Above-threshold ionization of atoms with allowance for multiple Coulomb scattering

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(Submitted 6 November 1992; resubmitted 3 December 1992)
Zh. Eksp. Teor. Fiz. **103**, 1192–1204 (April 1993)

We consider an analytic approach to the elucidation of the role played by Coulomb interaction in the formation of the spectrum and angular distributions of the electrons produced by above-threshold ionization of the atoms. Our approach is based on allowance for the multiple scattering of a photoelectron by the Coulomb potential of the residual ion, with pickup of a field photon in each scattering act. We determine the dependence of the photoelectron energy spectrum on the intensity and frequency of a laser wave. Criteria are established for the applicability of the proposed theory, and the possibility of verifying it in experiment is analyzed.

1. INTRODUCTION

Above-threshold ionization (ATI) of atoms in strong laser fields has been diligently studied in the past few years both experimentally and theoretically (see the reviews, Refs. 1 and 2). Above-threshold maxima in the photoelectron spectra appear in laser-field spectra of intensity $I > 10^{12}$ W/cm^2. The amplitudes of these peaks first increase rapidly from the threshold, reach a maximum whose position depends on the intensity and frequency of the employed wave, and the decrease slowly.

A number of attempts[3–6] were made to describe the ATI effect theoretically. Of greatest interest in this connection is a trend based on determining the role of Coulomb interaction between a photoelectron in the continuum and the ionic (or atomic) residue. In particular, in Ref. 2 was considered the matrix element for an $(n+1)$-photon transition from the ground state of an atom into a continuum, with an electron energy

$$\varepsilon_p = E_0 + (n+1)\hbar\omega$$

(E_0 is the ground-state energy, $\hbar\omega$ is the photon energy). This matrix element was estimated using a quasiclassical approximation of the electron wave functions, with allowance for the Coulomb interaction. The result was the dependence of the laser-wave intensity I on the frequency ω in the probabilities of transitions with pickup of $(n+1)$ and n photons become equalized ($I_{cr} \sim \omega^{10/3}$). This approach, however, provided no interpretation of the photoelectron energy spectrum.

From the standpoint of describing the ATI effects, interest attaches also to Refs. 7 and 8, in which the residual part of the Coulomb interaction of a photoelectron with the residual ion is accounted for by simply supplanting the plane wave $\exp(i\mathbf{p}\cdot\mathbf{r})$ in the Volkov solution of the Schrödinger equation by a properly asymptotic Coulomb wave function.

The spectrum of the photoelectrons and their angular distribution were interpreted for the hydrogen atom at fixed laser parameters by numerically solving the nonstationary Schrödinger equation.[1]

We consider in this paper an analytic elucidation of the role played by Coulomb interaction in a photoelectron + ion system in the formation of the spectrum and of the angular distribution of the photoelectrons resulting from multiphoton above-threshold ionization of an atom (hydrogen is considered for the sake of argument). The approach is based on allowance for multiple scattering of the photoelectron by the Coulomb potential of the residual ion, with pickup of one field quantum in each scattering act. We determine the dependence of the energy spectrum of the photoelectrons on the intensity and frequency of a laser wave. Criteria are established for the validity of the proposed model of multiple Coulomb scattering (MCS), and the possibility of its experimental verification are analyzed.

2. BASIC EQUATIONS

We describe the interaction of an electron with an electromagnetic wave by the operator ($\hbar = c = 1$)

$$\hat{V}(t) = e\mathbf{r}\mathbf{E}(t), \tag{1}$$

where $E(t)$ is the wave's electric-field intensity (we confine ourselves below to a linearly polarized monochromatic wave with a polarization unit vector $\mathbf{e}$ oriented along the polar axis z ($\mathbf{e} = \mathbf{e}_z$).

The Coulomb interaction in a photoelectron + ion system is treated as a perturbation in the iteration of the amplitude of the transition from the initial near-threshold state to the final highly-excited state of the photoelectron in the continuum. The basis function is the solution $\Psi_{\mathbf{p}}(\mathbf{r},t)$ of the nonstationary Schrödinger equation with account taken of the interaction (1) (the Keldysh method[9]). The nonrenormalized particle momentum $\tilde{\mathbf{p}}$ of the expression for the psi-function is given by

$$\tilde{\mathbf{p}} = \mathbf{p} - (eE_0/\omega)\mathbf{e}_z \sin\omega t,$$

where e is the unit charge, E_0 is the amplitude of the electric field intensity, and ω is the wave frequency.

Strictly speaking, the use of the functions of Ref. 9 to describe the states of an electron in a field of an ion core is not quite valid near the atom ionization threshold ($\varepsilon_p < \omega$). If $e^2/\hbar v \gg 1$ (v is the photoelectron velocity) it is more correct to use the quasiclassical functions derived in Ref. 10. The use of these functions, however, does not change in essence the main results of the present paper, but complicates considerably the formal description of the problem. We therefore develop an interaction procedure for the transition amplitude using the functions of Ref. 9. The justification for this approach is that the ensuing results are applicable and are of practical interest for the above-threshold maxima of photoelectrons with large $n \gg 1$, for which

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$$e^2/\hbar v = (\mathrm{Ry}/\varepsilon_p)^{1/2} < 1.$$

This makes the use of the functions of Ref. 9 valid (here $\mathrm{Ry} = m_e e^4/2\hbar^2 = 13.6$ eV).

The amplitude of a process with direct ionization of the atom [intermediate state $\Psi_{\mathbf{p}}(\mathbf{r},t)$] and subsequent scattering of the photoelectron in the continuum by the Coulomb potential of the residual ion [final state $\Psi_{\mathbf{p}'}(\mathbf{r},t)$ are described by the expression

$$A_{\mathbf{p}'}(t) = -i\int_{-\infty}^{t} dt' \int \frac{d\mathbf{p}V}{(2\pi)^3} \langle \Psi_{\mathbf{p}'}(\mathbf{r},t') \left| \frac{e^2}{r} \right| \Psi_{\mathbf{p}}(\mathbf{r},t') \rangle A_{\mathbf{p}}(t')$$

$$= -i\int_{-\infty}^{t} dt' \int \frac{d\mathbf{p}V}{(2\pi)^3} \frac{4\pi e^2}{|\mathbf{p}' - \mathbf{p}|^2 V}$$

$$\times \exp\{i[(\varepsilon_{\mathbf{p}'} - \varepsilon_{\mathbf{p}})t' - \frac{e(\mathbf{p}' - \mathbf{p})\mathbf{E}_0}{m_e\omega^2} \cos\omega t']\} A_{\mathbf{p}}(t'), \quad (2)$$

in which the initial amplitude $A_{\mathbf{p}}(t)$ is given by the equation[9]

$$A_{\mathbf{p}}(t) = \frac{(2z'\omega)^2}{\mathrm{Ry}} \left(\frac{\pi a_0^3}{V}\right)^{1/2} \sum_{k=0}^{\infty} (-1)^k J_{2k}(z' \sqrt{\frac{\varepsilon_p}{\omega}} \cos\theta)$$

$$\times J_{n_0/2-k-1}(z)\zeta^*(\varepsilon_p - \varepsilon)\exp[i(\varepsilon_p - \varepsilon - i\lambda)t]. \quad (3)$$

The following notation was used in (2) and (3): $\mathbf{p}$ and ε_p are the momentum and energy of the electron as the wave field is adiabatically switched off, V is the normalization volume of the functions $\Psi_{\mathbf{p}}(\mathbf{r},t)$;

$$z' = 2eE_0\lambda\hbar/(2m_e\omega)^{1/2}, \qquad z = (eE_0\lambda\hbar)^2/(8m_e\omega)$$

are dimensionless parameters contained in the phase of the basis functions and indicative of the intensity of the electron-wave interaction in accordance with the operator (1); $\lambda\hbar = 1/\omega$ is the wavelength of the light; θ is the angle between the vector $\mathbf{E}$ (the z axis) and the direction of the electron momentum $\mathbf{p}$; $n_0 = \langle \tilde{I}_0/\omega + 1 \rangle$ is the minimum number of field quanta needed to ionize the atom (assumed even to be specific); the symbol $\langle x \rangle$ denotes the integer part of the number x;

$$\tilde{I}_0 = I_0 + (eE_0)^2/(4m_e\omega^2)$$

is the binding energy of the electron in the ground state of the atom with allowance for the average vibrational energy of the electron in the wave field; $\varepsilon = n_0\omega - \tilde{I}_0$ is the excess of the energy of n_0 quanta over the ionization threshold; J_n is a Bessel function; $\zeta^*(x) = \mathcal{P}/x + i\pi\delta(x)$; the parameter $\lambda \approx +0$ corresponds to adiabatic switch-on of the wave at $t \to -\infty$; a_0 is the first Bohr orbit of hydrogen.

The photoelectron accumulates energy after ionization of the atom through scattering of the residual ion in the Coulomb field. Two alternatives are possible here: a) absorption of additional n quanta results from n-fold successive scattering of the photoelectron (Fig. 1a); b) n photons are picked up as a result of one act of photoelectron scattering by an ion (Fig. 1b).

If the photoelectron scattering by the residual ion is accompanied by absorption of an arbitrary number n of additional photons, the calculation of the integrals in (2) leads to the following result:

$$A_{\mathbf{p}_n}(t) = -\frac{2i}{\pi} z'^2 \omega \left(\frac{\varepsilon}{\mathrm{Ry}}\right)^{1/2} \left(\frac{\pi a_0^3}{V}\right)^{1/2} \sum_{k=0}^{\infty} (-1)^k J_{n_0/2-k-1}(z)$$

$$\times \zeta^*(\varepsilon_{\mathbf{p}_n} - \varepsilon - n\omega)\exp\{i(\varepsilon_{\mathbf{p}_n} - \varepsilon - n\omega - i\lambda)t\}$$

$$\times \int_{(4\pi)} d\Omega_p J_{2k}(z' \sqrt{\frac{\varepsilon}{\omega}} \cos\theta) J_n\left[z'\left(\sqrt{\frac{\varepsilon_{\mathbf{p}_n}}{\omega}} \cos\theta_n - \sqrt{\frac{\varepsilon}{\omega}} \cos\theta\right)\right]$$

$$\times \left[\frac{\varepsilon_{\mathbf{p}_n}}{\omega} + \frac{\varepsilon}{\omega} - \left(2\frac{\sqrt{\varepsilon_{\mathbf{p}_n}\varepsilon}}{\omega}\right)\right.$$

$$\left. \times [\cos\theta_n\cos\theta + \sin\theta_n\sin\theta\cos(\varphi_n - \varphi)]\right]^{-1}, \quad (4)$$

where the angles θ_n, φ_n and θ, φ determine respectively the directions of the photoelectron momenta $\mathbf{p}_n$ and $\mathbf{p}$ after and before the scattering; $n = 1, 2, 3, \ldots$. Equation (4) was derived using the pole approximation in the composite matrix element: $\zeta^*(x) \to i\pi\delta(x)$.

The validity of the pole approximation in the problem of two-photon ionization of an atom with highly excited levels, when $\omega > |E_k|$, was discussed in Refs. 2 and 11, where it was shown that at sufficiently large k and small ω the predominant contribution to the amplitude is made by the pole term. There are grounds for hoping that as the number of the quanta in the process increases the pole term will assume a greater role in the transition amplitude, since terms containing δ-functions make contributions of alternating sign to the amplitude.

Further integration in (4) over the azimuthal angle φ of the electron's intermediate state yields the equation

$$A_{\mathbf{p}_n}(t) = (2z')^2\omega(\varepsilon/\mathrm{Ry})^{1/2}(\pi a_0^3/V)^{1/2} \sum_k (-1)^k J_{n_0/2-k-1}(z)$$

$$\times \zeta^*(\varepsilon_{\mathbf{p}_n} - \varepsilon - n\omega)\exp[i(\varepsilon_{\mathbf{p}_n} - \varepsilon - n\omega - i\lambda)t]$$

$$\times \int_0^{\pi} J_{2k}(z'\sqrt{\varepsilon/\omega}\cos\theta) J_n[z'(\sqrt{\varepsilon_{\mathbf{p}_n}/\omega}\cos\theta_n - \sqrt{\varepsilon/\omega}\cos\theta)]$$

$$\times [\varepsilon_{\mathbf{p}_n}/\omega + \varepsilon/\omega - (2\sqrt{\varepsilon_{\mathbf{p}_n}\varepsilon}/\omega)\cos(\theta_n - \theta)]^{-1/2}$$

$$\times [\varepsilon_{\mathbf{p}_n}/\omega + \varepsilon/\omega - (2\sqrt{\varepsilon_{\mathbf{p}_n}\varepsilon}/\omega)\cos(\theta_n + \theta)]^{-1/2}\sin\theta d\theta. \quad (5)$$

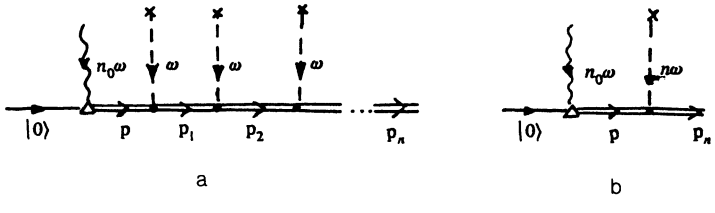

FIG. 1. The vertex represented as a triangle with a wavy line corresponds to the interaction between an electron and the wave field, leading to direct ionization from the ground state 0 to the intermediate state p of the continuous spectrum. The vertex represented as a broken line with a dot describes the interaction between a photoelectron in the continuum and the Coulomb field of the residual ion. The double line with an arrow represents the state of a photoelectron in the continuous spectrum arising in the wave field (the Keldysh method).

This equation is the basis of all the following calculation and has been derived under the most general assumptions concerning the problem parameters. Further analysis of the expressions is possible in the following limiting cases: a) a weak field, when the parameters $z \ll z' \ll 1$ (applicability of perturbation theory with respect to the interaction of the electron with the wave field, in the sense of Ref. 9); b) strong field, when $z' > z \sim 1$, and the n_0-quantum energy does not exceed greatly the ionization threshold, $\varepsilon < \omega$. We analyze hereafter the case of strong field, which is of interest for above-threshold ionization of atoms. In such fields, $(z' > z \sim 1)$, in the threshold region when the parameter $z'\sqrt{\varepsilon/\omega}$ can be regarded as small, $z'\sqrt{\varepsilon/\omega} < 1$, Eq. (5) can be greatly simplified. Retaining in the sum over k the principal term with $k = 0$ and putting $\varepsilon \approx n\omega$, we obtain after integrating in (5)

$$A_{\mathbf{p}_n}(t) = 4z'^2\,\omega \left(\frac{\varepsilon}{\mathrm{Ry}}\right)^{1/2}\left(\frac{\pi a_0^3}{V}\right)^{1/2} J_{n_0/2-1}(z) \times \frac{J_n(z'\sqrt{n}\cos\theta_n)}{n}\,\zeta^*(\varepsilon_{p_n} - n\omega)\exp[i(\varepsilon_{p_n} - n\omega - i\lambda)t], \qquad (6)$$

where, recall, n can be arbitrary: $n = 1, 2, 3, \ldots$.

We turn now to the alternative variant of the cascade process, when the specified number n of additional quanta is the result of n successive scatterings of the photoelectron in the Coulomb field of the residual ion (Fig. 1a). Under the most general assumptions concerning the parameters of the problem, the amplitude of such a transition of an electron to the final state satisfies the recurrence relation

$$\tilde{A}_{\mathbf{p}_n}(t) = [(\mathrm{Ry}\,\varepsilon_{p_{n-1}})^{1/2}/\omega]\pi\delta(\varepsilon_{p_n} - \varepsilon - n\omega) \times \exp[i(\varepsilon_{p_n} - \varepsilon - n\omega - i\lambda)t]\int_0^\pi \sin\theta_{n-1} \times \frac{J_1[z'(\sqrt{\varepsilon_{p_n}/\omega}\cos\theta_n - \sqrt{\varepsilon_{p_{n-1}}/\omega}\cos\theta_{n-1})]}{[\varepsilon_{p_n}/\omega + \varepsilon_{p_{n-1}}/\omega - (2\sqrt{\varepsilon_{p_n}\varepsilon_{p_{n-1}}}/\omega)\cos(\theta_n - \theta_{n-1})]^{1/2}} \times \left[\frac{\varepsilon_{p_n}}{\omega} + \frac{\varepsilon_{p_{n-1}}}{\omega} - \left(\frac{2\sqrt{\varepsilon_{p_n}\varepsilon_{p_{n-1}}}}{\omega}\right)\cos(\theta_n + \theta_{n-1})\right]^{-1/2} \times \tilde{A}_{\mathbf{p}_{n-1}}(\theta_{n-1})d\theta_{n-1}, \qquad (7)$$

where $\tilde{A}_{\mathbf{p}_n}(\theta_n)$ denotes the factor

$$\pi\delta(\varepsilon_{p_n} - \varepsilon - n\omega)\exp[i(\varepsilon_{p_n} - \varepsilon - n\omega - i\lambda)t]$$

in the expression for the amplitude $\tilde{A}_{\mathbf{p}_n}(t)$ (the amplitude of the transition to the energy surface); θ_n and θ_{n-1} are the angles between the electric field $\mathbf{E}$ of the wave and the vectors $\boldsymbol{\rho}_n$ and $\boldsymbol{\rho}_{n-1}$, respectively; the number n can take on the values 2, 3, 4, ...; the amplitude for $n = 1$ is given directly by the general equation (6).

Note that expression (7) is the result of summation of graphs of a definite type, shown in Fig. 1a. The restriction to graphs of just this type presupposes satisfaction of a definite condition imposed on the parameters z' and n, namely: $z' < 2\sqrt{n}$ (here $z' > 1$). Strictly speaking, in the calculation of the amplitude $\tilde{A}_{\mathbf{p}_n}(t)$ account must be taken of the possible processes involving absorption and emission in intermediate states of arbitrary numbers of photons, which lead in the long run to the same finite states of the photoelectron as the processes shown in Fig. 1a. As shown in Appendix III, if the inequality $z' < 2\sqrt{n}$ holds, allowance for graphs with transfer of an arbitrary number of photons in a single scattering act necessitates relatively small corrections to the transition amplitude $\tilde{A}_{\mathbf{p}_n}(t)$ [Eq. (7)].

In expanded form, the expression for the amplitude $\tilde{A}_{\mathbf{p}_n}(t)$ is (we have confined ourselves here to the case $z'\sqrt{\varepsilon/\omega} < 1$ for arbitrary $z' > 1$)

$$\tilde{A}_{\mathbf{p}_n}(t) = 4z'^2\,\omega \left(\frac{\varepsilon}{\mathrm{Ry}}\right)^{1/2}\left(\frac{\pi a_0^3}{V}\right)^{1/2} J_{n_0/2-1}(z)\left(\frac{\mathrm{Ry}}{\omega}\right)^{(n-1)/2} \times [(n-1)!]^{1/2}\pi\delta(\varepsilon_{p_n} - n\omega)\exp[i(\varepsilon_{p_n} - n\omega - i\lambda)t] \times \int_0^\pi \frac{\sin\theta_{n-1}J_1[z'(\sqrt{n}\cos\theta_n - \sqrt{n-1}\cos\theta_{n-1})]}{[(2n-1) - 2\sqrt{n(n-1)}\cos(\theta_n - \theta_{n-1})]^{1/2}} \times [(2n-1) - 2\sqrt{n(n-1)}\cos(\theta_n + \theta_{n-1})]^{-1/2}d\theta_{n-1} \ldots \int_0^\pi \frac{\sin\theta_1 J_1[z'(\sqrt{2}\cos\theta_2 - \sqrt{1}\cos\theta_1)]}{[3 - 2\sqrt{2}\cos(\theta_2 - \theta_1)]^{1/2}} \times [3 - 2\sqrt{2}\cos(\theta_2 + \theta_1)]^{-1/2}J_1(z'\cos\theta_1)d\theta_1. \qquad (8)$$

The general structure of (8) leads to a recurrence relation for the form factor that depends on the direction of the momentum $\mathbf{p}_n$ of the photoelectron in the final state, relative to the electric field-intensity vector $\mathbf{E}$:

$$F_n(\theta_n) = \frac{1}{2n-1} \times \int_0^\pi \frac{\sin\theta_{n-1}J_1[z'\sqrt{n-1}(\sqrt{n/(n-1)}\cos\theta_n - \cos\theta_{n-1})]}{[1 - a_n\cos(\theta_n - \theta_{n-1})]^{1/2}[1 - a_n\cos(\theta_n + \theta_{n-1})]^{1/2}} \times F_{n-1}(\theta_{n-1})d\theta_{n-1}, \qquad (9)$$

where $\alpha_n = 2\sqrt{n(n-1)}/(2n-1)$, $F_1(\theta_1) = J_1(z'\cos\theta_1)$.

In accordance with (8), the function $F_n(\theta_n)$ determines the spectral-angular distribution density of the intensity of the photoelectrons produced as a result of above-threshold ionization of the atoms (in the conventional units):

$$\frac{d\bar{N}}{d\varepsilon_p d\Omega_p} = \frac{1}{\pi\hbar}\,z'^4\,\varepsilon\left(\frac{\hbar\omega}{\mathrm{Ry}}\right)^{5/2} J^2_{n_0/2-1}(z)\sum_{n=1}(n-1)!n^{1/2} \times \left(\frac{\mathrm{Ry}}{\hbar\omega}\right)^{n-1} F_n^2(\theta_n)\delta(\varepsilon_{p_n} - n\hbar\omega). \qquad (10)$$

The expression for the form factor $F_n(\theta_n)$ for $\theta_n = 0$ or π (the angles corresponding to electrons emitted in the direction of the electric field of the wave) is derived in Appendix I. Substituting Eq. (I2) in (10) we obtain

$$\frac{d\bar{N}}{d\varepsilon_p d\Omega_p} \sim \sum_{n=1}\left(\frac{\mathrm{Ry}\,z'^2}{4\omega}\right)^{n-1} \frac{[\prod_{k=1}^{n}\ln(4k)]^2}{n^{1/2}[(2n-1)!!]} J_1^2(z')\delta(\varepsilon_p - n\omega). \qquad (11)$$

Recall that Eq. (11) was derived for $z' \lesssim 2\sqrt{n}$, and that

given z' this condition determines the minimum value of the above-threshold maximum $n_{\min} = \langle (z'/2)^2 + 1 \rangle$, starting with which Eq. (11) is valid.

It is of interest to compare the heights of neighboring above-threshold maxima in the electron-energy distribution as a function of the number n of peaks and of the intensity I of the wave field. For sufficiently large $n \gg 1$ it follows from (11), accurate to a numerical factor of order unity, that

$$\frac{d\bar{N}_n/d\varepsilon_p d\Omega_p}{d\bar{N}_{n-1}/d\varepsilon_p d\Omega_p} \approx \left(\frac{z'}{4}\right)^2 \left(\frac{\mathrm{Ry}}{\omega}\right) \left[\frac{\ln(4n)}{n}\right]^2. \tag{12}$$

Analysis of (12) shows readily that, starting with the minimum number $n_{\min}$ (for a given z') and up to numbers n satisfying the condition $n \leqslant n'_{\max}$, where

$$n_{max}/\ln(4n_{max}) = (z'/4)(\mathrm{Ry}/\omega)^{1/2} \tag{13}$$

the heights of the above-threshold maxima increases wtih increase of n. The succeeding increase of n to $n > n_{\max}$ is accompanied by a monotonic and rather slow decrease of the heights of the maxima. When the radiation intensity I increases, the locations of the highest peaks shift nonlinearly towards larger n, i.e., into the region of higher photoelectron energies corresponding to a plateau of the spectral distribution. The inversion of the heights of the above-threshold maxima, noted in the field of a high-intensity wave, has been well investigated experimentally and is discussed, for example in the reviews by Eberly *et al.*[1] and Fedorov.[2]

Of fundamental importance for the here-discussed mechanism for the onset of above-threshold multiphoton ionization of atoms by a strong field is the question of the relative values of the amplitudes given by Eqs. (6) and (8). Cancelling the common factors in these expressions, we obtain

$$A_{\mathbf{p}_n} \sim J_n(z'\sqrt{n}\cos\theta_n)/n,$$

$$\tilde{A}_{\mathbf{p}_n} \sim (\mathrm{Ry}/\omega)^{(n-1)/2}[(n-1)!]^{1/2} F_n(\theta_n). \tag{14}$$

Assuming the condition $1 < z' < \sqrt{n}$ to be satisfied and using the asymptotic representation of high-order Bessel functions, we obtain, accurate to a numerical factor of order unity,

$$\tilde{A}_{\mathbf{p}_n}/A_{\mathbf{p}_n} \approx nA(z')^{-3/2}(\mathrm{Ry}/4\omega n)^{n/2}. \tag{15}$$

An easily analyzed form of the coefficient A, containing the dependence on n explicitly, is

$$A \approx 1{,}5[\ln(4n)]^n \exp[-\mathrm{Ei}(\ln 4n)/4], \tag{16}$$

where $\mathrm{Ei}(x)$ is the integral exponential function.[12]

Substituting (16) in (15) we obtain for the amplitude ratio

$$\tilde{A}_{\mathbf{p}_n}/A_{\mathbf{p}_n} \approx 1{,}5n(z')^{-3/2}[\mathrm{Ry}(\ln 4n)^2/(4\omega n)]^{n/2} \times \exp[-\mathrm{Ei}(\ln 4n)/4]. \tag{17}$$

Numerical calculations for the real parameters of our problem, $\hbar\omega = 1.17$ eV; $\lambda = 1064$ nm; $E_0 = 1 \cdot 10^8$ V/cm ($I = 1.1 \cdot 10^{13}$ W/cm^2); $z' \approx 3.1$; $z \approx 0.61$ indicate that, starting with $n \lesssim n_0 = 12$ and all the way to $n \gtrsim n_{\max}$ the ratio (17) remains larger than unity. Subsequently, for $n > n_{\max}$, the growth of the absolute value of the photoelectron momentum increase in the act of Coulomb scattering by an ion, the amplitude $A_{\mathbf{p}n}$ begins to predominate over $\tilde{A}_{\mathbf{p}n}$. This, however, takes place in the region of small absolute heights of the above-threshold peaks.

From the standpoint of the theoretical interpretation of ATI effects, it is also of importance to compare the Coulomb amplitude $A_{\mathbf{p}n}$ with the amplitude $A_{\mathbf{p}n}^{(\mathrm{K})}$ obtained for direct photoionization of the atom in the framework of the Keldysh method. The ratio of these amplitudes is

$$\frac{A^{(\mathrm{K})}_{\mathbf{p}_n}}{A_{\mathbf{p}_n}} \approx \omega\,(\varepsilon\mathrm{Ry})^{-1/2} n^{3/2} (z')^{-1} \frac{J_{\tilde{n}-1}(z'\sqrt{n}) J_0(z)}{J_n(z'\sqrt{n}) J_{n_0/2-1}(z)}, \tag{18}$$

where $\tilde{n} = n_0 + n$ determines the total number of absorbed quanta needed to reach a maximum numbered n:

$$\varepsilon_{\mathbf{p}_n} = -I_0 + \tilde{n}\omega.$$

Assuming the factor $\omega/(\varepsilon\mathrm{Ry})^{1/2} \approx 1$ and $J_0(z) \approx 1$, we obtain from (18)

$$\frac{A^{(\mathrm{K})}_{\mathbf{p}_n}}{A_{\mathbf{p}_n}} \approx 10^7 \cdot \frac{(n/n_0)^{(n+n_0/2)}}{(1 + n/n_0)^{(n+n_0)}}. \tag{19}$$

As follows from (19), this relation is less than unity already starting with $n \approx n_0$. For peaks with numbers $n \gg n_0$ it decreases with increase of n like $\sim (n/n_0)^{-n_0/2}$.

3. DISCUSSION OF RESULTS

It follows from the foregoing that the proposed model of multiple Coulomb scattering (MCS) of an electron by a residual ion describes the main distinguishing features of the ATI of atoms.

1. An explanation is given of the photoelectron spectral distribution as a function of the intensity and frequency of the field of a linearly polarized laser wave. The heights of the ATI peaks increase with increase of n to a value $n_{\max}$ given by Eq. (14), and then decreases slowly with increase of n.

2. The value of $n_{\max}$ has a nonlinear dependence on the frequency ω and on the wave-field intensity I.

3. The angular distribution of the photoelectrons is almost isotropic if n is small. In the limit of large $n \gg 1$ the angular width of the peaks near $\theta \approx 0$ (or π) becomes small: $\delta\theta_n \sim 1/n$ (see Appendix II).

4. At low field intensities, when $z' \lesssim (\omega/\mathrm{Ry})^{1/2}$, the absolute peak height is small in proportion to the factor $(z')^{2n}$ and decreases monotonically starting with the lowest n.

5. It follows from (12) that for fixed n the ratio of the amplitudes of two neighboring maxima have the following dependence on the intensity and wavelength of the radiation:

$$\frac{d\bar{N}_n/d\varepsilon_p d\Omega_p}{d\bar{N}_{n-1}/d\varepsilon_p d\Omega_p} \sim I\lambda^4.$$

This dependence does not contradict the results of the numerical calculation for the hydrogen atom[1] and the result ($\sim I\lambda^{10/3}$) of the quasiclassical estimate.[2]

6. In our case of positive-ion ATI the maximum in the distribution of the heights of the ATI peaks is reached at larger values of $n_{\max}$ than for a neutral atom (at fixed parameters of the laser wave). In this case $n_{\max}$ is proportional to the charge of the ion. This theoretical result makes possible

an experimental verification of the MCS model proposed in the present paper.

Strictly speaking, the expressions obtained in this paper are valid for hydrogen or a hydrogenlike ion. For multielectron atoms, the Coulomb interaction of a photoelectron with a residual ion is generally not describable by the simple $\sim 1/r$ law. It seems to us, however, that the qualitative conclusions remain in force.

Note also that interaction between the electron and the wave field was chosen here of the form (1). Our main result [Eqs. (12) and (13)], however, does not depend on the calibration of the interaction.

We conclude by thanking F. Faisal, K. Monzhevskiĭ, and M. V. Fedorov for a discussion of the results.

APPENDIX I

We return to the calculation of the form factor $F_n(\theta_n)$, noting that it is a function symmetric about $\theta_n = \pi/2$. In the analysis that follows we confine ourselves to large numbers $n \gg 1$ (for lower numbers n of above-threshold maxima it is impossible to determine (9) analytically in general form, and numerical evaluations of the integrals is sufficient there). A distinguishing feature of the integrand of (9) is that for $n \gg 1$ the function $[1 - a_n \cos(\theta_n - \theta_{n-1})]$ changes radically in the vicinity of the angles $\theta_{n-1} \approx \theta_n$, and the characteristic width of the change region is $\Delta\theta \approx 1/n$. The remaining functions in the integrand are more continuous functions of the angle θ_{n-1} in the entire integration region $0 \leqslant \theta_{n-1} \leqslant \pi$. The question of the angular width of the form-factor $F_n(\theta_n)$ is not trivial and calls for a separate consideration (see Appendix II).

Attention is called to the particularity of the angle $\theta_n = 0$ and π, corresponding to photoelectron momentum direction along the electric field intensity in the wave. In fact, for $\theta_n = \pi/2$ and $n \gg 1$ the integrand of (9) has no poles at all. For arbitrary $\theta_n \sim 1$ the ensuing smallness of the denominator, of order $1/n$, is offset to a considerable degree by the smallness of the Bessel function $J_1 \sim z'/\sqrt{n}$. On the other hand if $\theta = 0$ (or π) a factor $1/(1 \mp a_n \cos\theta_{n-1})$ appears in the integrand of (9), and integration gives at the lower (upper) limit, in view of the poles, a logarithmic large value $\sim |\ln(1 - a_n)| \approx \ln(8n^2)$. As a result, the recurrence relation (9) takes, for example in the direction of the angle $\theta_n = 0$, the form

$$F_n(0) \approx \frac{z'}{2n^{1/2}(2n-1)^{1/2}} \ln(4n) F_{n-1}(0). \tag{I1}$$

Extending Eq. (I1) to include all the values of n, we obtain for the form factor in the direction of the angle $\theta_n = 0$

$$F_n(0) \approx \left(\frac{z'}{2}\right)^{n-1} J_1(z') \frac{\prod_{k=1}^{n} \ln(4k)}{(n!)^{1/2}(2n-1)!! \ln 4}. \tag{I2}$$

APPENDIX II

Let us estimate the angular width of the form factor $F_n(\theta_n)$ near the angle $\theta_n = 0$, using the expansion

$$F_n(\theta) \approx F_n(0) + \frac{dF_n}{d\theta}\Big|_{\theta=0}\theta + \frac{1}{2}\frac{d^2F_n}{d\theta^2}\Big|_{\theta=0}\theta^2. \tag{II1}$$

The derivatives follow from (9) and are given by

$$\frac{dF_n}{d\theta} = \frac{1}{2n-1}\int_0^\pi \sin\theta' F_{n-1}(\theta') \times \frac{d}{d\theta}\left\{\frac{J_1[z'\sqrt{n-1}(\sqrt{n/(n-1)}\cos\theta - \cos\theta')]}{[1 - a_n\cos(\theta - \theta')]^{1/2}[1 - a_n\cos(\theta + \theta')]^{1/2}}\right\} d\theta';$$

$$\frac{d^2F_n}{d\theta^2} = \frac{1}{2n-1}\int_0^\pi \sin\theta' F_{n-1}(\theta') \times \frac{d^2}{d\theta^2}\left\{\frac{J_1[z'\sqrt{n-1}(\sqrt{n/(n-1)}\cos\theta - \cos\theta')]}{[1 - a_n\cos(\theta - \theta')]^{1/2}[1 - a_n\cos(\theta + \theta')]^{1/2}}\right\} d\theta'.$$

It is easily seen that the derivative $\partial F_n/\partial\theta|_{\theta=0} = 0$ and the characteristic angular width of the form factor is estimated at

$$\delta\theta_n \sim \left[F_n(0)/\frac{d^2F_n}{d\theta^2}\Big|_{\theta=0}\right]^{1/2}. \tag{II2}$$

Calculation of the derivatives leads to the expression

$$\delta\theta_n \sim \left[\frac{J_1(z'/2\sqrt{n})I_1}{(-z'\sqrt{n})J_1'(z'/2\sqrt{n})I_1 + a_n^2 J_1(z'/2\sqrt{n})I_2 - a_n J_1(z'/2\sqrt{n})I_3}\right]^{1/2}, \tag{II3}$$

where J_1' corresponds to the first derivative of a Bessel function with respect to its argument, taken at $\theta = 0$; we denote by I_1, I_2, and I_3 the integrals

$$I_1 = \int_0^\pi \frac{\sin\theta}{1 - a_n\cos\theta}\, d\theta, \quad I_2 = \int_0^\pi \frac{\sin^3\theta}{(1 - a_n\cos\theta)^3}\, d\theta, \quad I_3 = \int_0^\pi \frac{\sin\theta\cos\theta}{(1 - a_n\cos\theta)^2}\, d\theta. \tag{II4}$$

Calculation of these integrals shows that the second and third terms in the denominator of (II3) cancel out, and as a result the value of $\delta\theta_n$ is given, accurate to small terms of order $1/n$, by

$$\delta\theta_n \sim \left[\frac{J_1(z'/2\sqrt{n})}{z'\sqrt{n}J_1'(z'/2\sqrt{n})}\right]^{1/2}. \tag{II5}$$

From (II5) follows (if $z' < 2\sqrt{n}$) the estimate $\delta\theta_n \sim 1/\sqrt{n}$.

APPENDIX III

Using Eq. (8), we compare in lowest order in the Coulomb interaction of a photoelectron with a residual ion, the

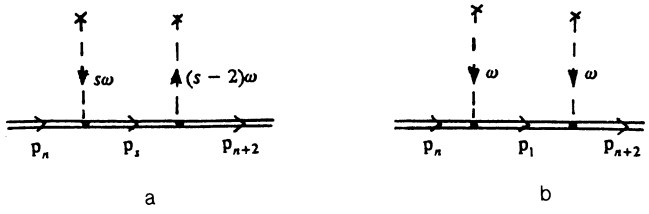

FIG. 2. The symbols are the same as in Fig. 1.

amplitudes of two processes that lead to the same final state of the electron:

$$\tilde{A}^{a)}_{\mathbf{p}_{n+2}} \sim \frac{(n+s)^{1/2}}{(2n+s+2)(2n+s)} \times \int_0^\pi \frac{\sin\theta_s J_{s-2}\left[z'\sqrt{n+s}\left(\sqrt{\frac{n+2}{n+s}}-\cos\theta_s\right)\right]}{1-\frac{2\sqrt{(n+2)(n-s)}}{2n+s+2}\cos\theta_s}\,d\theta_s$$

$$\times \int_0^\pi \frac{\sin\theta_n J_s\left[z'\sqrt{n}\left(\sqrt{\frac{n+1}{n}}-\cos\theta_n\right)\right]}{1-\frac{2\sqrt{(n+s)n}}{2n+s}\cos\theta_n}\,d\theta_n; \quad \text{(III1)}$$

$$\tilde{A}^{b)}_{\mathbf{p}_{n+2}} \sim \frac{(n+1)^{1/2}}{(2n+3)(2n+1)} \times \int_0^\pi \frac{\sin\theta_1 J_1\left[z'\sqrt{n+1}\left(\sqrt{\frac{n+2}{n+1}}-\cos\theta_1\right)\right]}{1-\frac{2\sqrt{(n+2)(n+1)}}{2n+3}\cos\theta_1}\,d\theta_1$$

$$\times \int_0^\pi \frac{\sin\theta_n J_1\left[z'\sqrt{n}\left(\sqrt{\frac{n+1}{n}}-\cos\theta_n\right)\right]}{1-\frac{2\sqrt{(n+1)n}}{2n+1}\cos\theta_n}\,d\theta_n. \quad \text{(III2)}$$

These amplitudes correspond to the graphs of Fig. 2.

The number s of photons picked up by an electron in an immediate state of amplitude $\tilde{A}^{a)}_{\mathbf{p}_n+2}$ can be arbitrary: $s = 3, 4, \ldots$, but must satisfy the condition $s < n$. Using the estimates obtained in Appendix I for the standard integral in the form factor (9), we get

$$\tilde{A}^{a)}_{\mathbf{p}_{n+2}} \sim \frac{1}{[s(s-2)]^{1/2}}\left(\frac{z'}{2\sqrt{n}}\right)^{2(s-1)}, \qquad \tilde{A}^{b)}_{\mathbf{p}_{n+2}} \sim \left(\frac{z'}{2\sqrt{n}}\right)^2 \quad \text{(III3)}$$

[in the derivation of (III3) we used for high-order Bessel functions an asymptotic representation valid if the inequality $z' < 2\sqrt{n}$ and the condition $s < n$ are valid].

It follows from (III3) that for $z' < 2\sqrt{n}$ and for arbitrary s the amplitude $\tilde{A}^{a)}_{\mathbf{p}_n+2}$ is small compared with $\tilde{A}^{b)}_{\mathbf{p}_n+2}$ and decreases rapidly with increase of s.

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Translated by J. G. Adashko