisfy the condition

$$|c_1|^2-|c_2|^2=1. \tag{22}$$

From (20)–(22) follow expansions for the other three functions:

$$\begin{aligned}&{}_-\varphi=c_1^{*}{}^-\varphi+c_2^{*}{}^+\varphi,\\ &{}^+\varphi=c_1^{*}{}_+\varphi-c_2{}_-\varphi,\\ &{}^-\varphi=c_1{}_-\varphi-c_2^{*}{}_+\varphi.\end{aligned} \tag{23}$$

Although the relations (20), (22), and (23) have kept the same form as in the nonrelativistic theory [see (25.6) in Ref. 12] the coefficients contain all the information about the scattering of particles and antiparticles and the creation of pairs by the field of the well.[10]

To determine the coefficients c_1 and c_2 in the WKB approximation we shall obtain equations relating the solutions to the left and to the right of the points z_j, $j=1,...,4$, where π_3^2 changes sign. By convention we shall regard the roots of positive quantities as real and positive. Then, according to Eq. (14), the quasiclassical solutions in the regions $\pi_3^2\lessgtr 0$ adjacent to the point z_j can be written in the form

$$\begin{aligned}\varphi_\pm&=\frac{e^{\mp i\pi/4}}{\sqrt[4]{-\pi_3^2}}\left[a_j\sqrt{\pi_\pm}\exp\left(-\int_{z_j}^{z}dz\sqrt{-\pi_3^2}\right)\right.\\ &\left.\quad+b_j\sqrt{\pi_\mp}\exp\left(\int_{z_j}^{z}dz\sqrt{-\pi_3^2}\right)\right],\quad \pi_3^2<0,\\ \varphi_\pm&=\frac{e^{\mp i\pi/4}}{\sqrt[4]{\pi_3^2}}\left[A_j\sqrt{\pi_\pm}\exp\left(i\int_{z_j}^{z}dz\sqrt{\pi_3^2}\right)\right.\\ &\left.\quad+B_j\sqrt{\pi_\mp}\exp\left(-i\int_{z_j}^{z}dz\sqrt{\pi_3^2}\right)\right],\quad \pi_3^2>0.\end{aligned} \tag{24}$$

Since Eqs. (11) are linear, the coefficients a_j, b_j and A_j, B_j are linearly related:

$$A_j=\alpha_ja_j+\beta_jb_j,\quad B_j=\gamma_ja_j+\delta_jb_j. \tag{25}$$

The corresponding formulas for the quasiclassical solutions of the simpler, scalar equation are found in Ref. 13. Repeating the arguments given there and assuming, for definiteness, that $\pi_3^2>0$ for $z>z_j$, we find without difficulty that

$$\delta_j=e^{i\pi/4}. \tag{26}$$

It follows from the relations (18a) that, for real a_j and b_j in the region $|\pi^0|<m_\perp$, the inequality

$$\varphi_\pm^*=\varphi_\mp \tag{27}$$

should be fulfilled. The functions $\varphi_\pm^*$ and $\varphi_\mp$ are described by the same equations (11), and, therefore, the equality (27), which is fulfilled in the region $|\pi^0|<m_\perp$, will also be valid in any other region. Thus, for real a_j and b_j the relation $A_j^*=\pm B_j$ is valid, whence we have

$$\alpha_j^*=\pm\gamma_j,\quad \beta_j^*=\pm\delta_j, \tag{28}$$

where the upper sign corresponds to the transitions $|\pi^0|<m_\perp\rightleftarrows\pi^0>m_\perp$, and the lower to the transitions $|\pi^0|<m_\perp\rightleftarrows\pi^0<-m_\perp$ [see (18b) and (18c)]. The coefficients α_j and γ_j can be found from the conservation law for the current density:

$$j_3=4\,\mathrm{Im}\,a_j^*b_j,\quad |\pi^0|<m_\perp, \tag{29}$$

$$j_3=\pm 2(|A_j|^2-|B_j|^2), \quad \pi^0 \gtrless \pm m_\perp . \tag{30}$$

Equating the right-hand sides of the expressions (29) and (30) and taking the relations (25), (26), and (28) into account, we obtain

$$\alpha_j=\tfrac{1}{2}e^{i\pi/4}+\rho_j e^{-i\pi/4}, \tag{31}$$

where ρ_j is a real parameter that cannot be determined by the WKB method. Thus, the formulas relating the solutions to the left and to the right of the point z_j have the form

$$\begin{aligned} A_j&=e^{i\pi/4}Q_j a_j \pm e^{-i\pi/4}b_j, \\ B_j&=\pm e^{-i\pi/4}P_j a_j+e^{i\pi/4}b_j, \end{aligned} \tag{32}$$

where

$$P_j=Q_j^*=\tfrac{1}{2}+i\rho_j, \tag{33}$$

and the upper signs correspond to the transitions $|\pi^0|<m_\perp \rightleftarrows \pi^0>m_\perp$, and the lower to the transitions $|\pi^0|<m_\perp \rightleftarrows \pi^0<-m_\perp$. In the case when $\pi_3^2>0$ for $z<z_j$, we must make in Eqs. (32) the replacement $a_j \rightleftarrows b_j$, $A_j \rightleftarrows B_j$. We note that the expressions (29) and (30) for the current density remain unchanged here.

We shall now make use of Eqs. (32) to find the coefficients c_1 and c_2. We introduce the notation

$$\begin{aligned} L&=\exp\left(-\int_{z_1}^{z_2} dz\sqrt{-\pi_3^2}\right), \\ R&=\exp\left(-\int_{z_3}^{z_4} dz\sqrt{-\pi_3^2}\right), \\ M&=\exp\left(i\int_{z_2}^{z_3} dz\sqrt{\pi_3^2}\right). \end{aligned} \tag{34}$$

The scalar components ${}_+\varphi_\pm$ of the function ${}_+\varphi$, which appear in the left-hand side of (20), have in the region $z<z_1$ the quasiclassical coefficients

$$A_1=1, \quad B_1=0. \tag{35}$$

In the region $z_1<z<z_2$, according to Eqs. (32), the coefficients of ${}_+\varphi_\pm$ are equal to

$$a_1=e^{-i\pi/4}Q_1, \quad b_1=-e^{i\pi/4}. \tag{36}$$

To go over into the region $z_2<z<z_3$ we rewrite the function ${}_+\varphi_\pm$ in the region $z_1<z<z_2$ in the form

$$\begin{aligned} {}_+\varphi_\pm=\frac{e^{\mp i\pi/4}}{\sqrt[4]{-\pi_3^2}}\Bigg[&a_1 L\sqrt{\pi_\pm}\exp\left(-\int_{z_2}^{z} dz\sqrt{-\pi_3^2}\right) \\ &+\frac{b_1}{L}\sqrt{\pi_\mp}\exp\left(\int_{z_2}^{z} dz\sqrt{-\pi_3^2}\right)\Bigg]. \end{aligned} \tag{37}$$

Then it follows from comparison of (37) with (24) that

$$a_2=a_1 L, \quad b_2=b_1/L, \tag{38}$$

and again we can use Eqs. (32). By repeating the above procedure several times, we find ourselves finally in the region $z>z_4$ and obtain for the coefficient A_4 of the wave moving away from the well to $+\infty$ the expression

$$\begin{aligned} A_4=\frac{1}{LR}\,[&M(1-Q_1Q_2L^2)(1-Q_3Q_4R^2) \\ &+M^{-1}(1+Q_1P_2L^2)(1+P_3Q_4R^2)], \end{aligned} \tag{39a}$$

and for the coefficient B_4 of the wave incident on the well from $+\infty$ the expression

$$B_4=-iA_4|_{Q_4\to -P_4}, \tag{39b}$$

which differs from A_4 by the replacement of Q_4 by $-P_4$ and by the factor $-i$. It is not difficult to verify that $|A_4|^2-|B_4|^2=1$, as should be the case by virtue of the conservation of the current density [see (30) and (32)]. An important point here is that for real E we have $M^{-1}=M^*$ and $Q_j=P_j^*$, while L and R are real quantities. It follows from Eqs. (20), (24), and (35) that A_4, B_4 coincide with the coefficients c_1, c_2 of the expansions (20) and (23), respectively. Thus, Eqs. (39) specify the coefficients c_1, c_2 for the potential well $U(z)$ in the WKB approximation. Only the real parameters ρ_j ($j=1,...,4$) remain undetermined here.

The results can be expressed compactly in matrix language. If we denote the 2×2 transformation matrix in Eq. (32) by $G_\pm(\rho_j)$, and that in Eq. (38) by $H_L=\mathrm{diag}(L,L^{-1})$ [H_R and H_M are defined analogously], the transformation of the coefficients A_1, B_1 of an arbitrary solution to the left of the well into the coefficients A_4, B_4 of the same solution to the right of the well will be implemented by the matrix

$$G_-(\rho_4)H_R G_+^+(-\rho_3)H_M G_+(\rho_2)H_L G_-^+(-\rho_1) =\begin{pmatrix} c_1 & c_2^* \\ c_2 & c_1^* \end{pmatrix}. \tag{40}$$

It is clear that the transformations determined by the matrix (40) and its inverse are equivalent to the relations (20), (23) in which the coefficients c_1 and c_2 in the WKB approximation are given by Eqs. (39). We note also that the relation between the WKB solutions to the left of the well and the solutions inside the well are implemented by the matrix

$$G_+(\rho_2)H_L G_-^+(-\rho_1)=\begin{pmatrix} c_1' & -c_2'^* \\ c_2' & -c_1'^* \end{pmatrix}, \tag{41}$$

$$c_1'=-L^{-1}(1-Q_1Q_2L^2), \quad c_2'=-iL^{-1}(1+Q_1P_2L^2), \tag{42}$$

which, like the matrix (40), has a determinant equal to 1:

$$-|c_1'|^2+|c_2'|^2=1. \tag{43}$$

The unimodularity of these matrices is ensured by the conservation of the current density. The different signs of the quantities $|c_1|^2-|c_2|^2$ and $|c_1'|^2-|c_2'|^2$ are caused by the fact that the matrix (40) relates solutions in regions with the same signs of π^0, while the matrix (41) relates solutions in regions with the opposite signs of π^0 [see (30)]. In the former case π^0 changes sign an even number of times between the regions, and in the latter case an odd number of times. The matrix (41) and its inverse lead to the relations (20), (23) with the replacements $c_{1,2} \to c'_{1,2}$ and $c^*_{1,2} \to -c'^*_{1,2}$, after which, together with Eq. (43), these relations coincide with Eqs. (5.17)–(5.18) of Nikishov.[10]

We note that the relativistic motion of an electron in a centrally symmetric field in the quasiclassical approximation has been considered by Mur, Popov, *et al.*[14–18] However, because of the restriction to particular solutions that are finite at $r=0$, expressions for the coefficients $c_{1,2}$ were not obtained by them.

Up to now we have considered stationary states, for which E is a real parameter. Quasistationary states are characterized by the presence at $\pm\infty$ of only diverging or only converging waves. We consider the latter first. For $z<z_1$ the function describing such a quasistationary state should coincide with $_+\varphi$, and for $z>z_4$ it should coincide with $^-\varphi$. It follows from the relation (20) that this is possible if

$$c_1(E)=0. \tag{44}$$

However, Eq. (44) cannot have real roots, since, if it had, the relation (22) would be violated. We shall show that it has complex roots with a negative imaginary part. In view of the complicated nature of this problem, we shall solve it in the quasiclassical approximation, and, therefore, for c_1 we shall use Eq. (39a) (we recall that A_4 in this formula coincides with c_1). We shall seek the solution of Eq. (44) in the form $E=E_0+i\Gamma/2$, assuming Γ to be small in comparison with E_0. Since L and R in the quasiclassical approximation are exponentially small, in the first approximation we can set $E=E_0$, $L=R=0$, and $M=M_0=M(E_0)$. Then Eq. (44) goes over into $M_0+M_0^{-1}=0$, whence we obtain an equation for E_0:

$$\int_{z_2}^{z_3} dz\,\sqrt{(E_0-U)^2-m_\perp^2}=\pi\left(n+\frac{1}{2}\right). \tag{45}$$

This is the relativistic analog of the Bohr–Sommerfeld quantization rule. In the second approximation we can retain in Eq. (44) only the terms linear in L^2 and R^2, after which it acquires the form

$$M+M^{-1}-Q_1(Q_2M-P_2M^{-1})L^2$$
$$-Q_4(Q_3M-P_3M^{-1})R^2=0. \tag{46}$$

In the terms containing L^2 or R^2 we can assume that $\Gamma=0$ and that E_0 satisfies (45). Then, in these terms, $M=M_0=i(-1)^n$, and $L^2=D_L$, $R^2=D_R$, where

$$D_{L,R}=\exp\left[-2\int_{z_{1,3}}^{z_{2,4}} dz\,\sqrt{m_\perp^2-(E_0-U)^2}\right] \tag{47}$$

are the real positive coefficients of transmission through the left and right walls of the potential well, respectively. In the first two terms of (46) we replace the function $M(E_0+i\Gamma/2)$ by the first two terms of the expansion in powers of Γ:

$$M+M^{-1}=M_0+M_0^{-1}-\frac{i}{2}(-1)^n\Gamma\tau+..., \tag{48}$$

$$\tau=\int_{z_2}^{z_3} dz\,\frac{E_0-U}{\sqrt{(E_0-U)^2-m_\perp^2}}. \tag{49}$$

The quantity τ coincides with the period of the oscillations of a classical relativistic particle with energy E_0 in a potential well. Now, using (45), from Eq. (46) we obtain

$$\Gamma=-\frac{1}{\tau}(D_L+D_R). \tag{50}$$

Thus, the quantity Γ has been found to be negative. We note a remarkable fact: The unknown quantities ρ_j have dropped out of the final result.

We shall consider now the quasistationary states that are characterized by the presence at $\pm\infty$ of only diverging waves $^+\varphi$ and $_-\varphi$. According to the second and third equations in (23), the complex energy of such states should be determined by the equation $c_1^*(E)=0$. In the WKB approximation under consideration, for real E the function $c_1^*(E)$ differs from $c_1(E)$ by the replacement of quantities in the square brackets in (39a): $L^2\to -L^2$, $R^2\to -R^2$, and $Q_{1,4}\to P_{1,4}$, and, therefore, the spectrum of the energy levels is determined by the previous equation (45) while their width is determined by Eq. (50) with the opposite sign, i.e., $\Gamma>0$. For more detail about quasistationary states with $\Gamma>0$ see Ref. 19.

Equations (12) contain the density of the conserved probability current. Since in the Klein region outside the well the directions of the current and momentum of the spinor wave are opposite [see (30)], in quasistationary states with converging waves the probability inside the well decreases, while in states with diverging waves it increases. This agrees with the signs $\Gamma \lessgtr 0$ in the first and second cases. On the other hand, a probability current opposite to the momentum of the wave for $\pi^0<-m_\perp$ should be regarded as a probability current for finding an antiparticle. Therefore, in the former case a pair is created: The antiparticle moves away from the well, and the particle occupies a state inside the well. In the latter case an antiparticle incident on the well is annhilated with a particle inside the well. The decrease of the probability inside the well in the former case and its increase in the latter case can be understood only if we assume it to be the probability of finding an unoccupied (vacant) state. Thus, the quasistationary solutions of the Dirac equation describe, inside the well, the evolution in time of unoccupied states: The creation of a pair implies the occupation (vanishing) of a vacant state, and the annihilation of a pair implies the formation of a vacant state. This interpretation of spinor quasistationary solutions implies that the particle and antiparticle obey Fermi statistics.

We turn now to the interpretation of stationary solutions in the language of particles and antiparticles. Here we use the formula $v_3=\pi_3/\pi^0$ for the velocity of the particles ($\pi^0>m_\perp$) and antiparticles ($\pi^0<-m_\perp$) and the fact that the particles and antiparticles arrive from the past and go off into the future. Then the relation

$$_+\varphi=c_1'\,^+\varphi+c_2'\,^-\varphi, \tag{51}$$

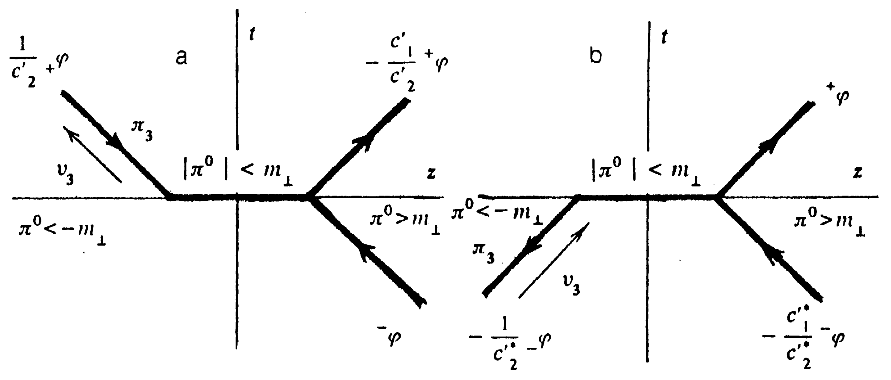

FIG. 2.

which expresses a solution with positive momentum to the left of the well in terms of solutions with positive and negative momentum inside the well, can be depicted by means of the t, z diagram of Fig. 2a and interpreted as follows. A state

$$^{-}\varphi=\frac{1}{c_2'}\,{}_{+}\varphi-\frac{c_1'}{c_2'}\,{}^{+}\varphi, \tag{52}$$

initially not occupied by a particle remains unoccupied with probability $w_0=|c_1'/c_2'|^2$, after reflection from a wall of the well and is occupied by the particle and antiparticle of a pair (generated by the field in the region where $|\pi^0|<m_\perp$) with probability $w_1=|1/c_2'|^2$; the particle of the pair is described by the function ${}^{+}\varphi$ in the region where $\pi^0>m_\perp$, and moves away from the wall into the well, while the antiparticle is described by the function ${}_{+}\varphi$ in the region where $\pi^0<-m_\perp$, and moves away to $-\infty$. It is clear that $w_0+w_1=1$. For the WKB solutions explicit expressions for the probabilities follow from (42): $w_0=1-D_L$ and $w_1=D_L$. The latter expression agrees with Eq. (50) for a quasistationary state with converging waves, according to which the probability of decay of the vacuum state as a result of the formation of a pair in the effective time of collision with the left and right walls is equal to $|\Gamma|\tau=D_L+D_R$ (we have considered above, collision only with the left wall). A negative sign of Γ corresponds to a decrease of the amplitude of the reflected wave in comparison with the amplitude of the incident wave, i.e., to a decrease of the probability of preservation of an unoccupied state as a result of the creation of a pair by the wall.

In an analogous way the relation

$${}_{-}\varphi=-c_1'^{*}\,{}^{-}\varphi-c_2'^{*}\,{}^{+}\varphi \tag{53}$$

can be depicted by the t, z diagram of Fig. 2b and interpreted as the formation (as a result of annihilation of a pair) of a state ${}^{+}\varphi=-c_1'^{*}/c_2'^{*}\,{}^{-}\varphi-1/c_2'^{*}\,{}_{-}\varphi$ (not occupied by a particle) that was, at first, unoccupied with probability $w_0=|c_1'/c_2'|^2$ and occupied by a pair with probability $w_1=|1/c_2'|^2$, the particle having been situated inside the well and the antiparticle to the left of the well. A positive sign of Γ in the corresponding quasistationary state with diverging waves (i.e., antiparticles incident on the well) implies an increase of the amplitude of the reflected wave inside the well in comparison with that of the incident wave, i.e., an increase, from w_0 to 1, of the probability of finding an unoccupied state because of annihilation of a pair.

Finally, the representations

$${}^{+}\varphi=-c_1'^{*}\,{}_{+}\varphi-c_2'\,{}_{-}\varphi,\quad {}^{-}\varphi=c_1'\,{}_{-}\varphi+c_2'^{*}\,{}_{+}\varphi \tag{54}$$

can be depicted by the t,z diagrams of Figs. 3a and 3b, respectively. The first of these describes the creation of a pair by the right wall of the potential barrier, as a result of which an initially unoccupied antiparticle state ${}_{-}\varphi$ has only a probability w_0 of remaining unoccupied and a probability w_1 of being occupied by a pair, with the particle going off to $+\infty$ and the antiparticle remaining under the barrier. The particle and antiparticle are described by the functions ${}^{+}\varphi$ and ${}_{+}\varphi$, respectively. The second representation (54) describes the annihilation of a pair at the right

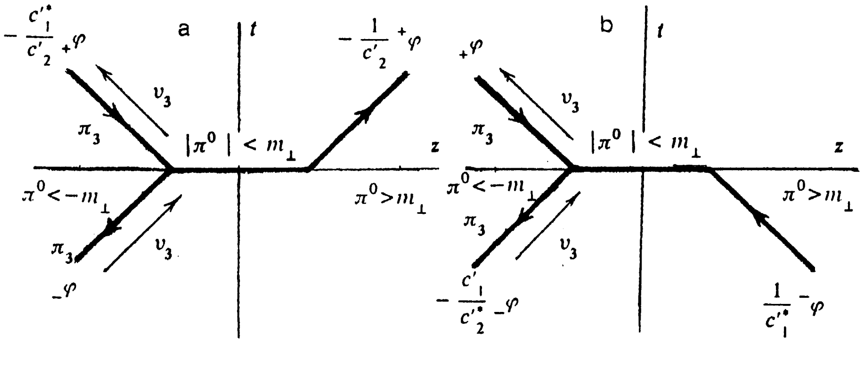

FIG. 3.

wall of the barrier, as a result of which an unoccupied antiparticle state ${}_{+}\varphi$ is formed that was previously unoccupied only with probability w_0 and occupied by a pair with probability w_1. It is possible to show that the quasistationary states corresponding to Figs. 3a and 3b have $\Gamma<0$ and $\Gamma>0$, respectively, implying a decrease or increase of the probability of finding an unoccupied antiparticle state under the barrier.

Thus, the relations (51), (53), and (54), after division by the coefficient c_2', can be given a probabilistic interpretation and it can be assumed that they describe the vanishing or formation of unoccupied states ${}^{-}\varphi$, ${}^{+}\varphi$, ${}_{-}\varphi$, ${}_{+}\varphi$. If, however, we assume, e.g., that the state ${}^{-}\varphi$ in (52) is occupied by a particle, then by virtue of the Pauli principle the creation, by the field, of a pair with an antiparticle in the state ${}_{+}\varphi$ is forbidden, since in this case the particle of the pair should be in the state ${}^{+}\varphi$, which is occupied by the incident particle reflected from the wall. This is confirmed by the zero probability of the transition ${}^{-}\varphi_n \to {}_{+}\varphi_{n'}$, with relative amplitude

$$M^{-}_{+n'n}=i\int_{z'<z} d^3\Omega' d^3\Omega\, {}_{+}\bar{\psi}_{n'}(x')\gamma_3 G(x',x)\gamma_3\, {}^{-}\psi_n(x)=0 \tag{55}$$

and by the unit probability of the transition ${}^{-}\varphi_n \to {}^{+}\varphi_{n'}$, with relative amplitude

$$M^{+-}_{n'n}=i\int_{z'>z} d^3\Omega' d^3\Omega\, {}^{+}\bar{\psi}_{n'}(x')\gamma_3 G(x',x)\gamma_3\, {}^{-}\psi_n(x) = -\frac{c'^*_{2n}}{c'^*_{1n}}\,\delta_{n'n}. \tag{56}$$

Here, $d^3\Omega=dxdydt$, $n=p_1, p_2, E, \lambda$. According to Feynman,[20] the relative amplitudes become absolute after they are multiplied by the amplitude for the vacuum to be maintained in its original state. In our case it is equal to c'_{1n}/c'_{2n}. The formulas (55) and (56) follow directly from the representation (7′) of Nikishov[21] for the causal Green function.

The parameters E_0 and Γ play the role of the energy and width of a level for the stationary states as well. We shall consider, e.g., a solution ${}^{-}\varphi$ with an energy E for which there are Klein regions to the left and right of the well. From the relation ${}_{-}\varphi=A\,{}^{-}\varphi+B\,{}_{+}\varphi$, where

$$A=\frac{1}{c_1},\quad B=\frac{c_2^*}{c_1},\quad |A|^2+|B|^2=1, \tag{57}$$

it follows that this solution describes an antiparticle incident on the well from $-\infty$, reflected from the well with probability $|B|^2$, and passing through it to $+\infty$ with probability $|A|^2$. Passage of the antiparticle through the well can be interpreted[1] as annihilation of the antiparticle with a particle inside the well and creation of a new pair, the particle of which remains in place of that which has been annihilated while the antiparticle goes away to $+\infty$. If the energy E of the antiparticle is close to the energy E_0 of the level determined by (44), then expanding the function $c_1(E)$ in a Taylor series about the point $E_0+i\Gamma/2$, where it vanishes, and using the condition $|\Gamma|\ll E_0$, we obtain $c_1(E)=\dot{c}_1(E_0)(E-E_0-i\Gamma/2)+\dots$, where $\dot{c}_1(E)\equiv dc_1(E)/dE$. Then the probabilities of transmission and reflection of the antiparticle depend on E in a resonant manner, with a resonance width Γ:

$$|A|^2\approx\frac{1}{|\dot{c}_1(E_0)|^2[(E-E_0)^2+1/4\,\Gamma^2]},\qquad |B|^2+1-|A|^2. \tag{58}$$

At energy $E=E_0$ the quantity $|A|^2$ can reach values almost equal to 1, so that reflection of the antiparticle from the well will be absent. In Sec. 4 it is shown that for a triangular well such resonance values of the energy, at which the well is completely transparent to antiparticles incident on it, really exist. These values are exponentially close to E_0 in a weak field. In nonrelativistic quantum mechanics, particles passing over a well or barrier may also fail to be reflected under certain conditions.[12]

3. PHYSICAL MEANING OF SOLUTIONS OF RELATIVISTIC WAVE EQUATIONS (THE SCALAR CASE)

A scalar particle is described by the Klein–Gordon equation $(\Pi^2+m^2)\psi=0$, whose solution we shall choose in the form (1). Then the function $\varphi(z)$ will obey the equation

$$\left[\frac{d^2}{dz^2}+(E-U)^2-m_\perp^2\right]\varphi=0. \tag{59}$$

The four-current densities and the scalar-field energy densities are given by the formulas

$$j^0=\frac{E-U}{m}|\varphi|^2,\quad j_3=-\frac{i}{2m}\varphi^*\frac{\overleftrightarrow{d}}{dz}\varphi, \tag{60}$$

$$T^{00}=\frac{1}{2m}\left\{[(E-U)^2+m_\perp^2]|\varphi|^2+\left|\frac{d\varphi}{dz}\right|^2\right\}. \tag{61}$$

Unlike in the spinor case, the energy density T^{00} of the scalar field is everywhere positive. However, in the Klein region the quantity j^0 is negative, and should therefore be interpreted not as a probability density but as a charge density (in units of e). The negativity of j^0 in the Klein region obviously implies that there the solutions of the Klein–Gordon equation describe an antiparticle.

The quasiclassical asymptotic form of the function φ is

$$\varphi=\frac{\sqrt{m}}{\sqrt[4]{\pi_3^2}}(Ae^{iS}+Be^{-iS}), \tag{62}$$

where π_3^2 and $S(z)$ are the same as in (15) and (16). The functions ${}^{\pm}\varphi$ and ${}_{\pm}\varphi$ are introduced in analogy with the corresponding spinor functions. The relationship between them is given by the same formulas (20), (22), and (23), if we take it into account that now, instead of (21), the relations

$${}^{+}\varphi^*={}^{-}\varphi,\quad {}_{+}\varphi^*={}_{-}\varphi \tag{63}$$

are fulfilled. In the quasiclassical approximation the coefficients c_1 and c_2 can be found, as in the spinor case, by means of the relationship between the solutions to the left and to the right of the turning points. The latter are given in Ref. 13 and coincide with Eqs. (32) with the upper sign. Therefore the following matrix equality is valid:

$$G_+(\rho_4)H_RG_+^\dagger(-\rho_3)H_MG_+(\rho_2)H_LG_+^\dagger(-\rho_1)$$
$$=\begin{pmatrix} c_1 & c_2^* \\ c_2 & c_1^* \end{pmatrix}. \tag{64}$$

It follows from (64) that the coefficients c_1 and c_2 differ from the analogous coefficients in the spinor case by a change of sign of L^2 and R^2 in the square brackets in Eqs. (39), and also by a change of sign of the right-hand side of Eq. (39b). Therefore, the spectrum of the energy levels of the quasistationary states is determined as before by Eq. (45), while the width of the levels is determined by Eq. (50) with the + sign ($\Gamma>0$) for states with converging waves and with the − sign ($\Gamma<0$) for states with diverging waves. Thus, the widths of the levels of the scalar and the spinor quasistationary states have opposite signs. This is due to the fact that a solution of the Klein–Gordon equation always describes occupied states, and j^μ is the electric-current density and not the probability density. The signs of the electric current $j_3=|A|^2-|B|^2$ and momentum π_3 always coincide [see (62)], but the antiparticle velocity $v_3=\pi_3/\pi^0$ is opposite to the electric current and momentum of the waves. Therefore, as in the spinor case, quasistationary states with converging or diverging waves describe the creation or annihilation of pairs. However, the creation of a scalar pair does not require the previous existence of a vacant state inside the well; it is considered as an increase of the charge inside the well.

The relation between the scalar WKB solutions to the left of the well and the solutions inside the well is implemented by the matrix

$$G_+(\rho_2)H_LG_+^\dagger(-\rho_1)=\begin{pmatrix} c_1' & c_2'^* \\ c_2' & c_1'^* \end{pmatrix}, \tag{65}$$

where

$$c_1'=L^{-1}(1+Q_1Q_2L^2),\quad c_2'=iL^{-1}(1-Q_1P_2L^2), \tag{66}$$

$$|c_1'|^2-|c_2'|^2=1. \tag{67}$$

It follows from (65) that the relations (51), (53), and (54) are valid for the scalar solutions, but with the opposite sign of $c_{1,2}'^*$. Therefore, the diagrams of Figs. 2 and 3 illustrate the processes described by the scalar solutions as well. Now, however, $|c_1'|^2>|c_2'|^2$ [see (67)], and, therefore, the quantities $|c_1'/c_2'|^2$ and $|1/c_2'|^2$, which we denote here by n_0 and n_1, can no longer be interpreted as probabilities. For creation (annihilation) processes n_0 has the meaning of the average number of particles or antiparticles emitted by (incident on) a wall, while n_1 has the meaning of the average number of pairs created (emitted) by the wall per incident (emitted) particle or antiparticle. The quantities that now have the meaning of probabilities are $w_0=|c_2'/c_1'|^2$ and $w_1=|1/c_1'|^2$. For creation (annihilation) processes w_1 is the probability that an emitted (incident) particle or antiparticle is created (absorbed) by the wall, and w_0 is the probability that the particle or antiparticle is not created (is not absorbed) but is reflected, i.e., w_0 coincides with the probability that the vacuum is preserved upon scattering in the state under consideration:

$$C_v=w_0. \tag{68}$$

On the other hand, in Nikishov's paper[21] it is shown that the squares of the moduli of the relative amplitudes of the transitions ${}^-\varphi_n\to{}^+\varphi_n$ and ${}^-\varphi_n\to{}_+\varphi_n$, which have the meaning of the relative probability of backward scattering of a particle and the relative probability of production of one pair by the field, are equal to w_0 and w_1. Then the relative probability of the production of 0, 1, 2,... pairs is equal to $1+w_1+w_1^2+...=(1-w_1)^{-1}=w_0^{-1}$, and the absolute probability is equal to $C_vw_0^{-1}=1$ (as it should be), if we take Eq. (68) into account. The absolute probability of scattering of a particle with the accompanying creation of 0, 1, 2,... pairs is also equal to 1 (Ref. 21).

The inequalities $|c_2'|^2 \lessgtr |c_1'|^2$ in the spinor and scalar cases [see (43) and (67)] express the spin-statistics connection for solutions of the relativistic wave equations. In fact, because of the Pauli principle, the creation of a pair (see Fig. 2a) in the spinor case must be regarded as occupation of a vacant state by a particle, as a result of which the probability current of the vacant state is decreased upon reflection, and in the scalar case must be regarded as an increase of the charge, so that upon reflection the flux of charge density is increased.

To conclude this section we note that in strong fields condensation of bosons into the ground state can cancel completely a field capable of creating pairs.[3] Therefore, quasistationary states of bosons in such a field have meaning if the condensation time is sufficiently long.

4. EXACT SOLUTIONS OF THE WAVE EQUATIONS WITH THE FIELD OF A WELL OF SPECIAL FORM

We shall consider the exact solutions of Eqs. (11) with the potential $A^0(z)=\varepsilon|z|$ $(e\varepsilon>0)$ of a triangular well. The general solution, continuous at $z=0$, can be expressed in terms of parabolic-cylinder functions and contains two arbitrary constants A_1 and A_2:

$$\varphi_+(z)$$
$$=\begin{cases} HA_1D_{i\nu}(e^{i\pi/4}\xi)-(FA_1+GA_2) \\ \quad\times D_{i\nu}(-e^{i\pi/4}\xi), \quad z>0 \\ -e^{-i\pi/4}\sqrt{\nu}[HA_2D_{i\nu-1}(e^{i\pi/4}\xi)+(FA_2-GA_1) \\ \quad\times D_{i\nu-1}(-e^{i\pi/4}\xi)], \end{cases}$$
$$z<0, \tag{69a}$$

$$\varphi_-(z)=\varphi_+(-z)|_{A_1\to A_2,A_2\to -A_1}. \tag{69b}$$

Here,

$$F(\varkappa)=D_{i\nu}(e^{i\pi/4}\varkappa)D_{i\nu}(-e^{i\pi/4}\varkappa)$$
$$+i\nu D_{i\nu-1}(e^{i\pi/4}\varkappa)D_{i\nu-1}(-e^{i\pi/4}\varkappa), \tag{70a}$$

$$H(\varkappa)=D_{i\nu}^2(e^{i\pi/4}\varkappa)-i\nu D_{i\nu-1}^2(e^{i\pi/4}\varkappa),$$

$$G=\frac{e^{i\pi/4}}{\Gamma(-i\nu)}\sqrt{\frac{2\pi}{\nu}}, \qquad (70b)$$

$$\nu=m_\perp^2/2e\varepsilon, \quad \varkappa=2\sqrt{\nu}E/m_\perp, \quad \xi=2\sqrt{\nu}\theta,$$

$$\theta=\frac{e\varepsilon|z|-E}{m_\perp}=-\pi^0/m_\perp. \qquad (71)$$

The quantities F, G, and H are connected with each other by the relation

$$F=e^{\pi\nu}H+\beta G, \quad F^2+G^2=H\widetilde{H}, \qquad (72)$$

$$\beta(\varkappa)=2\sqrt{\nu}e^{\pi\nu/2}\,\mathrm{Im}[e^{-i\pi/4}D_{i\nu}(e^{i\pi/4}\varkappa)D_{i\nu-1}^*(e^{i\pi/4}\varkappa)]. \qquad (73)$$

Here and below, $\widetilde{f}(\varkappa)\equiv f(-\varkappa)$. We note also the equality

$$|H|^2=e^{-\pi\nu}(1+\beta^2). \qquad (74)$$

These formulas are easily obtained from general relations for parabolic-cylinder functions.[22]

We shall find the functions ${}^\pm\varphi$ and ${}_\pm\varphi$. The action $S(z)$ for the potential under consideration has the form

$$S(z)=\mathrm{sgn}\,z\cdot\nu(\theta\sqrt{\theta^2-1}-\mathrm{Arch}\,\theta)+\mathrm{const}. \qquad (75)$$

The Klein region is determined by the inequality $\theta>1$, and the quasiclassical limit by the conditions

$$\nu=m_\perp^2c^3/2e\varepsilon\hbar\gg 1, \quad |\theta^2-1|\gg\nu^{-2/3}. \qquad (76)$$

The asymptotic forms of the functions $D_{i\nu-\mu}(e^{i\pi/4}\xi)$ for $\xi>2\sqrt{\nu}$ (more precisely, for $\theta-1\gg\nu^{-2/3}$, $\nu\gg 1$) are given by Darwin's formulas:[23]

$$D_{i\nu-\mu}(e^{i\pi/4}\xi)\approx\frac{\nu^{(i\nu-\mu/2)}(\theta+\sqrt{\theta^2-1})^{1/2-\mu}}{2^{1/2}(\theta^2-1)^{1/4}}\exp\left[-\frac{\pi\nu}{4}-\frac{i\pi\mu}{4}-i\nu(\tfrac{1}{2}+\theta\sqrt{\theta^2-1}-\mathrm{Arch}\,\theta)\right]. \qquad (77)$$

In the region $\theta+1\ll-\nu^{-2/3}$, $\nu\gg 1$ they can be obtained from the relation[23]

$$D_{i\nu-\mu}(-e^{i\pi/4}\xi)=e^{\pi\nu-i\pi\mu}D_{i\nu-\mu}(e^{i\pi/4}\xi)+\frac{\sqrt{2\pi}}{\Gamma(\mu-i\nu)}e^{\pi[\nu+i(\mu-1)]/2}\times D_{i\nu-1+\mu}^*(e^{i\pi/4}\xi) \qquad (78)$$

and from Eq. (77). Comparing the asymptotic forms of the parabolic-cylinder functions with the asymptotic forms (14), we can verify without difficulty that the functions ${}^-\varphi_+$ and ${}_-\varphi_+$ should be expressed in terms of $D_{i\nu}(e^{i\pi/4}\xi)$ and $D_{i\nu}^*(e^{i\pi/4}\xi)$, and the functions ${}^+\varphi_+$ and ${}_+\varphi_+$ in terms of $D_{i\nu-1}(e^{i\pi/4}\xi)$ and $D_{i\nu-1}^*(e^{i\pi/4}\xi)$. Thus, the general solution (69a) reduces to one of the functions ${}^\pm\varphi_+$, ${}_\pm\varphi_+$ if the coefficients A_1, A_2 satisfy one of the conditions

$$\begin{aligned}
{}^-\varphi_+&: \quad FA_1+GA_2=0,\\
{}^+\varphi_+&: \quad HA_1-e^{\pi\nu}(FA_1+GA_2)=0,\\
{}_-\varphi_+&: \quad HA_2-e^{\pi\nu}(FA_2-GA_1)=0,\\
{}_+\varphi_+&: \quad FA_2-GA_1=0.
\end{aligned} \qquad (79)$$

Expressing one of the coefficients A_1, A_2 in terms of the other, substituting the resulting expressions into (69a), and using the relations (72) and (74), we obtain

$$\begin{aligned}
&{}^-\varphi_+(z)=e^{i\delta}\chi_1(z), \quad {}^+\varphi_+(z)=e^{-i\delta}\chi_2'(-z),\\
&{}_-\varphi_+(z)=-e^{i\delta}\chi_1^*(-z), \quad {}_+\varphi_+(z)=e^{-i\delta}\chi_2(z),
\end{aligned} \qquad (80)$$

where

$$\chi_1(z)=A\begin{cases}GD_{i\nu}(e^{i\pi/4}\xi), & z>0,\\ \left[\begin{array}{l}e^{-i\pi/4}\sqrt{\nu}[FD_{i\nu-1}(e^{i\pi/4}\xi)\\ \quad+\widetilde{H}D_{i\nu-1}(-e^{i\pi/4}\xi)\end{array}\right], & z<0,\end{cases} \qquad (81a)$$

$$\chi_2(z)=A\begin{cases}FD_{i\nu}(e^{i\pi/4}\xi)-\widetilde{H}D_{i\nu}(-e^{i\pi/4}\xi), & z>0,\\ -e^{-i\pi/4}\sqrt{\nu}GD_{i\nu-1}(e^{i\pi/4}\xi), & z<0.\end{cases} \qquad (81b)$$

Here, A and δ are arbitrary complex and real constants. The functions ${}^\pm\varphi_-$ and ${}_\pm\varphi_-$ can be obtained from (80) by means of the relations (21). Substituting the resulting expressions for ${}_\pm\varphi$ and ${}^\pm\varphi$ into (20), we find the coefficients c_1 and c_2:

$$c_1=-\exp[2i\arg(AG)+\pi\nu/2]\widetilde{H}, \quad c_2=\exp(-2i\delta)\widetilde{\beta}. \qquad (82)$$

The normalization of the functions $\chi_{1,2}(z)$ is chosen in order that c_1, c_2 satisfy the relation (22) [see (74)]. We note that $c_1(E)$ is an entire function of E, while $c_2(E)$ is the real part of an entire function on the real axis. It follows from (82) that Eq. (44), which determines the spectrum of the complex energies of the quasistationary states with converging waves, has the form

$$\widetilde{H}(\varkappa)=0. \qquad (83)$$

Equation (83) can be decomposed into two simpler equations:

$$\chi^\pm(-\varkappa)\equiv D_{i\nu}(-e^{i\pi/4}\varkappa)\pm e^{i\pi/4}\sqrt{\nu}D_{i\nu-1}(-e^{i\pi/4}\varkappa)=0\,. \qquad (84s,a)$$

The functions $\chi^\pm$ are related by

$$\chi^\pm(z)=\frac{\mp i}{\sqrt{\nu\mp z/2}}\frac{d}{dz}\chi^\mp(z), \qquad (85)$$

which permits us to write Eqs. (84) in the form

$$D_{i\nu}'(-e^{i\pi/4}\varkappa)\mp e^{i\pi/4}\sqrt{\nu}D_{i\nu-1}'(-e^{i\pi/4}\varkappa)=0. \qquad (86s,a)$$

The equations (84) or (86) determine the s- and a-series of complex energy values of the quasistationary states. In the nonrelativistic limit

$$\nu=m_\perp^2c^3/2e\varepsilon\hbar\gg 1, \quad |\theta^2-1|\lesssim\nu^{-2/3} \qquad (87)$$

the roots of these equations are obtained in the Appendix. Using them, and also taking it into account that $m_\perp \approx m + p_\perp^2/2m$, we obtain

$$E_0^{(s,a)} = m + p_\perp^2/2m + e\varepsilon a k^{(s,a)} + \dots, \tag{88s,a}$$

$$\Gamma^{(s)} = -\frac{\pi e\varepsilon a e^{-2\pi\nu}}{2k^{(s)}\mathrm{Ai}^2(-k^{(s)})} + \dots, \tag{89s}$$

$$\Gamma^{(a)} = -\frac{\pi e\varepsilon a e^{-2\pi\nu}}{2\mathrm{Ai}'^2(-k^{(a)})} + \dots, \tag{89a}$$

where the length a is defined in (3), and $k^{(s,a)}$ are the roots of Eqs. (4). Thus, in the nonrelativistic approximation the real part of the energy of the quasistationary states coincides with the energy (3) of the nonrelativistic levels, while the imaginary part is exponentially small and vanishes as $\nu \to \infty$. By analogy with the nonrelativistic case, the quasistationary states with a complex energy satisfying Eqs. (84*s*) or (84*a*) will be called symmetric or antisymmetric.

From Eqs. (84) there follow the equalities

$$F = \pm iG, \tag{90}$$

corresponding to an antisymmetric and a symmetric state. Substituting (84) and (90) into (80) and (81), we obtain

$$\begin{aligned} {}^-\varphi_+(z) &= \pm i\,{}^-\varphi_-(-z) \\ &= AGe^{i\delta}\begin{cases} D_{i\nu}(e^{i\pi/4}\xi), & z>0 \\ \pm e^{i\pi/4}\sqrt{\nu}D_{i\nu-1}(e^{i\pi/4}\xi), & z<0. \end{cases} \end{aligned} \tag{91}$$

The parameter ξ in this formula is complex.

We shall show now that the wave functions of the quasistationary states in the nonrelativistic approximation (87) go over into the nonrelativistic wave functions (2). For this it is convenient to introduce in place of $u_\pm$ another basis:

$$u_{1,2} = \tfrac{1}{2}(u_+ \pm e^{2i\alpha}u_-), \quad e^{2i\alpha} = \frac{\lambda p_\perp + im}{m_\perp} \tag{92}$$

in which the function $\varphi = \varphi_+ u_+ + \varphi_- u_-$ is written in the form

$$\varphi = \varphi_1 u_1 + \varphi_2 u_2, \quad \varphi_{1,2} = \varphi_+ \pm e^{-2i\alpha}\varphi_-. \tag{93}$$

In the standard representation the bispinors u_1 and u_2 have the two lower components and the two upper components equal to zero, respectively. It is not difficult to see that in the nonrelativistic limit the two lower components of the relativistic bispinor φ vanish, and the two upper components form a nonrelativistic spinor. In fact, for $p_\perp \ll m$ we have $\alpha \approx \pi/4$, and for the symmetric solutions we obtain

$$\begin{aligned} &{}^-\varphi_1(z) \approx AGe^{i\delta}\chi^-(\xi), \\ &{}^-\varphi_2(z) \approx AGe^{i\delta}\,\mathrm{sgn}\,z\cdot\chi^+(\xi), \end{aligned} \tag{94s}$$

while for the antisymmetric solutions we have

$$\begin{aligned} &{}^-\varphi_1(z) \approx AGe^{i\delta}\,\mathrm{sgn}\,z\cdot\chi^-(\xi), \\ &{}^-\varphi_2(z) \approx AGe^{i\delta}\chi^+(\xi). \end{aligned} \tag{94a}$$

In the region (87), using the asymptotic forms given in the Appendix for the parabolic-cylinder functions we obtain

$$\chi^-(\xi) \approx \mathrm{const}\cdot\mathrm{Ai}(y), \quad \chi^+(\xi) \approx -\frac{i\,\mathrm{const}}{2\nu^{1/3}}\mathrm{Ai}'(y), \tag{95}$$

where $y = 2\nu^{2/3}(1+\theta) \approx |z|/a - k^{(s,a)}$, so that the quasistationary states (94) go over into the nonrelativistic stationary states (2). We note that these states are vacant.

We now consider the behavior of stationary solutions when their energy E is close to the energy E_0 of the quasistationary levels. As shown in the Appendix, for $e^{-2\pi\nu} \ll 1$ the spectrum of values E_0, given by the equation $c_1(E_0 + i\Gamma/2) = 0$ or $H(-\varkappa) = 0$, differs by an exponentially small amount from the spectrum of the resonance energies E_1, determined by the equation $c_2(E_1) = 0$ or $\beta(-\varkappa_1) = 0$, where $\varkappa_1 = 2\sqrt{\nu}E_1/m_\perp$. Therefore, for $E = E_1 \approx E_0$ it follows from (82) and (74) that $c_2 = 0$ and $|c_1| = 1$, i.e., the well is completely transparent for waves with this resonance energy. If we make use of Eqs. (72), (74), and (78) at $E = E_1$, we obtain for the combinations ${}^-\varphi_{1,2}(z) = {}^-\varphi_+(z) \mp i\,{}^-\varphi_-(z)$, which are important for the nonrelativistic limit, the expressions

$${}^-\varphi_{1,2}(z) = e^{i\delta}AG\begin{cases} \chi^\mp(\xi), & z>0 \\ e^{i\arg F}\chi^{\mp *}(\xi), & z<0 \end{cases}. \tag{96}$$

The solution (96) describes the following process: An antiparticle is incident from $-\infty$ on to the left wall of the well and, without reflection, is annihilated with a particle situated inside the well, while at the right wall a pair is created, the antiparticle of which goes away to $+\infty$. According to (A1)–(A3), in the nonrelativistic limit ${}^-\varphi_1(z)$ inside the well goes over into the nonrelativistic wave functions (2), while ${}^-\varphi_2(z)$ outside the well is exponentially small in comparison with ${}^-\varphi_1(z)$ inside the well. Thus, the incident and transmitted fluxes of antiparticles are exponentially small for $\nu \to \infty$ in comparison with the probability density inside the well. The probability of creation of a pair by the field of the well is then also exponentially small, and coincides exactly with the probability of annihilation of a particle incident on the well. We note that the stationary solution that we are considering describes, inside the well, an occupied state. Thus, the interpretation of the spinor solutions inside the well depends on the boundary conditions at $\pm\infty$. In the nonrelativistic limit one obtains from the quasistationary states and stationary states with a resonance energy the usual nonrelativistic stationary states, but in the former case they are vacant while in the latter case they are occupied.

We note that the resonance effects described require that the energy be fixed with accuracy $\Delta E \lesssim \Gamma$, whence for the spatiotemporal size of the wave packet we obtain $L \sim cT \gtrsim c\hbar\Gamma^{-1}$.

We shall compare Eqs. (89) for a triangular well with the transmission coefficient D_R at one of its walls (the right wall):

$$U_R(z) = \begin{cases} e\varepsilon z, & z>0 \\ 0, & z<0 \end{cases}. \tag{97}$$

The solution of the corresponding stationary problem leads to the result

$$D_R=\frac{2p_\parallel}{(E+p_\parallel)|K|^2}, \tag{98}$$

$$K=D_{i\nu}(-e^{i\pi/4}\varkappa)+e^{i\pi/4}\sqrt{\nu}\frac{m_\perp}{E+p_\parallel}D_{i\nu-1}$$
$$\times(-e^{i\pi/4}\varkappa), p_\parallel=\sqrt{E^2-m_\perp^2}. \tag{99}$$

In the nonrelativistic limit, using the asymptotic formulas in the Appendix, we obtain

$$D_R\approx\frac{\pi\sqrt{k}e^{-2\pi\nu}}{\mathrm{Ai}'^2(-k)+k\mathrm{Ai}^2(-k)},\quad k=\frac{E-m_\perp}{e\varepsilon a}\approx p_\parallel^2 a^2. \tag{100}$$

Assume that the energy E of the stationary problem with the wall (97) coincides with the energy E_0 of a quasistationary level in the triangular well, and that the width Γ of the level is sufficiently small. The quantity $\varkappa_1=2\sqrt{\nu E_0/m_\perp}$ satisfies then the equation $\beta(-\varkappa_1)=0$, and, according to Eqs. (A8)–(A10) and (98), (99), the relation

$$\Gamma=-\frac{2}{\tau}D_R,\quad \tau^{-1}=\frac{e\varepsilon}{4p_\parallel} \tag{101}$$

holds, where τ^{-1} is the frequency of collision with the walls by a classical particle with maximum longitudinal momentum $p_\parallel$. In the nonrelativistic approximation the validity of the relation (101) can be seen directly from comparison of the expressions (89) for Γ and Eqs. (100) taken at $k=k^{(s)}$ or $k^{(a)}$.

In the scalar case the general solution of Eq. (59) for the potential under consideration has the form

$$\varphi(z)=iHA_{1,2}D_{i\nu-1/2}(e^{i\pi/4}\xi)+(FA_{1,2}+iGA_{2,1})$$
$$\times D_{i\nu-1/2}(-e^{i\pi/4}\xi),\quad z\gtrless 0, \tag{102}$$

where the first and second subscripts on A correspond to $z>0$ and $z<0$. The quantities

$$F(\varkappa)=D_{i\nu-1/2}(e^{i\pi/4}\varkappa)D'_{i\nu-1/2}(-e^{i\pi/4}\varkappa)-D_{i\nu-1/2}$$
$$\times(-e^{i\pi/4}\varkappa)D'_{i\nu-1/2}(e^{i\pi/4}\varkappa),$$
$$H(\varkappa)=-2iD_{i\nu-1/2}(e^{i\pi/4}\varkappa)D'_{i\nu-1/2}(e^{i\pi/4}\varkappa), \tag{103}$$
$$G=\frac{i\sqrt{2\pi}}{\Gamma(-i\nu+1/2)}$$

satisfy the relations (72) and (74) if the function $\beta(\varkappa)$ is defined as

$$\beta(\varkappa)=2e^{\pi\nu/2}\mathrm{Re}[e^{-i\pi/4}$$
$$\times D_{i\nu-1/2}(e^{i\pi/4}\varkappa)D'^{*}_{i\nu-1/2}(e^{i\pi/4}\varkappa)]. \tag{104}$$

The asymptotic forms of the functions $D_{i\nu-1/2}(e^{i\pi/4}\xi)$ for $\theta-1\gg\nu^{-2/3}$ and $\theta+1\ll-\nu^{-2/3}$ can be obtained from (77) and (78). Using these asymptotic forms and the relations (72) and (74), we find expressions for the functions $_+\varphi$ and $^+\varphi$:

$$_+\varphi(z)=e^{i\delta}\chi(z),\quad ^+\varphi(z)=e^{i\delta}\chi^*(-z), \tag{105}$$

where

$$\chi(z)=A\begin{cases} iFD_{i\nu-1/2}(e^{i\pi/4}\xi)+\widetilde{H}D_{i\nu-1/2}(-e^{i\pi/4}\xi), & z>0,\\ GD_{i\nu-1/2}(e^{i\pi/4}\xi), & z<0.\end{cases} \tag{106}$$

The functions $_-\varphi$ and $^-\varphi$ can be found from (105) by means of the relations (63), and the coefficients c_1 and c_2 can be found by substituting $_+\varphi$ and $^\pm\varphi$ into the equality (20):

$$c_1=-\exp[2i\arg(AG)+\pi\nu/2+i\pi/4]\widetilde{H},$$
$$c_2=-i\exp(2i\delta)\widetilde{\beta}. \tag{107}$$

As in the spinor case, the function $c_1(E)$ is entire, while $c_2(E)$ is the real part of an entire function on the real axis. The energy levels of quasistationary states with converging waves are determined by the equation $\widetilde{H}=0$, which is equivalent to the two alternative equations

$$D'_{i\nu-1/2}(-e^{i\pi/4}\varkappa)=0,\quad D_{i\nu-1/2}(-e^{i\pi/4}\varkappa)=0. \tag{108s,a}$$

The first of these corresponds to symmetric states, and the second to antisymmetric states. In the nonrelativistic approximation these equations are solved in the Appendix. The result differs from (89) only that Γ is positive, in agreement with the results and considerations of Secs. 2 and 3. The wave functions $^-\varphi$ of the symmetric and antisymmetric quasistationary states have the following form:

$$^-\varphi(z)=AGe^{-i\delta}D_{i\nu-1/2}(e^{i\pi/4}\xi), \tag{109s}$$

$$^-\varphi(z)=AGe^{-i\delta}\,\mathrm{sgn}\, z\cdot D_{i\nu-1/2}(e^{i\pi/4}\xi). \tag{109a}$$

In the nonrelativistic limit they go over into the wave functions (2) (see the Appendix).

The stationary-state resonance energies, determined by the equation $c_2(E)=0$ or $\beta(-\varkappa_1)=0$, differ as in the spinor case from the energies of the quasistationary states, but this difference is exponentially small for $\nu\gg 1$ [see (A15)].

The transmission coefficient at the potential wall (97) is equal to

$$D_R=\frac{2\sqrt{\nu p_\parallel}\,e^{-\pi\nu/2}}{m_\perp|K|^2}, \tag{110}$$

$$K=D'_{-i\nu-1/2}(-e^{-i\pi/4}\varkappa)-e^{-i\pi/4}\frac{\sqrt{\nu p_\parallel}}{m_\perp}$$
$$\times D_{-i\nu-1/2}(-e^{-i\pi/4}\varkappa). \tag{111}$$

In the nonrelativistic approximation for D_R we again obtain the result (100). For energy values coinciding with a quasistationary-level energy in the triangular well and for a sufficiently small width of the levels, the relations (A11)–(A13), (110), and (111) lead to the relation

$$\Gamma=\frac{2}{\tau}D_R. \quad (112)$$

It is not difficult to see that the condition of sufficient smallness of the level width, which led to Eqs. (A8), (A11), (101), and (102), has the form

$$\frac{p_\parallel\Gamma}{e\varepsilon}\ll 1 \quad (113)$$

and is equivalent to the condition $D_R\ll 1$. There are grounds to suppose that an exact relationship between Γ and D_R, not restricted by the condition that D_R be small and taking the place of the relations (101) and (112) for the Fermi and Bose cases, is given, respectively, by constant-sign and alternating-sign series in powers of D_R (cf. Refs. 24 and 25).

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APPENDIX

Asymptotic expressions for the parabolic-cylinder functions of interest to us in the neighborhood of the points $\theta=\pm 1$ for $\nu\gg 1$ have been obtained by Nikishov and Ritus by the method described in Ref. 26:

$$D_{i\nu-\mu}(e^{i\pi/4}z)\approx C(\nu,\mu)\times\begin{cases}2e^{-\pi\nu-i\pi(\mu-1/3)}\mathrm{Ai}(e^{-2\pi i/3}y),\\ 2\mathrm{Ai}(y)+e^{-2\pi\nu-i2\pi(\mu-1/3)}\mathrm{Ai}(e^{2\pi i/3}y),\end{cases} \quad (A1)$$

where

$$C(\nu,\mu)=(2\pi)^{-1/2}e^{3\pi\nu(1-3\mu)/4-i\pi/4-i\nu/2}\nu^{1/6+(i\nu-\mu)/2}. \quad (A2)$$

Here, $z=2\sqrt{\nu}\theta$, $y=\nu^{2/3}(1-\theta^2)$, and the first and second rows pertain to the regions $|\theta\mp 1|\lesssim\nu^{-2/3}\ll 1$, respectively. We give only the leading terms of the dominant and recessive series, corresponding to the contributions of the high and the low saddle points in the contour integral for $D_{i\nu-\mu}$. We shall give also the asymptotic expression for the combination

$$\chi^-(z)\equiv D_{i\nu}(e^{i\pi/4}z)-e^{i\pi/4}\sqrt{\nu}D_{i\nu-1}(e^{i\pi/4}z)$$
$$\approx C(\nu,0)\cdot 2[2\mathrm{Ai}(y)+e^{-2\pi\nu+2\pi i/3}\mathrm{Ai}(e^{2\pi i/3}y)], \quad (A3)$$

valid for $\nu\gg 1$ near $\theta=-1$, when $|\theta+1|\lesssim\nu^{-2/3}\ll 1$.

Finally, the following formula is useful:[23]

$$2e^{\pm i\pi/3}\mathrm{Ai}(e^{\mp 2\pi i/3}y)=\mathrm{Ai}(y)\pm i\mathrm{Bi}(y).$$

It follows from (A1) and (A3) that the equations

$$\chi^-(-\varkappa)=0,\quad D_{i\nu-1/2}(-e^{i\pi/4}\varkappa)=0 \quad (A4)$$

for the complex energy levels of antisymmetric spinor and scalar quasistationary states in the nonrelativistic region ($\nu\gg 1$, $\theta\approx -1$) go over, respectively, into the equations

$$2\mathrm{Ai}(y)\mp\tfrac{1}{2}e^{-2\pi\nu}[\mathrm{Ai}(y)-i\mathrm{Bi}(y)]=0,$$
$$y=2\nu^{2/3}(1+\theta). \quad (A5)$$

The complex roots $y=y_1+iy_2$ of these equations possess a very small imaginary part ($|y_2|\ll 1$). We can therefore replace $\mathrm{Ai}(y)$ by $\mathrm{Ai}(y_1)+iy_2\mathrm{Ai}'(y_1)$ in the first term of the left-hand side of (A5) and we can replace y by y_1 in the second term. Then it follows from (A5) that

$$\mathrm{Ai}(y_1)=0,\quad y_2=\mp\frac{e^{-2\pi\nu}\mathrm{Bi}(y_1)}{4\mathrm{Ai}'(y_1)}=\pm\frac{\pi e^{-2\pi\nu}}{4\mathrm{Ai}'^2(y_1)}. \quad (A6)$$

In the latter equality we have used the value of the Wronskian $\mathrm{Ai}(y)\mathrm{Bi}'(y)-\mathrm{Ai}'(y)\mathrm{Bi}(y)=\pi$, which is independent of y.

The equations ($86s$) and ($108s$) for the complex energy levels of the symmetric spinor and scalar quasistationary states differ from (A4) by differentiation with respect to $\varkappa$, and therefore the corresponding nonrelativistic equations differ from (A5) by differentiation with respect to y. Then, in place of (A6), we obtain

$$\mathrm{Ai}'(y_1)=0,\quad y_2=\mp\frac{e^{-2\pi\nu}\mathrm{Bi}'(y_1)}{4\mathrm{Ai}''(y_1)}=\mp\frac{\pi e^{-2\pi\nu}}{4y_1\mathrm{Ai}^2(y_1)}. \quad (A7)$$

For the zeros of the Airy function and its derivative, and also their numerical values, see Ref. 23.

We note that by using certain exact relations for the parabolic-cylinder functions we can obtain for the complex energy levels equations whose validity is limited only by the condition that the field be small ($e^{-2\pi\nu}\ll 1$). In fact, setting $\varkappa=\varkappa_1+i\varkappa_2$ and assuming that $|\varkappa_2|$ is sufficiently small (a weak field), we replace the first of Eqs. (A4) by its linear expansion in $\varkappa_2$. Then

$$\varkappa_2=\frac{\chi^-(-\varkappa_1)}{i\dot{\chi}^-(-\varkappa_1)}$$
$$=-\left(\sqrt{\nu}+\frac{1}{2}\varkappa_1\right)\frac{e^{-\pi\nu/2}[1+i\beta(-\varkappa_1)]}{|\dot{\chi}^-(-\varkappa_1)|^2}, \quad (A8)$$

where the dot denotes the derivative with respect to $-\varkappa_1$ and we have used Eq. (85) and, for the Wronskian, the expression 8.2(11) from Ref. 22, according to which, for any real $\varkappa_1$,

$$\mathrm{Re}(\chi^-\chi^{+*})\equiv|D_{i\nu}(-e^{i\pi/4}\varkappa_1)|^2-\nu|D_{i\nu-1}\times(-e^{i\pi/4}\varkappa_1)|^2=e^{-\pi\nu/2}. \quad (A9)$$

At the same time, according to (73), $\mathrm{Im}(\chi^-\chi^{+*})\equiv e^{-\pi\nu/2}\beta(-\varkappa_1)$, and, therefore, the quantity $\varkappa_2$ will be real if $\varkappa_1$ satisfies the equation

$$\beta(-\varkappa_1)=0. \quad (A10)$$

Analogously, the second Eq. (A4) in the approximation linear in $\varkappa_2$ gives

$$\varkappa_2=\frac{D_{i\nu-1/2}(-e^{i\pi/4}\varkappa_1)\dot{D}^*_{i\nu-1/2}(-e^{i\pi/4}\varkappa_1)}{i|\dot{D}_{i\nu-1/2}(-e^{i\pi/4}\varkappa_1)|^2}$$

$$=\frac{e^{-\pi\nu/2}[1-i\beta(-\varkappa_1)]}{2|\dot{D}_{i\nu-1/2}(-e^{i\pi/4}\varkappa_1)|^2}. \qquad \text{(A11)}$$

In fact, for real $\varkappa_1$ the imaginary part of the numerator coincides with half the Wronskian 8.2(11) from Ref. 22:

$$\text{Im } D_{i\nu-1/2}(-e^{i\pi/4}\varkappa_1)\dot{D}^*_{i\nu-1/2}(-e^{i\pi/4}\varkappa_1)=\tfrac{1}{2}e^{-\pi\nu/2}, \qquad \text{(A12)}$$

and its real part is equal to $e^{-\pi\nu/2}\beta(-\varkappa_1)$. Therefore, the value of $\varkappa_2$ will be real if

$$\beta(-\varkappa_1)=0. \qquad \text{(A13)}$$

Using for D and χ^- and their derivatives in the nonrelativistic limit the dominant terms of the asymptotic forms (A1) and (A3) and their derivatives, we see that Eqs. (A8) and (A10) lead to Eqs. (A6) with the upper sign, while Eqs. (A11) and (A13) lead to Eqs. (A6) with the lower sign.

For the symmetric spinor and scalar quasistationary states, Eqs. (A8) and (A11) are replaced, respectively, by

$$\varkappa_2=\frac{e^{-\pi\nu/2}[1-i\beta(-\varkappa_1)]}{\sqrt{\nu}(1+\theta_1)|\chi^-(-\varkappa_1)|^2},$$

$$\varkappa_2=-\frac{e^{-\pi\nu/2}[1+i\beta(-\varkappa_1)]}{2\nu(1-\theta_1^2)|D_{i\nu-1/2}(-e^{i\pi/4}\varkappa_1)|^2}. \qquad \text{(A14)}$$

It can be seen that the quantities $\varkappa_2$ will be real under the same conditions (A10) and (A13) on $\varkappa_1$. In the nonrelativistic limit these conditions, together with (A14), reduce to (A7). If we solve Eqs. (A4) retaining the next terms of the expansion in $\varkappa_2$, we obtain in place of (A10) and (A13), respectively,

$$\beta(-\varkappa_1)=\frac{\varkappa_2}{4\sqrt{\nu+2\varkappa_1}}, \quad \beta(-\varkappa_1)=-\tfrac{1}{8}\varkappa_1\varkappa_2^3, \qquad \text{(A15)}$$

where $\varkappa_2$ is determined by Eqs. (A8) and (A11) for $\beta(-\varkappa_1)=0$.

We note that for the scalar case the function $\beta(x)$ can be expressed in terms of the real function $W(\nu,x)$ that is defined and tabulated in Ref. 23:

$$\beta(x)=qW(\nu,x)W'(\nu,x)-q^{-1}W(\nu,-x)W'(\nu,-x),$$

$$q=\sqrt{1+e^{2\pi\nu}}+e^{\pi\nu}. \qquad \text{(A16)}$$

It follows from the tables that for $\nu=1$ the first three negative zeros of the function $\beta(x)$ correspond to the resonance energies $E/m_\perp=1.47$, 2.06, and 2.43. With increase of ν these values approach 1.

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