The method of functional integration for one-dimensional localization, higher correlators, and the average current flowing in a mesoscopic ring in an arbitrary magnetic field

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The Abrikosov–Ryzhkin formulation of the problem of a particle in a one-dimensional random potential is used to obtain closed representations for the averaged physical quantities in the form of functional integrals, which are calculated exactly without using any expansions. Using these integrals, expressions are obtained for multipoint correlators of higher powers of the density, which are then used to find the mean-square dispersion in the size of a localized wave function and the mean value of the unattenuated current in a mesoscopic ring with an arbitrary magnetic flux. (In the limit of small fields this problem was recently solved by O. N. Dorokhov.[9]) The case of a finite correlation length for the random potential is also considered.

1. INTRODUCTION AND FORMULATION OF THE MODEL

The fundamental macroscopic quantum phenomena of Anderson localization has been studied in most detail for the case of one-dimensional systems, for which there exist exact results of very general character. For one-dimensional systems, all the eigenfunctions of the Hamiltonian

$$\hat{\mathscr{H}}=-\frac{d^2}{dx^2}+U(x), \tag{1}$$

with a random potential $U(x)$ are localized wave packets (detailed expositions of the techniques required for this problem, as well as rigorous formulations and a bibliography, can be found in Ref. 1). This assertion is also valid in the high-energy limit discussed in this paper.

Quantities that can be computed directly are averages over ensembles of the potentials $U(x)$. The measure for the averaging process is recovered from the spatial correlation properties present in the distribution of the realizations; in the simplest case, i.e., potentials of "white noise" type, it has the form

$$\mathscr{D}U\exp\left(-\frac{1}{2D}\int_{-L}^{L}U^2(x)dx\right),$$
$$\langle U(x)U(x')\rangle=D\delta(x-x'). \tag{2}$$

Here $(-L,L)$ is the interval on the real line occupied by the system. Anderson[2] showed that one choice of a criterion for localization of a state with energy E is a nonzero value for the density-density correlator in the thermodynamic limit:

$$p_E(x,x')=\lim_{L\to\infty}\left\langle\sum_n\delta(E-E_n)\left|\Psi_n(x)\right|^2\left|\Psi_n(x')\right|^2\right\rangle$$
$$=\lim_{L\to\infty}\lim_{\varepsilon\to+0}\frac{\varepsilon}{\pi}\langle|G(x,x'|E+i\varepsilon)|^2\rangle, \tag{3}$$

where $\Psi_n(x)$ are eigenfunctions of $\hat{\mathscr{H}}$:

$$\hat{\mathscr{H}}\Psi_n(x)=E_n\Psi_n(x)$$

and $G(x,x'|E+i\varepsilon)$ is the resolvent of $\hat{\mathscr{H}}$:

$$(\hat{\mathscr{H}}-E)G(x,x'|E+i\varepsilon)=\delta(x-x'). \tag{4}$$

In fact, the wave functions of the continuous spectrum at each point are $\propto 1/L^{1/2}$, while the sum over n gives a factor L, so that $p_E(x,x')\sim 1/L\to 0$. For a potential that is spatially uniform on the average the probability of observing a state localized in the neighborhood of a given point is $\propto 1/L$; however, in this case $\Psi_n(x)$ itself does not depend on L in the limit $L\to\infty$, so that only normalizable states can contribute to $p_E(x,x')$. [It is understood that the boundary conditions at the ends of the interval $(-L,L)$ ensure that $\hat{\mathscr{H}}$ is Hermitian.]

Two approaches have been developed to compute quantities of the type (3). The first, the so-called "phase formalism," in principle allows us to represent all possible averages with respect to the ensemble (2) for arbitrary energies E (Refs. 3 and 4) in the form of solutions of partial differential equations of the Fokker–Planck type (a review can be found in Ref. 1). However, it gives explicit answers only in a regime that is quasiclassical from the kinetic point of view, i.e., when

$$D/E^{3/2}\ll 1. \tag{5}$$

In this regime the second approach[5] can be used: isolation and summation of infrared-singular terms in a perturbation series with respect to the parameter (5) (see also the review in Ref. 6). The direct implementation of such a program requires rather refined mathematical constructions and lengthy calculations.

In Refs. 7 and 8 it was noted that the sum of the leading terms in perturbation theory with respect to the potential $U(x)$ is equivalent to a certain average for a spin-1/2 particle in a random magnetic field with Gaussian sta-

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tistics. (In essence, this representation was used in Ref. 9). In this paper we will derive this spin model by a somewhat more transparent argument.

We apply the following boundary conditions to the wave function of a particle on the segment $(-L,L)$:

$$\frac{d}{dx}\Psi_n(x=-L)=\Psi_n(x=L)=0. \tag{6}$$

The Green's function can be constructed in terms of the solutions $u(x)$, $\tilde{u}(x)$ of the Cauchy problem for the equation

$$(\hat{\mathscr{H}}-E)u=(\hat{\mathscr{H}}-E)\tilde{u}=0,$$
$$u'(-L)=0,\quad u(-L)=1,\quad \tilde{u}'(L)=1,\quad \tilde{u}(L)=0, \tag{7}$$
$$G(x,x')=\frac{1}{W}\begin{cases}u(x)\tilde{u}(x'), & x<x',\\ u(x')\tilde{u}(x), & x'<x.\end{cases}$$

Here W is the Wronskian of the solutions u and $\tilde{u}$:

$$W=-u'(x)\tilde{u}(x)+u(x)\tilde{u}'(x). \tag{8}$$

All the physical quantities of interest to us can be written in terms of only one solution, for instance $u(x)$ (see below). For the function $u(x)$ it is natural to define "plane-wave components" $v_1(x)$ and $v_2(x)$:

$$v_1(x)=e^{-ikx}[u'(x)+iku(x)],$$
$$v_2(x)=-e^{ikx}[u'(x)-iku(x)],\quad E=k^2, \tag{9}$$
$$u(x)=\frac{1}{2ik}[v_1(x)e^{ikx}+v_2(x)e^{-ikx}],$$

so that $v_1=0$ ($v_2=0$) for the case of a plane wave traveling from left to right (right to left). Equation (7) is equivalent to the following first-order matrix equation:

$$\frac{d}{dx}\begin{pmatrix}v_1(x)\\ v_2(x)\end{pmatrix}=\begin{pmatrix}U(x)/2ik & U(x)e^{-2ikx}/2ik\\ -U(x)e^{2ikx}/2ik & -U(x)/2ik\end{pmatrix}\times\begin{pmatrix}v_1(x)\\ v_2(x)\end{pmatrix} \tag{10}$$

and the reduction

$$v_1(x)=v_2^*(x). \tag{11}$$

From (10) it is clear that the derivatives of v_1 and v_2 with respect to x are small when $U(x)$ is small, that is $v_1(x)$ and $v_2(x)$ vary slowly in comparison with $\exp(\pm ikx)$. Let us rewrite (10) in a more compact notation:

$$\dot{\hat{v}}=[i\varphi(x)s^z+\zeta^+(x)s^-+\zeta^-(x)s^+]\hat{v}, \tag{12}$$

where

$$\hat{v}=\begin{pmatrix}v_1(x)\\ v_2(x)\end{pmatrix},$$
$$\varphi(x)=-U(x)/k,\quad \zeta^{\pm}(x)=\pm iU(x)\exp(\pm 2ikx), \tag{13}$$

and $s^z=\sigma^z/2$, $s^{\pm}=(\sigma^x\pm i\sigma^y)/2$ are the ordinary spin operators; here and in what follows, the dot denotes differentiation with respect to x. The formal solution to Eq. (12) can be written

$$\hat{v}(x)=\mathscr{T}(x,-L)\hat{v}(-L),$$
$$\mathscr{T}(x,-L)=T\exp\left(\int_{-L}^{x}[i\varphi(t)s^z+\zeta^+(t)s^-+\zeta^-(t)s^+]dt\right), \tag{14}$$

where the symbol T implies "time" ordering along the interval $(-L,L)$. By representing the T-exponential (14) in the form of a series with respect to the fields $\varphi(t)$, $\zeta^{\pm}(t)$, we find that averaging of any functional of the components $v_1(x)$, $v_2(x)$ with respect to $\mathscr{D}U(x)$ leads to expressions that are combinations of integrals over certain intervals of the correlation functions for the fields $\varphi(t)$, $\zeta^{\pm}(t)$:

$$\int dt\,dt'\langle\varphi(t)\varphi(t')\rangle,\quad \int dt\,dt'\langle\zeta^+(t)\zeta^-(t')\rangle,$$
$$\int dt\,dt'\langle\varphi(t)\zeta^{\pm}(t')\rangle,\quad \int dt\,dt'\langle\zeta^+(t)\zeta^+(t')\rangle,$$
$$\int dt\,dt'\langle\zeta^-(t)\zeta^-(t')\rangle. \tag{15}$$

In our case the last three kinds of integrals will be integrals over rapidly oscillating functions, whose values do not increase as the interval increases and which decrease as k increases. However, the first two kinds of integrals correspond to infrared-singular contributions which grow linearly with the length of the integration interval. The identification of those terms in the perturbation series that are nonzero in the thermodynamic limit thus reduces to neglecting the correlators $\langle\zeta^+\zeta^+\rangle$, $\langle\zeta^-\zeta^-\rangle$, and $\langle\varphi,\zeta^{\pm}\rangle$, which is equivalent to the assumptions of statistical independence of the fields φ and $\zeta^{\pm}$ and phase invariance of the averaging weight with respect to $\mathscr{D}\zeta^{\pm}$. For "white noise" potentials the corresponding integration measure with respect to the fields φ and $\zeta^{\pm}$ has the form

$$\mathscr{D}\varphi(x)\mathscr{D}\zeta^{\pm}(x)\times\exp\left\{-\frac{2}{\alpha}\int_{-L}^{L}[a\varphi^2(x)+\zeta^+(x)\zeta^-(x)]dx\right\}, \tag{16}$$

where

$$\alpha=\frac{D}{2k^2},\quad a=\frac{1}{8}. \tag{17}$$

In what follows we will consider the parameter a to be arbitrary (it corresponds to a random jump in the phase and, as we will verify below, nothing depends on its specific value in the thermodynamic limit).

Expressions (14) and (16) were first proposed in Ref. 7 for the problem of a particle on a line in a random potential. Our derivation, which does not turn on the presence of a Fermi level, leads us to assert that the infrared

behavior of this Abrikosov–Ryzhkin model has universal features. The model can easily be generalized to a potential with finite correlation length and can be used to study the spectral properties of operators that are not in a literal sense random (see Conclusion).

The terms that were discarded in deriving Eqs. (14) and (16) are smaller by a factor $\sim 1/(kL)$ than those we retain. Consequently, this model is also applicable to mesoscopic systems (see the review in Ref. 5), since for large k the inequality $1/kL \ll 1$ is compatible with $l \geqslant L$, where l is the mean free path.

Abrikosov *et al.*[7] calculated the conductivity of a one-dimensional metal using the relations (14) and (16). Unfortunately, they resorted once more to perturbation methods, whose awkward and difficult-to-visualize constructions were not appropriate given the simplicity of the original model. In this paper we will show that the Abrikosov–Ryzhkin model is exactly solvable using the method of functional integration; we will calculate the multipoint correlators of higher powers of the density, and find the mean-square dispersion of the size of a localized wave function and the dependence of the localization length on the correlation length of the potential.

As a physical application of the method we calculate the average absolute value of the unattenuated current flowing in a mesoscopic loop with an arbitrary magnetic flux Φ (in the limit of small Φ this problem was solved recently[9]).

2. FUNCTIONAL REPRESENTATION OF AVERAGES OF $\hat{v}(x)$

Explicit calculation of the function $\hat{v}(x)$ using Eqs. (12) and (14) in the form of a functional of the fields $\varphi(x)$, $\zeta^{\pm}(x)$ is impossible. Exactly the same problem arises when we attempt to write down a closed functional representation for the partition function of a quantum ferromagnet. This latter problem was solved in Refs. 10–12, and the methods described in these papers will be used here as well.

The "time"-ordered exponential operator $\mathcal{T}(x,-L)$ is defined by the equation

$$\dot{\mathcal{T}} = [i\varphi(t)s^z + \zeta^+(t)s^- + \zeta^-(t)s^-]\mathcal{T} \tag{18}$$

and the initial condition

$$\mathcal{T}(x=-L,-L)=1. \tag{19}$$

Let us consider the following operator given in the form of a product of ordinary matrix exponentials:

$$\tilde{\mathcal{T}}(x,-L)=\exp[s^+\psi^-(x)]\exp\left(is^z\int_{-L}^{x}\rho dt\right)\times\exp\left[s^-\int_{-L}^{x}dt\psi^+(t)\exp\left(\int_{-L}^{t}\rho d\tau\right)\right]\times\exp[-s^+\psi^-(-L)], \tag{20}$$

where $\psi^{\pm}(x)$, $\rho(x)$ are certain new fields. Here $\tilde{\mathcal{T}}$ satisfies the equation

$$\dot{\tilde{\mathcal{T}}}=\{(i\rho+2\psi^+\psi^-)s^z+\psi^+s^- + [\dot{\psi}^- - i\rho\psi^- - \psi^+(\psi^-)^2]s^+\}\tilde{\mathcal{T}}, \tag{21}$$

and the last factor in (20) satisfies the initial condition

$$\tilde{\mathcal{T}}(-L,-L)=1. \tag{22}$$

Thus, by transforming in the functional integral over the measure (16) according to

$$\begin{aligned} i\varphi &= i\rho + 2\psi^+\psi^-, \\ \zeta^- &= \dot{\psi}^- - i\rho\psi^- - \psi^+(\psi^-)^2, \\ \zeta^+ &= \psi^+. \end{aligned} \tag{23}$$

We reduce the "time"-ordered exponential $\mathcal{T}(x,-L)$ to the form (20):

$$\mathcal{T}(x,-L)=\tilde{\mathcal{T}}(x,-L), \tag{24}$$

allowing us to construct an explicit functional representation for any averaged quantity [a parametrization of fields with values in the group $SL(2,C)$ analogous to (23), was also used in Ref. 13]. An important feature of this calculation is that it is sufficient to specify the variable change (22) in only one direction in order to compute the Jacobian: from $(\phi,\zeta^{\pm})$ to $(\rho,\psi^{\pm})$. The Jacobian $\mathcal{J}[\rho,\psi^{\pm}]$:

$$\mathcal{D}\varphi\mathcal{D}\zeta^+\mathcal{D}\zeta^- = \mathcal{J}[\rho,\psi^{\pm}]\mathcal{D}\rho\mathcal{D}\psi^+\mathcal{D}\psi^- \tag{25}$$

depends on regularization of the mapping (23), and on whether we impose initial or boundary value conditions on the field ψ^-. Ordinary periodic boundary conditions make the transformation (23) noninvertible. Following Refs. 11 and 12, we will assume that the field $\psi^-(x)$ is subject to the initial condition:

$$\psi^-(-L)=\psi_0, \tag{26}$$

however, in contrast to (11) and (12), we will choose the specific value of ψ_0 depending on the situation.

The choice of a regularization of the mapping (23) is determined by the physical meaning of the model. The δ-function correlator (2) is the limiting value of a symmetric correlation function with a finite correlation length. For any method used to define the δ-function, the limiting value of the correlators

$$\left\langle \zeta^+(t)\int_0^t \zeta^-(t')dt' \right\rangle = \left\langle \zeta^-(t)\int_0^t \zeta^+(t')dt' \right\rangle \tag{27}$$

will equal $\alpha/4$, which agrees with the definition of the step-function $\theta(x)$:

$$\theta(0)=1/2. \tag{28}$$

The discretization of the transformation (23) that ensures that Eq. (27) is satisfied has the following form ($\zeta_n^{\pm}=\zeta^{\pm}(t_n)$, $\rho_n=\rho(t_n),\ldots$, $n=1,\ldots,M$, $t_n=-L+2nL/M$, $h=2L/M\to 0$, $M\to\infty$):

$$
\begin{aligned}
&i\phi_n=i\rho_n+\psi_n^+(\psi_n^-+\psi_{n-1}^-),\\
&\zeta_n^-=\frac{1}{h}(\psi_n^--\psi_{n-1}^-)-\tfrac{1}{2}i\rho_n(\psi_n^-+\psi_{n-1}^-)\\
&\qquad-\tfrac{1}{4}\psi_n^+(\psi_n^-+\psi_{n-1}^-)^2, \qquad (29)\\
&\zeta_n^+=\psi_n^+.
\end{aligned}
$$

The differential of the mapping (29) is a lower-triangular matrix, and its determinant—the Jacobian $\mathscr{J}$—equals the product of the diagonal elements:

$$
\mathscr{J}=\text{const}\cdot\exp\left(-\frac{i}{2}\int_{-L}^{L}\rho\,dt\right). \qquad (30)
$$

Substituting the variable change (23) and the Jacobian (25), (30) into the measure (16), we obtain the averaging weight over the fields $(\rho,\psi^{\pm})$ in the form

$$
\begin{aligned}
&N\mathscr{D}\rho\mathscr{D}\psi^+\mathscr{D}\psi^-\exp[-S(\rho,\psi^{\pm})],\\
&S(\rho,\psi^{\pm})=\frac{2}{\alpha}\int_{-L}^{L}dx[a\rho^2+\psi^+\dot{\psi}^--(1+4a)i\rho\psi^+\psi^-\\
&\qquad-(1+4a)(\psi^+\psi^-)^2]+\frac{i}{2}\int_{-L}^{L}dx\rho. \qquad (31)
\end{aligned}
$$

Here N is a normalization factor that depends on L and α.

In calculating the Jacobian (25) we will treat $(\rho,\psi^{\pm})$ and $(\phi,\zeta^{\pm})$ as sets of independent complex variables; i.e., as different systems of coordinates in the space $\mathscr{C}^{3M}$ of field configurations. However, the condition

$$
\text{Im}\,\varphi=0,\quad \zeta^+=(\zeta^-)^*, \qquad (32)
$$

initially imposed in the model determines a surface Σ in $\mathscr{C}^{3M}$, over which the form $\mathscr{D}\varphi\wedge\mathscr{D}\zeta^+\wedge\mathscr{D}\zeta^-$ or $\mathscr{D}\rho\wedge\mathscr{D}\psi^+\wedge\mathscr{D}\psi^-$ is integrated. Equation (32) for Σ is implicit in the variables $(\rho,\psi^{\pm})$; however, if we integrate a holomorphic function in the space $\mathscr{C}^{3M}$, according to the Cauchy–Poincaré theorem this surface of integration can be deformed into the region of convergence in an arbitrary way. There exists a continuous family of surfaces, including Σ and the "standard" surface Σ':

$$
\Sigma'=\{\text{Im}\,\rho=0,\psi^+=(\psi^-)^*\}, \qquad (33)
$$

which lies entirely in the region of "perturbative" convergence (i.e., to each order of perturbation theory) of the integral over the measure (31) (the explicit form of this homotopy is given in Ref. 10). Thus, by treating the functional integral as a sum of a perturbation series,[14] we can replace the surface of integration Σ by the standard surface Σ'.

However, in order for the transformation from Σ to Σ' to be possible it is necessary that the quantity to be averaged (and not just the action) be represented in a form that admits analytic continuation to the surface Σ. Constructively this implies that no definition of a physical quantity in terms of matrix elements of $\mathscr{T}(x,-L)$ should require the use of complex conjugation.

3. EXPRESSION FOR THE DENSITY-DENSITY CORRELATOR IN TERMS OF THE FUNCTION $\hat{v}(x)$

Equation (3) expresses the correlator $p_E(x,x')$ in terms of the part of the Green's function $G(x,x'|E+i\varepsilon)$ that is singular as $\varepsilon\to+0$. In the representation (8), this singularity arises due to the vanishing Wronskian $W(E)$ on the real axis. Neglecting the quantity ε in the numerator of (18) and substituting

$$
W(E\pm i\varepsilon)=W(E)\pm i\varepsilon W'(E) \qquad (34)
$$

in the denominator, we find

$$
p_E(x,x')=\left\langle\frac{u^2(x)\tilde{u}^2(x')}{W'(E)}\delta(W)\right\rangle,\quad x'>x. \qquad (35)
$$

The Wronskian $W(E)$ does not depend on x, and, consequently, we can set $x=L$ in (8):

$$
W=u(L). \qquad (36)
$$

In the product with $\delta(W)=\delta[u(L)]$, the solution $\tilde{u}(x)$ is proportional to $u(x)$, with a coefficient of proportionality determined by condition (7). Thus, for an arbitrary functional $\mathscr{F}[\tilde{u}(x)]$ we have the equation

$$
\mathscr{F}[\tilde{u}(x)]\delta[u(L)]=\mathscr{F}\left[\frac{u(x)}{u'(L)}\right]\delta[u(L)]. \qquad (37)
$$

The quantity $W'(E)$, according to (36), can be written in terms of the derivative of the solution $u(x)$ with respect to energy E:

$$
W'(E)=\frac{\partial u(L)}{\partial E}. \qquad (38)
$$

The function $g(x)=\partial u(x)/\partial E$ satisfies the equation

$$
\left(-\frac{d^2}{dx^2}+U(x)-E\right)g(x)=u(x), \qquad (39)
$$

and the initial condition

$$
g(x=-L)=g'(x=-L)=0. \qquad (40)
$$

The substitution $g(x)=q(x)u(x)$ leads to a first-order linear equation for $q'(x)$; solving it we obtain

$$
g(x)=u(x)\int_{-L}^{x}\frac{dy}{u^2(y)}\int_{-L}^{y}dy_1u^2(y_1) \qquad (41)
$$

and

$$
\begin{aligned}
\frac{1}{W'(E)}\delta[u(L)]&=\frac{1}{g(L)}\delta[u(L)]\\
&=\frac{u'(L)}{\int_{-L}^{L}u^2(y)dy}\delta[u(L)]. \qquad (42)
\end{aligned}
$$

Thus, the correlator $p_E(x,x')$ is expressed in terms of the solution $u(x)$ of the Cauchy problem (7) in the following way:

$$p_E(x,x')=\left\langle \frac{u^2(x)u^2(x')}{u'(L)\int_{-L}^{L}u^2(y)dy}\,\delta[u(L)]\right\rangle, \quad x'>x. \tag{43}$$

However, in the high-energy limit (5) we can get rid of the δ-function and obtain a simple expression for $p_E(x,x')$ in terms of the slowly varying amplitudes $v_{1,2}(x)$. Actually, in the neighborhood of any given point x_0 the function $u(x)$ can be written in the form

$$u(x)=u_{sl}(x)\sin(kx+\delta), \tag{44}$$

where the envelope $u_{sl}(x)$ and the phase δ vary slowly on a scale $\sim 1/k$. Let us average expression (43) over an interval ΔL of positions at the right edge of our segment $(-L,L)$:

$$\tilde{p}_E(x,x')=\frac{1}{\Delta L}\int_{L}^{L+\Delta L}p_E(x,x')dL, \tag{45}$$

$$\frac{1}{k}\ll\Delta L\ll\frac{2}{\alpha}\equiv l. \tag{46}$$

(Here we have introduced the standard notation l for the localization length.) In the thermodynamic limit $\tilde{p}_E(x,x')$ and $p_E(x,x')$ coincide. On the other hand, by construction the value of the function $u(x)$ at a given point does not depend on the position of the left-hand edge L. The integral in the denominator (43) is determined only by the envelope $u_{sl}(x)$, and it varies by a relatively small amount as L varies from L to $L+\Delta L$. The average (45) is important only for the factor $\delta[u(L)]/u'(L)$:

$$\frac{1}{\Delta L}\int_{L}^{L+\Delta L}dL\,\frac{1}{u'(L)}\,\delta[u(L)]=\frac{1}{\pi k u_{sl}^2(L)}, \tag{47}$$

since condition (46) is equivalent to the fact that $u_{sl}(x)$ can be considered to be constant over the averaging interval. We can obtain an analogous relation between $u_{sl}^2(x)$ and $u^2(x)$, specifically:

$$u_{sl}^2(L)\approx\frac{2}{\Delta L}\int_{L}^{L+\Delta L}dL u^2(L). \tag{48}$$

Substituting the expression for $u(x)$ into (42)–(48) in terms of the amplitude $\hat{v}(x)$, neglecting contributions that are small as $k\to\infty$, and retaining only the "resonant" terms in the denominator of Eq. (43), we obtain

$$p_E(x,x')\approx\tilde{p}_E(x,x')\approx\frac{1}{2\pi k}\left\langle\frac{v_1(x)v_2(x)v_1(x')v_2(x')}{v_1(L)v_2(L)\int_{-L}^{L}v_1(y)v_2(y)dy}\right\rangle, \quad x'>x. \tag{49}$$

It can be verified that the discarded "nonresonant" terms contain the oscillatory factors $\exp[\pm 2ik(x-x')]$, and therefore give contributions that are exponentially small with respect to αL after averaging over the random field. For this reason we can neglect them.

4. FUNCTIONAL INTEGRALS FOR THE DENSITY-DENSITY CORRELATORS AND THEIR EVALUATION

Expression (49) has a form that admits analytic continuation with respect to the functions $v_{1,2}(x)$ to the surface $v_1=(v_2)^*$. For simplicity we will set $\exp(ikL)=1$ (in the thermodynamic limit this is not a restriction). Then the initial condition for $\hat{v}(z)$ has the form

$$\hat{v}(-L)=ik\begin{pmatrix}1\\1\end{pmatrix}. \tag{50}$$

In order to determine $\hat{v}(x)$, we substitute the evolution operator $\mathcal{T}(x,-L)$ in the form (20) into (14), in this case choosing the value ψ_0 to be 1:

$$\psi_0=\psi^-(-L)=1. \tag{51}$$

As a result we obtain

$$\hat{v}(x)=\exp\left(-\frac{i}{2}\int_{-L}^{x}\rho dt\right)\begin{pmatrix}\psi^-(x)\\1\end{pmatrix}, \tag{52}$$

and the expression for $p_E(x,x')$ is

$$p_E(x,x')\approx\frac{1}{2\pi k}\left\langle\frac{\psi^-(x)\psi^-(x')\exp(-i\int_{-L}^{x}\rho dt-i\int_{-L}^{x'}\rho dt+i\int_{-L}^{L}\rho dt)}{\psi^-(L)\int_{-L}^{L}\psi^-(y)\exp(-i\int_{-L}^{y}\rho dt)dy}\right\rangle, \quad x'>x, \tag{53}$$

where the average over the measure $\mathcal{D}\rho\mathcal{D}\psi^+\mathcal{D}\psi^-$ is carried out with the weights (31). In order to calculate this functional integral, we use an approach analogous to the so-called "bosonization" in the field theory models of Ref. 15. Using the identity

$$\exp[-S(\rho,\psi^{\pm})]=\int\mathcal{D}\eta\exp[-\tilde{S}(\eta,\rho,\psi^{\pm})],$$

$$\tilde{S}(\eta,\rho,\psi^{\pm})=\frac{2}{\alpha}\int_{-L}^{L}dx[(1+4a)\eta^2+a\rho^2+\psi^+\dot{\psi}^-$$

$$+(1+4a)(2\eta-i\rho)\psi^+\psi^-]$$

$$+\frac{i}{2}\int_{-L}^{L}dx\rho \tag{54}$$

and the gauge transformation

$$\psi^{\pm}(x)=\chi^{\pm}\exp\left[\pm(1+4a)\int_{-L}^{x}dt(2\eta-i\rho)\right], \tag{55}$$

we get rid of the nonlinear terms in the action. The Jacobian of the rotation (55), taking into account the regularization (29), equals

$$\mathcal{J}_R=\text{const}\cdot\exp\left(-\frac{1+4a}{2}\int_{-L}^{L}(2\eta-i\rho)dt\right). \quad (56)$$

The fields η and ρ enter into the averaged expression (53) only in the combination

$$\int_{-L}^{x}[2(1+4a)\eta-4ia\rho]dt,$$

which we naturally treat as a new variable of integration:

$$\dot{\xi}=2(1+4a)\eta-4ia\rho,$$

$$\xi(-L)=0, \quad (57)$$

$$\mathcal{D}\rho\mathcal{D}\eta=\text{const}\,\mathcal{D}\rho\mathcal{D}\xi,$$

after which the Gaussian $\mathcal{D}\rho$ integration is easily carried out, and we are led to an expression for the measure:

$$\text{const}\,\mathcal{D}\xi\mathcal{D}\chi^{+}\mathcal{D}\chi^{-}\exp\left(-\frac{1}{2\alpha}\int_{-L}^{L}dx\dot{\xi}^2-\frac{2}{\alpha}\int_{-L}^{L}dx\chi^{+}\dot{\chi}^{-}-\frac{\xi(L)}{2}\right) \quad (58)$$

and for the average quantities:

$$p_E(x,x')=\frac{1}{2\pi k}\left\langle\frac{\chi^{-}(x)\chi^{-}(x')\exp[-\xi(x)-\xi(x')+\xi(L)]}{\chi^{-}(L)\int_{-L}^{L}\chi^{-}(y)e^{-\xi(y)}dy}\right\rangle, \quad x'>x. \quad (59)$$

[The equality is understood to hold in the limit (5).] The initial condition for the field $\chi^{-}(x)$ following from (51):

$$\chi^{-}(-L)=1 \quad (60)$$

implies that $\chi^{-}(x)$ contains both a fluctuating part $\chi_f^{-}(x)$ and a regular part:

$$\chi^{-}(x)=1+\chi_f^{-}(x), \quad \chi_f^{-}(-L)=0. \quad (61)$$

The component $\chi_f^{-}(x)$ does not contribute to $p_E(x,x')$, since (59) does not contain the conjugate field. Thus, there remains only the average with respect to the field $\xi(x)$ with the weight:

$$\exp\left(-\frac{\alpha L}{4}\right)N'\mathcal{D}\xi\exp\left(-\frac{1}{2\alpha}\int_{-L}^{L}dx\dot{\xi}^2-\frac{\xi(L)}{2}\right). \quad (62)$$

The normalization constant N' is determined by terms that are quadratic in $\dot{\xi}$:

$$N'\mathcal{D}\xi\exp\left(-\frac{1}{2\alpha}\int_{-L}^{L}dx\dot{\xi}^2\right)=1, \quad (63)$$

and the factor $\exp(-\alpha L/4)$ ensures that the equation $\langle 1\rangle=1$ is satisfied when we average over the total measure (62). According to what we have said above, the correlator $p_E(x,x')$ can be represented in the form of an integral over $\mathcal{D}\xi$ in the following way:

$$p_E(x,x')=\frac{1}{2\pi k}N'\exp\left(-\frac{\alpha L}{4}\right)\int_{\xi(-L)=0}\mathcal{D}\xi\times\exp\left(-\frac{1}{2\alpha}\int_{-L}^{L}dt\dot{\xi}^2-\frac{\xi(L)}{2}\right)\times\exp[-\xi(x)-\xi(x')+\xi(L)]\left\{\int_{-L}^{L}\times\exp[-\xi(t)]dt\right\}^{-1}$$

$$=\frac{N'}{4\pi k\alpha}\exp\left(-\frac{\alpha L}{4}\right)\int_0^{\infty}d\lambda\int_{\xi(-L)=0}\mathcal{D}\xi\times\exp\left(-\frac{1}{2\alpha}\int_{-L}^{L}dt(\dot{\xi}^2+\lambda e^{-\xi})+\frac{\xi(L)}{2}\right)\times e^{-\xi(x)-\xi(x')}$$

$$=N'(4\pi k\alpha)^{-1}\int_{-\infty}^{+\infty}\int d\sigma d\sigma'\exp\left(\frac{\sigma+\sigma'}{2}\right)\times\int_{\xi(-L)=\sigma',\xi(L)=\sigma}\mathcal{D}\xi\exp\left(-\frac{1}{2\alpha}\times\int_{-L}^{L}dt(\dot{\xi}^2+e^{-\xi})-\frac{\alpha L}{4}\right)e^{-\xi(x)-\xi(x')}. \quad (64)$$

We are led to this latter expression once we have made a change of variables:

$$\lambda=e^{-\sigma'}, \quad \xi\to\xi-\sigma' \quad (65)$$

and have isolated the integrals over the quantities $\xi(t)$ at the end points $t=L$ and $t=-L$ in explicit form. The last path integral is a standard Feynmann–Kac integral,[16] and equals the following matrix element:

$$p_E(x,x')=\exp\left(-\frac{\alpha L}{4}\right)(4\pi k\alpha)^{-1}\langle e^{\xi/2}|\times\exp[-(L-x')\hat{H}]e^{-\xi}\exp[-(x'-x)\hat{H}]e^{-\xi}\times\exp[-(x+L)\hat{H}]|e^{\xi/2}\rangle, \quad (66)$$

with the Hamiltonian

$$\hat{H}=\frac{\alpha}{2}\partial_\xi^2-\frac{1}{2\alpha}e^{-\xi}. \tag{67}$$

The function $e^{\xi/2}$ grows as $\xi\to\infty$, and therefore we are not permitted to write it in the form of a linear superposition of eigenfunctions of $\hat{H}$:

$$\hat{H}f_\nu(\xi)=-\frac{\alpha}{2}\nu^2 f_\nu(\xi),$$

$$f_\nu(\xi)=\frac{2}{\pi}\sqrt{\nu\,\mathrm{sh}\,2\pi\nu}K_{2i\nu}\left(\frac{2}{\alpha}e^{-\xi/2}\right), \tag{68}$$

$$\langle f_\nu|f_{\nu'}\rangle=\delta(\nu-\nu').$$

However, explicit solution of the corresponding evolution equation leads to the following asymptotic relation:

$$\exp(-T\hat{H})e^{\xi/2}\to\exp\left(\frac{\alpha T}{8}\right)\mathbf{Y}_0(\xi)$$

$$=\frac{2}{\alpha}\exp\left(\frac{\alpha T}{8}\right)K_1\left(\frac{2}{\alpha}e^{-\xi/2}\right). \tag{69}$$

Here $K_\mu(z)$ is the standard notation for the modified Bessel function. The function $K_1(2/\alpha e^{-\xi/2})e^{-\xi}$ is already expanded with regard to the complete set (68). Thus, the correlator $p_E(x,x')$ equals

$$p_E(x,x')=\frac{1}{4\pi k\alpha}\exp\left(-\frac{\alpha|x-x'|}{8}\right)\langle\mathbf{Y}_0(\xi)e^{-\xi}|$$

$$\times\exp(-|x-x'|\hat{H})|\mathbf{Y}_0(\xi)e^{-\xi}\rangle$$

$$=\frac{\alpha}{\pi^3 k}\exp\left(-\frac{\alpha|x-x'|}{8}\right)\int_0^\infty d\nu\nu\,\mathrm{sh}\,2\pi\nu$$

$$\times\exp\left(-\frac{\alpha\nu^2}{2}|x-x'|\right)$$

$$\times\left\{\int_0^\infty dyyK_1(y)K_{2i\nu}(y)\right\}^2$$

$$=\frac{\alpha\pi}{2k}\exp\left(-\frac{\alpha|x-x'|}{8}\right)\int_0^\infty\frac{d\nu\nu\,\mathrm{sh}\,\pi\nu}{\mathrm{ch}^3\,\pi\nu}$$

$$\times\left(\nu^2+\frac{1}{4}\right)^2\exp\left(-\frac{\alpha\nu^2}{2}|x-x'|\right). \tag{70}$$

Equation (70), to within the redefinition $\alpha/2=l^{-1}$, coincides with known results of Refs. 1 and 6. Using this method, we also can calculate multipoint correlators of higher powers of the density, e.g., ($x_1<x_2<\ldots<x_{2m}$, $m>1$):

$$2\pi k p_E^{(q,m)}(x_1,x_2,\ldots x_{2m-1},x_{2m})=2\pi k\lim_{L\to\infty}\left\langle\sum_n\delta(E-E_n)\left|\Psi_n(x_1)\right|^{2q}\left|\Psi_n(x_2)\right|^{2q}\ldots\left|\Psi_n(x_{2m-1})\right|^{2q}\left|\Psi_n(x_{2m})\right|^{2q}\right\rangle$$

$$=\lim_{L\to\infty}\lim_{\varepsilon\to+0}\frac{2^{2qm}(qm-1)!\varepsilon^{2qm-1}k}{(2qm-2)!}\langle|G(x_1,x_2|E+i\varepsilon)\ldots G(x_{2m-1},x_{2m}|E+i\varepsilon)|^{2q}\rangle$$

$$=2\pi k\left\langle\left(\prod_{j=1}^m u^{2q}(x_{2j-1})\tilde{u}^{2q}(x_{2j})\right)(W'(E))^{-2qm+1}\delta(W(E))\right\rangle$$

$$=2\pi k\left\langle\left(\prod_{j=1}^m u^{2q}(x_{2j-1})u^{2q}(x_{2j})\right)\left(\int_{-L}^L u^2(y)dy\right)^{-2qm+1}(u'(L))^{-1}\delta(u(L))\right\rangle$$

$$\approx\left\langle\prod_{j=1}^{2m}(v_1(x_j)v_2(x_j))^q\left(\int_{-L}^L v_1(y)v_2(y)dy\right)^{-2qm+1}(v_1(L)v_2(L))^{-1}\right\rangle$$

$$=\frac{N'}{(2\alpha)^{2qm-1}(2qm-2)!}\exp\left(-\frac{\alpha L}{4}\right)\int\int_{-\infty}^{+\infty}d\sigma d\sigma' e^{(\sigma+\sigma')/2}\int_{\xi(-L)=\sigma',\xi(L)=\sigma}\mathscr{D}\xi$$

$$\times\exp\left(-\frac{1}{2\alpha}\int_{-L}^L dt(\dot{\xi}^2+e^{-\xi})\right)\exp\left(-\sum_{j=1}^{2m}q\xi(x_j)\right)$$

$$=\frac{1}{(2\alpha)^{2qm-1}(2qm-2)!}\exp\left(-\frac{\alpha(x_{2m}-x_1)}{8}\right)$$

$$\times\langle\mathbf{Y}_0(\xi)\exp(-q\xi)|\exp[-(x_2-x_1)\hat{H}]\exp(-q\xi)$$

$$\times\exp[-(x_3-x_2)\hat{H}\exp(-q\xi)\ldots\exp(-q\xi)$$

$$\times\exp[-(x_{2m}-x_{2m-1})\hat{H}]|\mathbf{Y}_0(\xi)\exp(-q\xi)\rangle$$

$$=\left(\frac{\alpha}{8}\right)^{2mq-1}2^{4(q+m)-6}\frac{\pi^{m+1}}{\Gamma(2mq-1)\Gamma^2(2q)}\left(\frac{\Gamma(q)}{\Gamma(q+1/2)}\right)^{2m-2}$$

$$\times\exp\left(-\frac{\alpha(x_{2m}-x_1)}{8}\right)\int_0^\infty\int_0^\infty\prod_{j=1}^{2m-1}d\nu_j\exp\left(-\frac{\nu_j^2\alpha}{2}\Delta x_j\right)\nu_j\,\mathrm{sh}\,2\pi\nu_jP^{(q)}$$

$$\times(\nu_1)P^{(q)}(\nu_{2m-1})\prod_{s=1}^{2m-1}Q^{(q)}(\nu_s,\nu_{s+1}). \tag{71}$$

Here $\Delta x_j=x_{j+1}-x_j$ and the functions $p^{(q)}(\nu)$ and $Q^{(q)}(\nu,\nu')$ are defined as follows:

$$P^{(q)}(\nu)=\frac{1}{\mathrm{ch}^2\,\pi\nu}[(q-1/2)^2+\nu^2]\prod_{j=1}^{q-1}[(j-1/2)^2+\nu^2]^2, \tag{72}$$

$$Q^{(q)}(\nu,\nu')=\frac{\nu^2-\nu'^2}{\mathrm{ch}\,2\pi\nu-\mathrm{ch}\,2\pi\nu'}\prod_{j=1}^{q-1}$$

$$\times\left[1+\frac{2}{j^2}(\nu^2+\nu'^2)+\frac{1}{j^4}(\nu^2-\nu'^2)\right]. \tag{73}$$

The expression for $p_E^{(q,1)}(x_1,x_2)$ is obtained from (71) by formally replacing the product over s from 1 to $2m-2$ by unity and setting $m=1$ in the remaining integral.

As one possible application of Eqs. (71) and (72) we consider the dispersion of the size of localized wave functions. It is clear from (71) that the remote exponential asymptotic behavior of the probability density does not fluctuate. On the other hand, it is natural to use some integral characteristic to define the size of the packet R_E, e.g.:

$$R_E^{-1}=\frac{4}{3}\int_{-L}^{L}dx|\psi(x)|^4. \tag{74}$$

The factor 4/3 cancels out the average value of the rapidly oscillating factor $\sin^4(kx+\delta)$ [see (44)]. Then

$$\langle R_E^{-1}\rangle=\frac{4}{3\rho(E)}\int_{-L}^{L}dx p_E(x,x)=\frac{2\alpha}{9}=\frac{4}{9l}. \tag{75}$$

In Eq. (75), $\rho(E)$ denotes the density of states in the region of large E:

$$\rho(E)=L/\pi k. \tag{76}$$

The mean square value of R_E^{-1} is determined by the correlation function $\langle|\psi(x)\psi(x')|^4\rangle$:

$$\langle R_E^{-2}\rangle=\frac{32}{9\rho(E)}\int_{-L}^{L}dx\int_{-L}^{x}dx'p_E^{(2,1)}(x',x). \tag{77}$$

Substituting the explicit expression for $p_E^{(2,1)}(x',x)$ into this expression,

$$p_E^{(2,1)}(x',x)=\frac{\pi\alpha^3}{576}\exp\left(-\frac{\alpha(x-x')}{8}\right)\int_0^\infty d\nu\exp\left(-\frac{\alpha\nu^2}{2}\right.$$

$$\left.\times(x-x')\right)\frac{\mathrm{sh}\,\pi\nu}{\mathrm{ch}^3\,\pi\nu}\nu\left(\frac{9}{4}+\nu^2\right)^2\left(\frac{1}{4}+\nu^2\right)^4, \tag{78}$$

and calculating the integrals over dx' and $d\nu$, we obtain

$$\langle R_E^{-2}\rangle\approx 0.23/l^2. \tag{79}$$

Together with (75), this gives the value of the mean square dispersion of the quantity R_E^{-1}:

$$\frac{\langle R_E^{-2}\rangle-\langle R_E^{-1}\rangle^2}{\langle R_E^{-2}\rangle}\approx 0.13. \tag{80}$$

5. THE AVERAGE CURRENT IN A ONE-DIMENSIONAL MESOSCOPIC RING WITH MAGNETIC FLUX Φ

Let us consider a one-dimensional metallic ring in a transverse magnetic field. The average value of the current operator for a single electron with respect to the stationary state in an arbitrary potential becomes nonzero, a T-odd term appears in the energy, and the Fermi surfaces for the left- and right-hand directions of average velocity are found to be shifted with respect to one other. As a result, an unattenuated current will flow around the ring in the ground state.[17] The arguments of Refs. 18 and 9 advanced the idea that the total current in the ring is of order the average current I transported by a single electronic state at the Fermi level. We will assume that the size of the ring $2L$ is comparable to the mean free path. In this case, effects of localization do not completely suppress I; however, they make the dependence of I on magnetic field very nontrivial (see below). (The case of ordered nonuniform conductors was treated in Ref. 19.)

There is a gauge in which the wave function of an electron in a ring with magnetic flux Φ is subject to the boundary condition

$$\psi(L)=\exp(2\pi i\Phi)\psi(-L), \tag{81}$$

while the Hamiltonian, as before, has the form (1). The average absolute value of the current transported by a state with energy E in the limit (5) can be written in the form[20,9]:

$$I=\left\langle\frac{\pi k}{L}\sum_n\delta(E-E_n)\left|j_n\right|\right\rangle,\quad j_n=-\frac{1}{2\pi}\frac{\partial E_n}{\partial\Phi} \tag{82}$$

(here $h=c=e=1$, so that the magnetic flux quantum equals unity). Condition (81) is nonlocal, and therefore Eq. (82) for I cannot be rewritten in terms of the functions of the form $u(x)$, $\tilde{u}(x)$ given in the previous paragraph. However, as was shown in Ref. 9, I can be expressed directly in terms of elements of the $\mathscr{T}$ matrix (14), for which, strictly speaking, we also have a functional representation.

The fact is, by construction the matrix $\mathscr{T}\equiv\mathscr{T}(-L,L)$ satisfies the relations

$$\sigma^z\mathscr{T}^\dagger\sigma^z=\mathscr{T}^{-1},\quad \det\mathscr{T}=1, \tag{83}$$

and consequently can be parametrized in the following way:

$$\mathscr{T}=\begin{pmatrix} \operatorname{ch}\Gamma\exp(i\alpha_s) & \operatorname{sh}\Gamma\exp(i\beta_s) \\ \operatorname{sh}\Gamma\exp(-i\beta_s) & \operatorname{ch}\Gamma\exp(-i\alpha_s)\end{pmatrix}, \tag{84}$$

where Γ, α_s, and β_s are slowly varying real functions of L. The mapping of the space of initial data at the point $x=-L$ for Eq. (7) into the space of solutions at the point $x=L$ in the basis $u'\pm iku$ is carried out by the transfer matrix T:

$$T=\exp(ikL\sigma^z)\mathscr{T}\exp(ikL\sigma^z)$$

$$=\begin{pmatrix} \operatorname{ch}\Gamma\exp[i(\alpha_s+kL)] & \operatorname{sh}\Gamma\exp(i\beta_s) \\ \operatorname{sh}\Gamma\exp(-i\beta_s) & \operatorname{ch}\Gamma\exp[-i(\alpha_s+kL)]\end{pmatrix}. \tag{85}$$

Condition (81) is equivalent to the matrix T having an eigenvalue $e^{i\theta}$ with $\theta=2\pi\Phi$, that is,

$$\det(T-e^{i\theta})=1,$$

or

$$\tau(E)\equiv\operatorname{ch}\Gamma\cos(\alpha_s+kL)=\cos\theta. \tag{86}$$

The latter equation determines eigenvalues with energy $E=k^2$, and according to Eq. (82) we have for j_n

$$\sum_n \delta(E-E_n)|j_n| = \sum_n \delta(E-E_n)\left|\frac{\sin\theta}{\tau'(E)}\right|$$

$$=\delta(\tau(E)-\cos\theta)|\sin\theta|. \tag{87}$$

Proceeding as in the previous section, we can get rid of the δ-function by averaging over a small range ΔL of possible lengths L ($1/k\ll\Delta L\ll l$):

$$I=\left\langle\frac{\pi k}{L}\delta(\tau(E)-\cos\theta)\right\rangle|\sin\theta|$$

$$\approx\left\langle\frac{1}{\sqrt{\operatorname{sh}^2\Gamma+\sin^2\theta}}\right\rangle\left|\frac{k}{L}\sin\theta\right|$$

$$=\left|\frac{k}{L}\sin\theta\right|\frac{1}{\sqrt{\pi}}\left\langle\int_{-\infty}^{\infty}d\mu\exp[-\mu^2(\operatorname{sh}^2\Gamma+\sin^2\theta)]\right\rangle. \tag{88}$$

It is important the quantity $\operatorname{sinh}^2\Gamma$ can be expressed in terms of elements of the matrix T in a form that admits analytic continuation from the surface Σ:

$$\operatorname{sh}^2\Gamma=(1,0)\mathscr{T}^t s^-\mathscr{T}\begin{pmatrix}0\\1\end{pmatrix}. \tag{89}$$

Here, as usual, the letter t implies transposition (without complex conjugation!). The simplest way to express $\operatorname{sinh}^2\Gamma$ in terms of the fields $\rho,\psi^{\pm}$ is to choose the zero initial condition for the field ψ^-:

$$\psi^-(-L)=0. \tag{90}$$

In this case, after substituting (20) into (89) we obtain

$$\operatorname{sh}^2\Gamma=\psi^-(L)\int_{-L}^{L}dt\psi^+(t)\exp\left(-i\int_t^L\rho d\tau\right). \tag{91}$$

The right-hand portion of (91) is bilinear in the fields $\psi^{\pm}$, but is nonlocal. Therefore, rather than carrying out the integration with respect to $\mathscr{D}\psi^{\pm}$, it is convenient to transform $\exp(-\mu^2\operatorname{sinh}^2\Gamma)$ into an integral that involves linear combinations of the $\psi^{\pm}$:

$$I=\left|\frac{k\sin\theta}{\sqrt{\pi}L}\right|\int_{-\infty}^{+\infty}d\mu\exp(-\mu^2\sin^2\theta)$$

$$\times\left\langle\exp\left[-\mu^2\psi^-(L)\int_{-L}^{L}dt\psi^+(t)\right.\right.$$

$$\left.\left.\times\exp\left(-i\int_t^L\rho d\tau\right)\right]\right\rangle$$

$$=\left|\frac{k\sin\theta}{\sqrt{\pi}L}\right|\int_{-\infty}^{\infty}d\mu\int dzdz^*\exp(-\mu^2\sin^2\theta-|z|^2)$$

$$\times\left\langle\exp\left[-i\mu z\psi^-(L)-i\mu z^*\int_{-L}^{L}dt\psi^+(t)\right.\right.$$

$$\left.\left.\times\exp\left(-i\int_t^L\rho d\tau\right)\right]\right\rangle. \tag{92}$$

Repeating steps (54) to (56) of the previous paragraph, introducing a variable $\xi(t)$ analogous to (57):

$$\dot{\xi}=-2(1+4a)\eta+4ia\rho,\quad \xi(L)=0,$$
$$\mathscr{D}\rho\mathscr{D}\eta=\text{const}\,\mathscr{D}\rho\mathscr{D}\xi \tag{93}$$

and carrying out the Gaussian integration with respect to $\mathscr{D}\rho$ and $\mathscr{D}\chi$, we are led to the following expression for I:

$$I=\exp\left(-\frac{\alpha L}{4}\right)\left|\frac{k}{L}\sin\theta\right|\frac{1}{\pi^{3/2}}N'\int_{-\infty}^{\infty}d\mu\int dzdz^*$$

$$\times\exp(-\mu^2\sin^2\theta-|z|^2)\int_{\xi(L)=0}\mathscr{D}\xi$$

$$\times\exp\left(-\frac{1}{2\alpha}\int_{-L}^{L}dx(\dot{\xi}^2+\alpha^2\mu^2|z|^2e^{-\xi})\right.$$

$$\left.-\frac{\xi(-L)}{2}\right). \tag{94}$$

We now change the variable of integration from μ to a new variable σ:

$$\alpha^2\mu^2|z|^2=e^{-\sigma}, \tag{95}$$

and shift the path $\xi(x)$ by $-\sigma$:

$$\xi(x)\to\xi(x)-\sigma.$$

Then after integrating over $dzdz^*$ we obtain a representation for I in the form of a matrix element:

$$I=\exp\left(-\frac{\alpha L}{4}\right)\left|\frac{k}{\sqrt{\pi}\alpha L}\sin\theta\right|$$

$$\times\langle\mathbf{Y}_2(\xi)|\exp(-2L\hat{H})|\mathbf{Y}_1(\xi)\rangle, \tag{96}$$

where $\hat{H}$ is defined by Eq. (67), and the functions $\mathbf{Y}_{1,2}(\xi)$ equal

$$\mathbf{Y}_1(\xi)=\exp\left(-\frac{\xi}{2}\right) \tag{97}$$

and

$$\mathbf{Y}_2(\xi)=\frac{\sqrt{\pi}}{2}\exp\left\{-\frac{2}{\alpha}\exp\left(-\frac{\xi}{2}\right)\left|\sin\theta\right|\right\}. \tag{98}$$

Expanding in the complete set (68) of eigenfunctions of $\hat{H}$, and using the integral representation for $K_{2iv}(y)$, we are led to the following expression for the current I:

$$I=\frac{k}{L}\left(\frac{l}{L}\right)^{1/2}\exp\left(-\frac{L}{4l}\right)$$
$$\times\sin^2\theta\int_{-\infty}^{+\infty}\frac{dt\,\mathrm{ch}\,t}{\sin^2\theta+\mathrm{sh}^2\,t}\exp\left(-\frac{l}{L}t^2\right) \tag{99}$$

(here $l^{-1}=\alpha/2$). As $\sin^2\theta$ varies from 0 to 1, I increases monotonically from 0 to $I_{\max}$. When this expression is expanded to first order in θ as $\theta\to 0$, we recover the results of Ref. 9. Note, however, that the parameter l/L determines not only the absolute value of I but also the dependence of I on magnetic field:

$$\frac{I}{I_{\max}}=\frac{\sin^2\theta}{\int_{-\infty}^{\infty}\frac{dy}{\mathrm{ch}\,y}\exp\left(-\frac{l}{L}y^2\right)}\int_{-\infty}^{+\infty}\frac{dt\,\mathrm{ch}\,t}{\sin^2\theta+\mathrm{sh}^2\,t}$$
$$\times\exp\left(-\frac{l}{L}t^2\right), \tag{100}$$

which in principle allows us to determine l/L from the shape of the experimental curve $I(\theta)$.

6. CONCLUSION

If the random potential $U(x)$ has a small but finite correlation length $\varkappa^{-1}$:

$$\langle U(x)U(x')\rangle=\tfrac{1}{2}D\varkappa\exp(-\varkappa|x-x'|), \tag{101}$$

so that we have $1\ll\varkappa l$ but $k/\varkappa\sim 1$, in the limit of large energies its influence is taken into account by simply renormalizing the parameter α, or, what is the same thing, the localization length l:

$$\alpha_\varkappa=\frac{\alpha_0}{1+4k^2/\varkappa^2}. \tag{102}$$

Actually, the correlator (101) corresponds to the averaging measure with respect to the fields $\zeta^{\pm}$ [see (13)]:

$$\mathscr{D}\zeta^{\pm}\exp\left\{-\frac{2}{\alpha}\int_{-L}^{L}\left(\frac{1}{\varkappa^2}\left|\dot{\zeta}\right|^2+2i\frac{k}{\varkappa^2}(\dot{\zeta}^+\zeta^- -\zeta^+\dot{\zeta}^-)\right.\right.$$
$$\left.\left.+\left(1+\frac{4k^2}{\varkappa^2}\right)\left|\zeta\right|^2\right)dx\right\}. \tag{103}$$

After "bosonization" and transforming to the variable ξ, we are led to a certain effective action, which contains terms with the same number of derivatives of the field ξ as the terms involving derivatives of $\dot{\zeta}^{\pm}$ that appear in (103). Since the unperturbed problem contains a single parameter with dimensions of length, i.e., l, the contributions from these "non-Markovian" terms to the various correlators will be suppressed by the corresponding powers of the quantity $(\varkappa l)^{-1}$, and we may neglect them.

The variable σ' appearing in (64) and (65) can be treated as a global order parameter which characterizes the phenomenon of localization. Actually, in view of Eqs. (64) and (65), the nonzero value of the correlator $p_E(x,x')$ in the thermodynamic limit is a consequence of

$$\lim_{L\to\infty}\frac{\sigma'}{\alpha L}=\frac{1}{2}>0. \tag{104}$$

On the other hand, the quantity $\lambda=\exp(-\sigma')$ is associated with the norm of the wave function, so that inequality (104) corresponds to exponential growth of the solutions to Eq. (7).

In conclusion, we note that the infrared-singular terms in the perturbation series, which determine the behavior of the wave function at large distances, depend only on certain average characteristics of the potential $U(x)$. In particular, if the quantity

$$b_\Delta(x,d)=\int_{x-\Delta}^{x+\Delta}dx'\exp[ik(x-x')]U(x)U(x') \tag{105}$$

ceases to depend on x starting with a certain Δ such that $k^{-1}\ll\Delta\ll l$, the properties of the wave function in this potential in the thermodynamic limit can be studied using the Abrikosov–Ryzhkin model (14)–(16) with effective parameters a and α.

We note that use of methods analogous to those described here can lead to explicit functional representations for the average of any combination of Green's functions over the ensemble (2) for arbitrary energies E.[21] Unfortunately, effective methods for calculating the resulting path integrals have been found only in the simplest cases.

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