The magnetohydrodynamics of rotating superfluid solutions

A. D. Sedrakyan and D. M. Sedrakyan,

Byurakan Astrophysical Observatory, Armenian Academy of Sciences

G. Röpke

Rostock University, 0-2500 Rostock, FRG

(Submitted 27 July 1992; revised 9 February 1993)
Zh. Eksp. Teor. Fiz. **104**, 2754–2760 (August 1993)

Proceeding from conservation laws, we derive phenomenologically the equations that describe the unsteady magnetohydrodynamics of rotating two-condensate solutions. We also derive the equations of motion of quantum vortices of the neutral and charged superfluid components of the solution.

1. As is well known, superfluid two-condensate systems can form in certain conditions. Such a system constitutes a mixture of neutral and charged superfluid liquids in the presence of a third normal charged component, which ensures the local neutrality of the system.[1]

An example is the solution of protons and neutrons in matrices of heavy metals below the point of phase transition of the neutron–proton liquid to the superconducting state.[2,3]

Another system with two superfluid condensates is the *n–p–e* phase of neutron stars, where Cooper pairs of neutrons and protons form as a result of nuclear interaction. The proton number density in this phase of the star amounts to 1–10% of the neutron number density ($n_n \approx 1.7 \times 10^{38}$ cm^{-3}), and the proton charge is balanced by the electron charge ($n_p = n_e$) (Ref. 4). There is probably no neutron–proton pairing[5] because of the great difference between the chemical potentials of these particles.

Three-fluid magnetohydrodynamics equations describing superconducting two-condensate systems in the absence of dissipation have been suggested by Vardanyan and Sedrakyan.[1] The same paper demonstrated that there exists an effect in which the proton condensate is entrained by the neutron condensate, in analogy with the effect in which He^4 atoms entrain the He^3 component in superfluid solutions of these liquids.[6] As a result, when the system is uniformly rotated, in addition to the quantum vortices of the neutron component a magnetic vorticity lattice of the proton component arises (in the absence of an external magnetic field) generated by proton drag currents.[1,7]

The purpose of this paper is to generalize the equations of Ref. 1 to the case of unsteady rotation of superfluid solutions. This will, among other things, provide the equations of motion of quantum vortices of both superfluid components and also the forces with which these vortices interact with a solid surface, which is especially interesting in studying the unsteady rotational dynamics of the *n–p–e* phase of a neutron star.[8]

The first to derive the phenomenological equations of the dissipative hydrodynamics of a rotating superfluid (helium II) were Bekarevich and Khalatnikov,[9] who used a method based on the differential form of the conservation laws (see Ref. 10). Various aspects of the dissipative hydrodynamics of a rotating superfluid liquid were also studied in Refs. 11–14; for instance, effects caused by deformation of the vorticity lattice. Below we follow the Bekarevich–Khalatnikov method in deriving the equations of time-dependent magnetohydrodynamics.[9]

Note that the nondissipative magnetohydrodynamics of superfluid solutions was also studied by Holm and Kupershmidt[15] and by Mendell and Lindblom.[16] For instance, Mendell and Lindblom,[16] using the framework of the Hamiltonian approach of Ref. 14, obtained the form of the nondissipative forces acting on the vortices in the general case of N superfluid condensates. They demonstrated that the equations of nondissipative three-fluid magnetohydrodynamics of Ref. 1 are a particular case of their equations.

Before we derive the equations, we note that under the conditions of the *n–p–s* phase of neutron stars, the elementary excitations of the superfluid condensates are coupled with each other and with the normal (nonsuperfluid) plasma, with the exchange between these excitations and the normal plasma occurring quite rapidly. This is due to the strong nuclear interaction of the neutron and proton excitations and the Coulomb interaction of the two types of excitation with relativistic electrons (see, e.g., Ref. 7). Thus, we can assume in the given case that the normal components of the solution are completely entrained.

We also note that at typical temperatures of the system the electron number density usually greatly exceeds the number density of the elementary excitations and the effects of mutual friction of vortices arise from by the interaction of the vortices with the electrons.

2. The system of magnetohydrodynamic equations of a rotating solution, allowing for dissipation, contains the equations of mass and momentum conservation and the entropy equation which we write as

$$\frac{\partial \rho_\alpha}{\partial t} + \operatorname{div} \mathbf{j}_\alpha = 0, \tag{1}$$

$$\frac{\partial \rho_e}{\partial t} + \operatorname{div} \mathbf{j}_e = 0, \tag{2}$$

 1063-7761/93/080262-04$10.00

$$\frac{\partial j_i}{\partial t}+\frac{\partial}{\partial x_k}(\Pi^0_{ik}+\pi_{ik})=0, \tag{3}$$

$$T\left[\frac{\partial S}{\partial t}+\operatorname{div}\left(S\mathbf{v}+\frac{\mathbf{q}}{T}\right)\right]=R; \tag{4}$$

here the subscripts $\alpha,\beta=1,2$ number the superfluid condensates, the subscript e stands for the electron liquid, ρ_α designates the number density of particles of the α species, ρ_e the electron number density (the total density of the solution ρ is equal to $\Sigma_\alpha(\rho_\alpha+\rho_e)$, $\mathbf{j}=\Sigma_\alpha\mathbf{j}_\alpha+\mathbf{j}_e$ is the solution momentum per unit volume, S and T are the entropy density and the temperature of the normal component of the system (electrons and elementary excitations of the condensates), and $\mathbf{v}$ is the velocity of the center of mass of the system of electrons and normal excitations. Summation over indices is assumed only if a summation sign appears in front of the respective expression.

The mass flux of particles of the α species in the laboratory system of coordinates is given by the following expression:[1,6]

$$\mathbf{j}=\mathbf{p}^{(s)}_\alpha+\rho_\alpha\mathbf{v}=\sum_\beta \rho^{(s)}_{\alpha\beta}(\mathbf{v}_\beta-\mathbf{v})+\rho_\alpha\mathbf{v}, \tag{5}$$

where the $\mathbf{p}^{(s)}_\alpha$ are the relative momenta of the superfluid condensates, and $\mathbf{v}_\alpha$ are the velocities of these components. The density matrix $\rho^{(s)}_{\alpha\beta}$ of the superfluid condensates, is symmetric, and its off-diagonal elements describe the drag on each component of the solution by the two superfluids. Note that the superfluid velocities are determined, as in Ref. 1, in terms of the phases of the order parameters and the vector potential as follows: $\mathbf{v}_\alpha=(\hbar/2m_\alpha)\nabla\phi_\alpha-(e_\alpha/m_\alpha c)\mathbf{A}$.

The quantities π_{ik}, $\mathbf{q}$, and $R\geqslant 0$ in Eqs. (1)–(4) are nonequilibrium correction terms caused by dissipative processes and are usually determined by the requirement that energy be conserved, or

$$\frac{\partial E}{\partial t}+\operatorname{div}(\mathbf{Q}_0+\mathbf{Q})=0, \tag{6}$$

and by the condition that the dissipation function R be a positive definite quadratic form.[10] Here $\mathbf{Q}_0$ stands for the steady-state energy flux per unit volume.

The solution energy E per unit volume in the laboratory system of coordinates is related to the internal energy ϵ of that unit volume in the system of coordinates where $\mathbf{v}=0$ holds by the following expression:

$$E=\frac{\rho v^2}{2}+\sum_\alpha \mathbf{p}^{(s)}_\alpha\cdot\mathbf{v}+\epsilon+E_{\mathrm{em}}, \tag{7}$$

where $E_{\mathrm{em}}=(8\pi)^{-1}(E^2+B^2)$ is the energy of the electromagnetic field, and ϵ is determined by the thermodynamic identity

$$d\epsilon=T\,dS+\sum_\alpha\left[\mu_\alpha d\rho_\alpha+\mu_e d\rho_e+\mathbf{p}^{(s)}_\alpha\cdot d(\mathbf{v}_\alpha-\mathbf{v})+\lambda_\alpha\nu_\alpha d\left(\omega_\alpha+\frac{e_\alpha}{m_\alpha c}\mathbf{B}\right)\right], \tag{8}$$

with μ_α and μ_e the chemical potentials of particles of species α and of electrons, respectively.

The last term on the right-hand side of Eq. (8) is the change in the energy of the superfluid solution because the curl of the superfluid condensates is nonzero,

$$\operatorname{curl}\mathbf{v}_\alpha=\frac{\pi\hbar}{m_\alpha}\nu_\alpha n_\alpha\delta(\mathbf{r}_\alpha-\mathbf{r}_{k\alpha})-\frac{e_\alpha}{m_\alpha c}\mathbf{B}\equiv\omega_\alpha, \tag{9}$$

where e_α and m_α respectively are the electric charge and mass of individual particles forming a Cooper pair, n_α is the vortex number density in the plane perpendicular to the unit vector $\nu_\alpha\equiv\omega_\alpha/\omega_\alpha$, and λ_α is determined by the energy per unit length of a vortex. Note that $e_\alpha=0$ for neutrons, $e_\alpha=e$ for protons, and $e_\alpha=-e$ for electrons.

The equilibrium momentum-flux tensor has the form

$$\Pi^0_{ik}=\sum_\alpha(\rho v_i v_k+p^{(s)}_{\alpha k}v_{\alpha i}+v_k p^{(s)}_{\alpha_i}+v_k p^{(s)}_{\alpha i})+P\delta_{ik}-\frac{1}{4\pi}\left(B_iB_k-\frac{1}{2}\delta_{ik}B_iB_k\right), \tag{10}$$

where $P=-\epsilon+\Sigma_\alpha\mu_\alpha\rho_\alpha+\mu_e\rho_e+TS$ is the pressure, and the last term in parentheses is the Maxwell stress tensor.

We now write the equation of motion of the superfluid components in the form

$$\dot{\mathbf{v}}_\alpha+(\mathbf{v}_\alpha\cdot\nabla\,\mathbf{v}_\alpha)+\nabla\,[\mu_\alpha-\tfrac{1}{2}(\mathbf{v}_\alpha-\mathbf{v})^2]=\mathbf{f}_\alpha, \tag{11}$$

where $\mathbf{f}_\alpha$ is the yet-to-be-determined additional term caused by both vortex motion and the action of the electromagnetic field. This equation can be written in a form more convenient for the discussion below:

$$\dot{\mathbf{v}}_\alpha=\mathbf{f}^0_\alpha+(\mathbf{v}_\alpha\times\omega^0_\alpha)-\nabla(\mu_\alpha-\tfrac{1}{2}v^2+\mathbf{v}\cdot\mathbf{v}_\alpha), \tag{12}$$

where we have introduced the notation

$$\mathbf{f}^0_\alpha=\mathbf{f}_\alpha-\frac{e_\alpha}{m_\alpha c}\mathbf{v}_\alpha\times\mathbf{B},$$

$$\omega^0_\alpha=\omega_\alpha+\frac{e_\alpha}{m_\alpha c}\mathbf{B}.$$

Equations (1)–(12) together with the Maxwell equations constitute a closed system that determines the dissipative terms in equations.

To find the unknown quantities entering into the system of equations, we proceed in the usual manner. That is, we differentiate Eq. (6) with respect to time and express the time derivatives in terms of Eqs. (1)–(4) and (12). In the process we transform the derivatives $\dot{\mathbf{B}}$ and $\dot{\omega}_\alpha$ using the Maxwell equation $\operatorname{curl}\mathbf{E}=-c^{-1}(\partial\mathbf{B}/\partial t)$ and Eq. (12) in the form

$$\lambda_\alpha\nu_\alpha\operatorname{curl}\dot{\mathbf{v}}_\alpha=\lambda_\alpha\nu_\alpha\operatorname{curl}[\mathbf{f}^0_\alpha+\omega^0_\alpha\times(\mathbf{v}-\mathbf{v}_\alpha)]-\lambda_\alpha\nu_\alpha\omega^0_\alpha\times\mathbf{v}. \tag{13}$$

After simple transformations we get

$$\dot{E}=-\text{div}\left\{\mathbf{Q}_0+\mathbf{q}+\pi_{ik}\mathbf{v}_i+\lambda\nu_\alpha\times\left[\mathbf{f}_\alpha-\frac{e_\alpha}{m_\alpha}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_\alpha\times\mathbf{B}\right)\right.\right.$$

$$\left.\left.+\omega_\alpha^0\times(\mathbf{v}-\mathbf{v}_\alpha)\right]\right\}+R-\mathbf{j}\cdot\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)+\mathbf{q}\cdot\frac{\nabla T}{T}$$

$$+\left[\pi_{ik}+\sum_\alpha\left(-\lambda_\alpha\omega_\alpha^0\delta_{ik}+\frac{\lambda_\alpha\omega_{\alpha i}^0\omega_{\alpha k}^0}{\omega_\alpha^0}\right)\right]\frac{\partial v_i}{\partial x_k}$$

$$+\sum_\alpha(\mathbf{p}_\alpha^{(s)}+\text{curl}\,\lambda_\alpha\nu_\alpha)$$

$$\times\left[\mathbf{f}_\alpha-\frac{e_\alpha}{m_\alpha}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_\alpha\times\mathbf{B}\right)+\omega_\alpha^0\times(\mathbf{v}-\mathbf{v}_\alpha)\right], \quad (14)$$

where

$$\mathbf{Q}_0=(\mu_\alpha-\tfrac{1}{2}v^2)\mathbf{j}_\alpha+(\mu_e+\tfrac{1}{2}v^2)\rho_e\mathbf{v}+ST\,\mathbf{v}+\mathbf{v}(\mathbf{j}_\alpha\cdot\mathbf{v})$$

$$+\mathbf{p}_\alpha^{(s)}(\mathbf{v}\cdot\mathbf{v}_\alpha)+\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}. \quad (15)$$

According to Eqs. (4) and (6),

$$R=\frac{\mathbf{j}^2}{\sigma}-\mathbf{q}\cdot\frac{\nabla T}{T}-\left[\pi_{ik}+\sum_\alpha\left(-\lambda_\alpha\omega_\alpha^0\delta_{ik}+\frac{\lambda_0\omega_{\alpha i}^0\omega_{\alpha k}^0}{\omega_\alpha^0}\right)\right]$$

$$\times\frac{\partial v_i}{\partial x_k}-\sum_\alpha\left\{(\mathbf{p}_\alpha^{(s)}+\text{curl}\,\lambda_\alpha\nu_\alpha)\right.$$

$$\left.\cdot\left[\mathbf{f}_\alpha-\frac{e_\alpha}{m_\alpha}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_\alpha\times\mathbf{B}\right)+\omega_\alpha^0\times(\mathbf{v}-\mathbf{v}_\alpha)\right]\right\}. \quad (16)$$

As is well known, in the case of small deviations of the system from equilibrium, the dissipation function must be a positive definite quadratic form of the velocity gradients, thermodynamic quantities, and derivatives of field potentials. The first and second terms in (16) correspond to ohmic dissipation and dissipation related to the heat flux due to the temperature gradient, with $\mathbf{j}=\sigma(\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B})$ and $\mathbf{q}=-\kappa\nabla T$, where σ and κ are the electric and thermal conductivity coefficient, respectively.

Also,

$$\pi_{ik}=\sum_\alpha\left(\lambda_\alpha\omega_\alpha^0\delta_{ik}-\frac{\lambda_\alpha\omega_{\alpha i}^0\omega_{\alpha k}^0}{\omega_\alpha^0}\right)\frac{\partial v_i}{\partial x_k}+\eta_{iklm}^\alpha\frac{\partial v_l}{\partial x_m}, \quad (17)$$

where η_{iklm}^α is the viscosity tensor, and the expression in parentheses is correction to the momentum flux tensor related to the vortices, with the first term corresponding to pressure renormalization and the second to vortex filament stretching. Finally, the most general form of the expression for the force acting on the superfluid condensate of the α species is

$$\mathbf{f}_\alpha=\frac{e_\alpha}{m_\alpha}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_\alpha\times\mathbf{B}\right)-\omega_\alpha^0\times(\mathbf{v}-\mathbf{v}_\alpha)+\alpha_\alpha\omega_0^0$$

$$\times\left[\sum_\beta\rho_{\alpha\beta}^{(s)}(\mathbf{v}_\beta-\mathbf{v})+\text{curl}\,\lambda_\alpha\nu_\alpha\right]+\beta_\alpha\nu_\alpha$$

$$\times\left\{\omega_\alpha^0\times\left[\sum_\beta\rho_{\alpha\beta}(\mathbf{v}_\beta-\mathbf{v})+\text{curl}\,\lambda_\alpha\nu_\alpha\right]\right\}$$

$$-\gamma_\alpha\nu_\alpha\left[\omega_\alpha^0\cdot\sum_\beta\rho_{\alpha\beta}(\mathbf{v}_\beta-\mathbf{v}+\text{curl}\,\lambda_\alpha\nu_\alpha)\right]. \quad (18)$$

For the force acting on the normal component we have

$$\mathbf{f}=-\sum_\alpha\mathbf{f}_\alpha. \quad (19)$$

The coefficients α_α, β_α, and γ_α are determined from the microscopic theory of scattering of particles of the normal component on the vortex strings of the condensates.

3. Let us find the equations of motion of the vortex strings in a two-condensate solution. We ignore longitudinal drag and put $\gamma=0$.[1)] Substituting the expression for $\mathbf{f}_\alpha$ into the equation of superfluid velocities (12), finding the curl of both sides of the resulting equation, and employing the Maxwell equation curl $\mathbf{E}=-c^{-1}(\partial\mathbf{B}/\partial t)$, we get

$$\frac{\partial\omega_\alpha^0}{\partial t}=\text{curl}\left[\mathbf{f}_\alpha-\frac{e_\alpha}{m_\alpha}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_\alpha\times\mathbf{B}\right)+\mathbf{v}_\alpha\times\omega_\alpha^0\right]$$

$$=\text{curl}(\mathbf{v}_{L\alpha}\times\omega_\alpha^0), \quad (20)$$

where

$$\mathbf{v}_{L\alpha}=\mathbf{v}-\alpha_\alpha\rho_\alpha\left(\frac{\rho_{\alpha\alpha}}{\rho_\alpha}\mathbf{v}_\alpha+\frac{\rho_{\alpha\beta}}{\rho_\alpha}\mathbf{v}_\beta+\frac{1}{\rho_\alpha}\text{curl}\,\lambda_\alpha\nu_\alpha-\mathbf{v}\right)$$

$$-\beta_\alpha\rho_\alpha\nu_\alpha\times\left(\frac{\rho_{\alpha\alpha}}{\rho_\alpha}\mathbf{v}_\alpha+\frac{\rho_{\alpha\beta}}{\rho_\alpha}\mathbf{v}_\beta+\frac{1}{\rho_\alpha}\text{curl}\,\lambda_\alpha\nu_\alpha-\mathbf{v}\right) \quad (21)$$

is, obviously, the velocity with which a vortex filament of condensate α moves. The equation of vortex motion, (21), can be reduced to

$$\rho_\alpha\left(\frac{\rho_{\alpha\alpha}}{\rho_\alpha}\mathbf{v}_\alpha+\frac{\rho_{\alpha\beta}}{\rho_\alpha}\mathbf{v}_\beta+\frac{1}{\rho_\alpha}\text{curl}\,\lambda_\alpha\nu_\alpha-\mathbf{v}_{L\alpha}\right)$$

$$\times\omega_\alpha^0-\eta(\mathbf{v}_L-\mathbf{v})+\eta'(\mathbf{v}_L-\mathbf{v})\times\omega_\alpha^0=0, \quad (22)$$

where the coefficients α_α and β_α are related to η and η' by

$$\alpha_\alpha=\frac{\rho_\alpha(\omega_\alpha^0)^2(1-\eta'/\rho_\alpha)}{\eta^2+(\rho_\alpha\omega_\alpha^0)^2(1-\eta'/\rho_\alpha)^2}, \quad (23)$$

$$\beta_\alpha=\frac{\eta\omega_\alpha^0}{\eta^2+(\rho_\alpha\omega_\alpha^0)^2(1-\eta'/\rho_\alpha)^2}. \quad (24)$$

The first two terms on the right-hand side of Eq. (22) correspond to the interaction of the free and entrained superfluid fluxes with a vortex via the Magnus effect. The third term reflects the variations in the flux surrounding a vortex due to vortex filament oscillations. Equation (22) also implies an absence of forces caused by the interaction between the ambient condensate of the α species and a vortex of the condensate of the other species.

Interestingly, the interaction force is proportional to the mechanical circulation ω_α^0 of a vortex both for the charged condensate and for the neutral one. This agrees with the assertion that in charged superfluid systems (say, in type-II superconductors) the force of interaction of a vortex with the ambient flux (the transport current), known as the Lorentz force, is the exact analog of the Magnus force in neutral superfluid liquids.

Let us consider the boundary conditions that the superfluid components must satisfy on a free surface and on the surface of a solid. Proceeding from Eqs. (14) and (17) and reasoning in a manner similar to that of Bekarevich and Khalatnikov,[9] we conclude that the vortices of both condensates are perpendicular to the free surface.

The boundary conditions on a solid surface moving with a velocity **u** have the form

$$\mathbf{v}_{L\alpha}-\mathbf{u}=\zeta\nu_\alpha\times(\mathbf{N}\times\nu_\alpha)+\zeta'\mathbf{N}\times\nu_\alpha, \qquad (25)$$

where **N** is a unit vector normal to the surface, and the coefficients ζ and ζ' are determined from microscopic theory. The limit of $\zeta\to 0$ and $\eta'\to 0$ corresponds to pinning the vortices to the surface, and the limit of $\zeta\to\infty$ and $\zeta'\to\infty$ to their free slippage. The boundary conditions (25) can be reduced to a form similar to Eq. (22):

$$\lambda_\alpha\mathbf{N}\times\omega_\alpha^0-\xi(\mathbf{v}_L-\mathbf{u})-\xi'(\mathbf{v}_L-\mathbf{u})\times\nu=0, \qquad (26)$$

where the coefficients ζ_α and ζ'_α are related to ξ and ξ' by the following formulas:

$$\zeta_\alpha=\frac{\lambda_\alpha\omega_\alpha^0\zeta'_\alpha}{\xi_\alpha^2+\xi_\alpha^{*2}}, \qquad (27)$$

$$\zeta'_\alpha=\frac{\lambda_\alpha\omega_\alpha^0\zeta_\alpha}{\xi_\alpha^2+\xi_\alpha'^2}. \qquad (28)$$

In the particular case of a steadily rotating spherical vessel, the neutron and proton vortices, which have a straight structure in the vessel's main volume, change their form in the narrow layer adjacent to the solid surface and emerge at the surface at right angles to the surface,[7] that is, we have the identities $\mathbf{N}\times\nu_\alpha=0$ and $\mathbf{v}_{L\alpha}=\mathbf{u}$.

One of the authors (A. D. S.) is grateful to the American Astronomical Society for financial help.

[1)] In the n–p–e phase of a neutron star this approximation is justified because of the smallness of the deviations of the proton and neutron vortices from their average direction owing to the very large force of attraction.

[1] G. A. Vardanyan and D. M. Sedrakyan, Zh. Eksp. Teor. Fiz. **81**, 1731 (1981) [Sov. Phys. JETP **54**, 919 (1981)].
[2] A. F. Andreev and I. M. Lifshits, Zh. Eksp. Teor. Fiz. **56**, 2057 (1969) [Sov. Phys. JETP **29**, 1107 (1969)].
[3] B. T. Geĭlikman, Fiz. Tverd. Tela (Leningrad) **15**, 3293 (1973) [Sov. Phys. Solid State **15**, 2194 (1974)].
[4] V. A. Ambartsumyan and G. S. Saakyan, Astron. Zh. **37**, 193 (1960) [Sov. Astron. **X**, XXX (1960)].
[5] T. Alm, G. Röpke, and M. Schmidt, Z. Phys. A **337**, 355 (1990).
[6] A. F. Andreev and E. M. Bashkin, Zh. Eksp. Teor. Fiz. **69**, 319 (1975) [Sov. Phys. JETP **42**, 164 (1975)].
[7] D. M. Sedrakyan and K. M. Shakhabasyan, Usp. Fiz. Nauk **161**, 3 (1991) [Sov. Phys. Usp. **34**, 555 (1991)].
[8] A. D. Sedrakyan and D. M. Sedrakyan, Zh. Eksp. Teor. Fiz. **102**, 721 (1992) [Sov. Phys. JETP **75**, 395 (1992)].
[9] I. M. Bekarevich and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **40**, 923 (1961) [Sov. Phys. JETP **13**, 652 (1961)].
[10] I. M. Khalatnikov, *Theory of Superfluidity*, Nauka, Moscow (1971) [in Russian].
[11] A. F. Andreev and M. Yu. Kagan, Zh. Eksp. Teor. Fiz. **86**, 546 (1984) [Sov. Phys. JETP **59**, 318 (1984)].
[12] G. Baym and E. Chandler, J. Low Temp. Phys. **50**, 57 (1983).
[13] G. E. Volovik and V. S. Dotsenko, Zh. Eksp. Teor. Fiz. **78**, 132 (1980) [Sov. Phys. JETP **51**, 65 (1980)].
[14] E. B. Sonin, Rev. Mod. Phys. **59**, 87 (1987).
[15] D. D. Holm and B. A. Kupershmidt, Phys. Rev. A **36**, 3947 (1987).
[16] G. Mendell and L. Lindblom, Ann. Phys. (N.Y.) **205**, 110 (1991).

Translated by Eugene Yankovsky

This article was translated in Russia, and it is reproduced here the way it was submitted by the translator, except for the stylistic changes by the Translator Editor.