New logarithmic contributions in muonium and positronium

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The leading logarithmic contributions to the hyperfine splitting and to the decay widths of positronium, of relative order $\alpha^3\ln^2(1/\alpha)$, and to the hyperfine splitting in muonium, of relative order $\alpha^4\ln^2(1/\alpha)$ and $\alpha^3(m/M)\ln^2(1/\alpha)$, are found. The contribution to the Lamb shift in hydrogen of order of magnitude $\alpha^2(Z\alpha)^6mc^2\ln^3(1/Z\alpha)$ is also obtained.

At present, the theoretical uncertainty in the hyperfine splitting in positronium and the decay width of orthopositronium are due to the uncalculated corrections of relative order α^2, while the theoretical uncertainty in the hyperfine splitting in muonium is due to uncalculated corrections of order α^3. However, contributions of higher order, containing the squared logarithm $\ln^2(1/\alpha)$, may also turn out to be important. These corrections are of order $\alpha^3\ln^2(1/\alpha)$ for positronium and $\alpha^4\ln^2(1/\alpha)$ and $\alpha^3(m/M)\ln^2(1/\alpha)$ in muonium, and the present paper is devoted to calculating them.

First we discuss the derivation of the familiar contributions[1,2] of order $\alpha^3\ln^2(1/\alpha)$ in hydrogen-like systems, and then proceed to a discussion of these corrections in positronium and of the $\alpha^4\ln^2(1/\alpha)$, and $\alpha^3(m/M)\ln^2(1/\alpha)$ contributions in muonium. In conclusion, we consider the corrections to the Lamb shift that have order of magnitude $\alpha^2(Z\alpha)^6mc^2\ln^3(1/Z\alpha)$.

LOGARITHMIC CONTRIBUTION OF ORDER $\alpha^3\ln^2(1/\alpha)$ TO THE HYPERFINE SPLITTING IN HYDROGEN-LIKE ATOMS

The contribution of relative order $\alpha(Z\alpha)^2\ln^2(1/Z\alpha)$ to the hyperfine splitting in the ground state of the hydrogen and muonium atoms has been found in the external-field limit (i.e., in leading order in relation to the masses) in Refs. 1 and 2. It can be shown that it arises from the diagram of Fig. 1, in which the thick line denotes the electron Coulomb Green's function and the dashed line denotes Coulomb exchange. This can be seen most explicitly in the Fried–Yennie gauge.[3] At the same time, the contribution of this graph can also be calculated without performing an expansion in the ratio of the masses of the electron and nucleus:

$$\delta\nu_1 = -\frac{8}{3}\frac{\alpha}{\pi}(Z\alpha)^2\left(\frac{m_R}{m}\right)^2\ln^2\left(\frac{1}{Z\alpha}\right)\nu_F. \qquad (1)$$

Here we have used the standard notation for the electron mass (m), the reduced mass (m_R), the nuclear charge in units of the electron charge (Z), and the frequency corresponding to the energy of the Fermi hyperfine splitting:[1)]

$$\nu_F = \frac{16}{3}(Z\alpha)^2(Z^2\mathrm{Ry})\frac{\mu}{\mu_B}\left(\frac{m_R}{m}\right)^3. \qquad (2)$$

The magnetic moment μ of the nucleus includes the Dirac and the anomalous magnetic moments, and the Rydberg constant, here and below, in in frequency units. In muonium, hydrogen, and positronium the constant Z is equal to unity, but its presence indicates the origin of the contributions and so it is traditionally retained.

We shall discuss certain stages in the derivation of Eq. (1). We represent the Coulomb Green's function of the electron in the form of a sum over states:

$$G_c(E_1) = \sum_{n\neq 1}\frac{|nlm\rangle\langle nlm|}{E_1 - E_n} + \sum_{lm}\int\frac{dk}{2\pi}\frac{|klm\rangle\langle klm|}{E_1 - E_k},$$

in which we shall be interested in the nonrelativistic range of momenta so that $|nlm\rangle$ and $|klm\rangle$ are wave functions of the Schrödinger equation with the reduced mass, multiplied by free spinors of the electron and the nucleus. The Green's function is calculated with the energy equal to the value of the ground-state energy. We divide the total contribution into two terms, arising respectively from the continuous and discrete parts of the spectrum. The leading logarithmic contribution arises from the continuous part, and this is the part that we shall consider below. The discrete part of the spectrum leads, as is easily seen, only to the first power of the logarithm.

The correction to the energy of the splitting of the ground state has the form

$$\delta\nu_{ext} = 2\sum_{lm}\int\frac{dk}{2\pi}\frac{1}{E_1 - E_k}\langle 100|\Lambda_e|klm\rangle\langle klm|T|100\rangle, \qquad (3)$$

where Λ_e is the contribution of the electric form factor of the electron to the vertex and T is the kernel of the transverse-photon exchange. It is not difficult to calculate the matrix elements in (3), the first of which is analogous to the expression for the Lamb shift while the second is analogous to the expression for the energy of the Fermi hyperfine splitting:

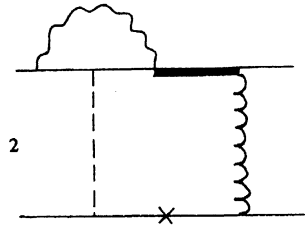

FIG. 1. Graph leading to a correction of order $\alpha(Z\alpha)^2\nu_F\ln^2(1/Z\alpha)$ to the hyperfine splitting in a hydrogen-like atom.

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$$\langle 100|\Lambda_e|klm\rangle = -\frac{8}{3\pi}\alpha(Z\alpha)^4\frac{m_R^3}{m_2}\ln\left(\frac{k}{m}\right)\delta_{l0}\delta_{m0}\frac{|\phi_k(0)|}{|\phi_1(0)|}, \tag{4}$$

$$\langle klm|T|100\rangle = \nu_F\delta_{l0}\delta_{m0}\frac{|\phi_k(0)|}{|\phi_1(0)|}, \tag{5}$$

where $\phi_k(0)$ is the value at zero of a wave function from the continuous spectrum:

$$\phi_k^2(0) = \frac{\gamma_R^3}{\pi}\frac{k/\gamma_R}{1-\exp(-2\pi\gamma_R/k)}\frac{2\pi}{\gamma_R},$$

and $\phi_1(0)$ is the value at zero of the ground-state wave function:

$$\phi_1^2(0) = \gamma_R^3/\pi.$$

Explicit substitution of these values, moderately large nonrelativistic momenta k, i.e., for

$$\gamma_R = Z\alpha m_R \ll k \ll m,$$

leads to the integral

$$\delta\nu_{ext} = \frac{8}{3\pi}\alpha(Z\alpha)\nu_F\frac{m_R}{m}\int_\gamma^m dk\left(\frac{2}{\pi m}+\frac{2\gamma_R}{km}\right)\ln\left(\frac{k}{m}\right). \tag{6}$$

The first term, which arises upon expansion of the Coulomb Green's function in the integrand, corresponds to the free electron propagator, and to the familiar contribution of order $\alpha(Z\alpha)\nu_F$ (see Ref. 4). The second term, containing γ_R, arises from the inclusion of one Coulomb exchange in the Coulomb Green's function, and, after integration over the logarithmic region, leads to the result (1). Thus, one of the logarithms is the standard infrared logarithm in the electric form factor while the other arises, upon logarithmic integration, from the block represented graphically in Fig. 2, in which the heavy points denote a delta-function potential. Diagrams of this kind are responsible in positronium for logarithmic contributions of relative order $\alpha^2\ln(1/\alpha)$ (Refs. 5–7).

The dependence of the contribution of order $\alpha(Z\alpha)^2\ln^2(1/Z\alpha)$ on the mass ratio m/M can be restored using dimensional considerations. In fact, the "normal" mass in the nonrelativistic problem, which, in particular, arises in the calculation of the leading infrared logarithms, is precisely the reduced mass m_R, and powers of the electron mass m or nuclear mass M can be easily traced, as a rule. For example, the Fermi energy (2) explicitly contains the product of the magnetic moments of the electron and nucleus, and, consequently, is proportional to $(mM)^{-1}$. The contribution (1) arises from the product of the electric form factor ($\sim m^{-2}$) and the contact interaction of the magnetic moments [$\sim(mM)^{-1}$]. The second factor leads finally to the Fermi energy, while the first gives an additional factor $(m_R/m)^2$, which corresponds to (1).

Besides the purely radiative contribution of order $\alpha(Z\alpha)^2\ln^2(1/Z\alpha)$ there are corrections of relative order $(Z\alpha)^3(m/M)\ln^2(1/Z\alpha)$ and $Z^2\alpha(Z\alpha)^2(m/M)^2\ln^2(1/Z\alpha)$, which appear when recoil effects are taken into account. As is well known, corrections for recoil, unlike purely radiative contributions, are not universal. However, the leading infrared logarithmic corrections of order $(Z\alpha)^3(m/M)\ln^2(1/Z\alpha)$ and $Z^2\alpha(Z\alpha)^2(m/M)^2\ln^2(1/Z\alpha)$ nevertheless have the same form for atoms with leptonic and hadronic nuclei.

Corrections of order $(Z\alpha)^3(m/M)\ln^2(1/Z\alpha)$ arise when one takes into account two-photon interaction (Fig. 3), which leads to corrections to the Lamb shift[8] of order $(Z\alpha)^5(m^2/M)\ln(1/Z\alpha)$. The corresponding contribution to the hyperfine splitting can be calculated by making a simple substitution in the matrix element (4) for the Lamb shift (cf. Refs. 7 and 8):

$$\langle 100|\Lambda_e|klm\rangle \to \langle 100|Recoil|klm\rangle$$
$$= -\frac{2}{3\pi}(Z\alpha)^5\left(\frac{m^2}{M}\right)\ln\left(\frac{k}{m}\right)\delta_{l0}\frac{|\phi_k(0)|}{|\phi_1(0)|},$$

after which it is not difficult to find the contribution to the hyperfine splitting:

$$\delta\nu_{rec} = -\frac{2}{3}\frac{(Z\alpha)^3}{\pi}\frac{m_R^2}{mM}\ln^2\left(\frac{1}{Z\alpha}\right)\nu_F. \tag{7}$$

Corrections of order $Z^2\alpha(Z\alpha)^2(m/M)^2\ln^2(1/Z\alpha)$ appear when one takes into account the simplest electrodynamic form-factor correction to the vertex function of the nucleus (Fig. 4). It is obvious that this contribution differs from (1) by the trivial replacement

$$\frac{\alpha}{m^2}\to\frac{Z^2\alpha}{M^2},$$

and, consequently

$$\delta\nu_{nucl} = -\frac{8}{3}\frac{\alpha}{\pi}(Z\alpha)^2\left(\frac{m_R}{M}\right)^2\ln^2\left(\frac{1}{Z\alpha}\right)\nu_F. \tag{8}$$

The total contribution of relative order $\alpha^3\ln^2(1/\alpha)$ is obtained by summing the corrections (1), (7), and (8) found above:

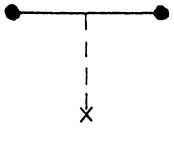

FIG. 2. Two-loop block leading to a lorgarithmic integration.

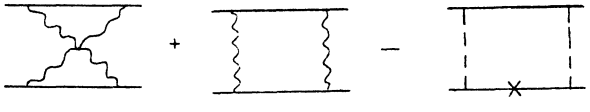

FIG. 3. Two-photon graphs for the Lamb shift.

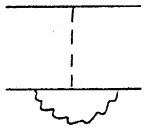

FIG. 4. Electrodynamic correction to the form factor of the nucleus.

$$\delta\nu_{log} = -\frac{8}{3}\frac{\alpha}{\pi}(Z\alpha)^2\left(\frac{m_R}{m}\right)^2 \times\left[1+\frac{1}{4}Z\frac{m}{M}+Z^2\left(\frac{m}{M}\right)^2\right]\ln^2\left(\frac{1}{Z\alpha}\right)\nu_F. \quad (9)$$

It is important to note that, since we have not performed expansions in the mass ratio, the results (1) and (7)–(9) obtained above are also valid for positronium, although the form of the diagrams is somewhat asymmetric when the electron is replaced by the positron. The point is that, in effect, the Coulomb Green's function corresponds to a certain effective particle (with the reduced mass) which, in an "ordinary" atom with a heavy nucleus, is "almost" an electron. Nevertheless, to emphasize the analogy between hydrogen and positronium, we shall use the asymmetric representation of the Coulomb Green's function. Henceforth, for brevity in discussing the corrections, we shall give only diagrams with form-factor contributions to the Lamb matrix element, and, for positronium, taking into account the equality of the contributions (1) and (8), we shall double the corresponding coefficient.

POSITRONIUM

The radiative contributions for the ground state of positronium are obtained from the expression (9) by the substitution $m = M$. In this way the contributions of the exchange interaction are taken into account. In addition, it is necessary also to consider the annihilation diagrams.

The contribution of the exchange graphs (see Fig. 1) leads in the case of positronium to the result

$$\delta\nu^T_{log} = -\frac{3}{2}\frac{\alpha^3}{\pi}\ln^2\left(\frac{1}{\alpha}\right)\nu^T_F. \quad (10)$$

The right-hand side of this equality is that part of the Fermi hyperfine splitting which arises from the exhange channel [cf. (2)]:

$$\nu^T_F = \frac{2}{3}\alpha^2\mathrm{Ry}. \quad (11)$$

It is easy to find the contribution of the annihilation diagrams (see Fig. 5). For this it is sufficient to replace the second matrix element in the expression (5) for the magnitude of the splitting by the contribution of the annihilation kernel to the Fermi energy of positronium

$$\nu^A_F = \frac{1}{2}\alpha^2\mathrm{Ry}, \quad (12)$$

and this leads quickly to the result

$$\delta\nu^A_{\log} = -\frac{3}{2}\frac{\alpha^3}{\pi}\ln^2\left(\frac{1}{\alpha}\right)\nu^A_F. \quad (13)$$

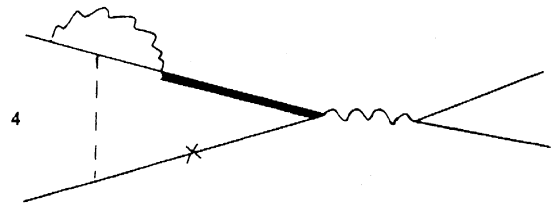

FIG. 5. Annihilation diagram for the contribution of order $\alpha^3\ln^2(1/\alpha)\nu_F^4$ to the hyperfine splitting in positronium.

In fact, the integration over the nonrelativistic momentum k in the derivation of the expressions (9), (10), and (13) has the same structure [see (3)], and contains a product of two hard matrix elements, the first of which is the same in all three cases [see (4)]. The second matrix element [see (5)] in the nonrelativistic approximation is a delta function. Replacement of the contact interaction of the magnetic moments by the annihilation kernel, which is also a delta function in the nonrelativistic limit, does not change the logarithmic character of the integration over the momentum k. We note also that the annihilation diagrams (see Fig. 5) also make a contribution to the Lamb shift of the s-levels that differs from the quantity (13) only by the obvious factor $3/4n^3$.

We now compare the radiative corrections for annihilation that are represented in the graphs of Fig. 5 and Figs. 6 and 7. In the derivation of the contribution (13) from one-photon annihilation the matrix element from the hard annihilation kernel [see (12)] was not sensitive to the internal structure of this kernel, and so the relative magnitudes of the corrections for the diagrams with one-, two-, and three-photon annihilation also coincide:

$$\delta\Gamma_{O-Ps} = -\frac{3}{2}\frac{\alpha^3}{\pi}\ln^2\left(\frac{1}{\alpha}\right)\Gamma^{(0)}_{3\gamma}, \quad (14)$$

$$\delta\Gamma_{P-Ps} = -\frac{3}{2}\frac{\alpha^3}{\pi}\ln^2\left(\frac{1}{\alpha}\right)\Gamma^{(0)}_{2\gamma}, \quad (15)$$

where, in the right-hand sides of the expressions (14) and (15), we have introduced obvious notation for the leading contributions to the decay width of orthopositronium and parapositronium.

Adding to the results (12)–(15) obtained above the well known[5,9] logarithmic contributions, we obtain expressions for the logarithmic terms:

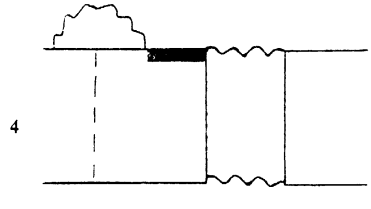

FIG. 6. Typical diagram with imaginary part corresponding to a contribution to the decay width of parapositronium.

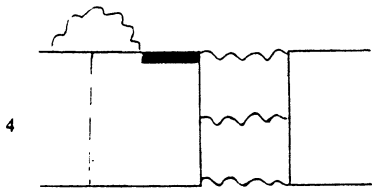

FIG. 7. Characteristic graph for the logarithmic correction of order $\alpha^3\ln^2(1/\alpha)$ to the decay width of orthopositronium.

$$\nu_{hfs} = \alpha^2 \mathrm{Ry}\left[\frac{7}{6} + \frac{1}{12}\alpha^2 \ln\left(\frac{1}{\alpha}\right) - \frac{7}{4}\frac{\alpha^3}{\pi}\ln^2\left(\frac{1}{\alpha}\right)\right], \quad (16)$$

$$\Gamma_{O-Ps} = \Gamma^{(0)}_{3\gamma}\left[1 - \frac{1}{3}\alpha^2 \ln\left(\frac{1}{\alpha}\right) - \frac{3}{2}\frac{\alpha^3}{\pi}\ln^2\left(\frac{1}{\alpha}\right)\right], \quad (17)$$

$$\Gamma_{P-Ps} = \Gamma^{(0)}_{2\gamma}\left[1 + 2\alpha^2 \ln\left(\frac{1}{\alpha}\right) - \frac{3}{2}\frac{\alpha^3}{\pi}\ln^2\left(\frac{1}{\alpha}\right)\right], \quad (18)$$

in which the leading contributions are also written out explicitly.

The corrections of relative order $\alpha^2\ln(1/\alpha)$ can also be calculated by the method developed above. As an example we shall discuss the derivation of the transverse-exchange correction of this order to the hyperfine splitting (16). In this case there are two somewhat different methods of calculation. First, we can take both matrix elements, and, after this, the integral over the momentum. This calculation is completely analogous to those performed above for the corrections of order $\alpha^3\ln^2(1/\alpha)$. Second, it is possible to find first the second matrix element [see (3) and (5)] and immediately take the integral over k:

$$\sum_{lm}\int \frac{dk}{2\pi}\,|klm\rangle \frac{1}{E_1 - E_k}\langle klm|T|100\rangle \equiv \delta_T\phi_1(0),$$

where $\delta_T\phi_1(0)$ is the effective correction made by the transverse exchange to $\phi_1(0)$ (the value of the wave function at the coordinate origin). Calculation of the quantity $\delta\phi_1(0)$ with subsequent determination of the contribution to the energy or width of the decay leads to the formalism developed in Refs. 5 and 7 for the logarithmic corrections of order $\alpha^2\ln(1/\alpha)$ in positronium.

The above-determined corrections (10), (13)–(15) of order $\alpha^3\ln^2(1/\alpha)$ to the hyperfine splitting and the decay widths can be represented conveniently in units of $(\alpha/\pi)^2\nu_F$ and $(\alpha/\pi)^2\Gamma^{(0)}$, respectively. The corrections to the frequency and to the widths amount in these units to $-0.83...$, and are commensurate with the corrections of relative order $(\alpha/\pi)^2$ that have not been calculated to completion.

HYPERFINE SPLITTING IN MUONIUM

It is also possible to discuss the leading logarithmic corrections of relative order $\alpha^2(Z\alpha)^2\ln^2(1/Z\alpha)$, $\alpha(Z\alpha)^2(m/M)\ln^2(1/Z\alpha)$, and $(Z\alpha)^3(m/M)\ln^2(1/Z\alpha)$ to the hyperfind splitting in muonium. The latter have already been found in the expression (9), which it is now sufficient to expand in the ratio of the electron and muon masses. The determination of the contribution of order $\alpha^2(Z\alpha)^2\ln^2(1/Z\alpha)$ is related to the diagram of Fig. 8, the desired correction being picked up in the case when one of the vertex insertions is the electric form factor and the other is the anomalous magnetic moment. The correction differs only by the obvious factor $\alpha/2\pi$ from the known contribution (1). Summing all the corrections with the logarithm squared, we obtain

$$\delta\nu_{log} = -\frac{8}{3}\frac{\alpha}{\pi}(Z\alpha)^2\left(1 + \frac{\alpha}{2\pi} - 2\frac{m}{M} + \frac{1}{4}Z\frac{m}{M}\right)\ln^2\left(\frac{1}{Z\alpha}\right)\nu_F. \quad (19)$$

At the present time the principal theoretical uncertainty is due to contributions of order α^3/π (Refs. 10–13), and, therefore, the corrections found can be usefully represented in units of $(\alpha^3/\pi)\nu_F$. The contributions of order $\alpha^2(Z\alpha)^2\ln^2(1/Z\alpha)$, $\alpha(Z\alpha)^2(m/M)\ln^2(1/Z\alpha)$, and $(Z\alpha)^3(m/M)\ln^2(1/Z\alpha)$ amount, in these units, to $-0.075...$, $0.62...$, and $-0.08...$, respectively. We also give the values of these corrections to the hyperfine splitting directly in kilohertz:

$$\delta\nu_{AMM} = -0{,}04 \text{ kHz}, \quad \delta\nu_{rec} = -0{,}04 \text{ kHz},$$

$$\delta\nu_{ext} = 0{,}34 \text{ kHz},$$

where the last quantity exceeds the error in the numerical integration in the calculation of the contribution of relative order $\alpha(Z\alpha)^2$ (see below).[10] We note also that, as well as the leading logarithmic contributions, containing low-energy logarithms $\ln(1/Z\alpha)$ (16) or mass-ratio logarithms $\ln(M/m)$ (Ref. 14), nonlogarithmic corrections can also turn out to be important. For example, we can write an exact expression for the magnitude of the splitting in the Dirac atoms (see, e.g., Ref. 15). The Fermi energy (2) is multiplied by the factor

$$1 + \frac{3}{2}(Z\alpha)^2 + \frac{17}{8}(Z\alpha)^4 + \ldots,$$

in which the terms of order $(Z\alpha)^4$ make a contribution commensurate with the leading logarithmic corrections of order $\alpha^4\ln^2(1/\alpha)$ (16) and of order $\alpha^3(m/M)\ln^3(M/m)$ and $\alpha^3(m/M)\ln^2(M/m)$ (Ref. 14).

We shall examine briefly the theoretical and experimental uncertainties for the hyperfine splitting of the ground state in muonium. The error in the measurement of the hy-

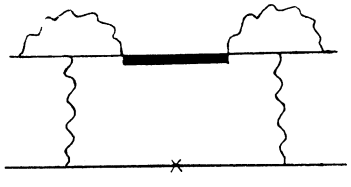

FIG. 8. Leading logarithmic diagram for two-loop insertions into an electron line.

perfine-splitting frequency amounts to 0.16 kHz (Ref. 16). More important is the error in the determination of the Fermi energy (2), corresponding to measurement of the muon magnetic moment in Bohr magnetons; in frequency units this error is equal to 1.3 kHz (Refs. 16, 17). Both these errors will be reduced in the near future by an order of magnitude.[18] The theoretical uncertainties also arise from several sources. The error in the numerical integration in the calculation of the corrections of relative order $\alpha(Z\alpha)^2$ amounts to 0.2 kHz (Ref. 10). The terms of order $\alpha^2(Z\alpha)$ have not been completely determined; the contributions of only five of the six gauge-invariant sets of diagrams have been calculated.[11-13] In the Appendix we give a preliminary estimate of the contribution of the sixth set of diagrams, which corresponds to two-loop insertions in the electron line (Fig. 9):

$$\delta\nu_6 = -1{,}47(48)\frac{\alpha^2(Z\alpha)}{\pi}\nu_F \approx -0{,}83(27)\ \text{kHz}, \qquad (20)$$

which, together with the contributions of order $\alpha(Z\alpha)^2(m/M)\ln^2(1/Z\alpha)$ and $(Z\alpha)^3(m/M)\ln^2(1/Z\alpha)$ found above [see (19)], leads to the following theoretical value of the frequency of the hyperfine splitting of the ground state in muonium:

$$\nu_{th} = 4463303{,}2(1{,}3)(0{,}3)\ \text{kHz}, \qquad (21)$$

where the first error relates to the measurement of the muon magnetic moment and the second relates to the theoretical calculations.

It is also possible to estimate the uncertainty introduced by the corrections of order $\alpha(Z\alpha)^2(m/M)\ln(1/Z\alpha)$ and $\alpha(Z\alpha)^2(m/M)$. Starting from the known values of the contributions of order $\alpha(Z\alpha)^2\ln(1/Z\alpha)$ (Refs. 1, 2), $\alpha(Z\alpha)^2$ (Ref. 10), $(Z\alpha)^2(m/M)\ln(1/Z\alpha)$, and $(Z\alpha)^2(m/M)$ (Ref. 19), we can conclude that these corrections amount to no more than 0.1 kHz.

The experimental result[16]

$$\nu_{exp} = 4463302{,}88(0{,}16)\ \text{kHz} \qquad (22)$$

is in excellent agreement with the theoretical result (21).

THE LAMB SHIFT

We convinced ourselves above that the square of the logarithm can compensate one power of the small parameter α or m/M. A similar situation also builds up in the Lamb shift. Using the spin-independent terms of the Breit Hamiltonian (for $l=0$) as a perturbation in the calculation of the second matrix element [cf. (3)] we can find the doubly logarithmic corrections. In particular, the contribution induced by the correction to the wave function (cf. Refs. 20 and 21) is, in the external-field limit,

$$\delta\nu_{ext}(n) = \nu_L(n)(Z\alpha)^2\ln^2\left(\frac{1}{Z\alpha}\right) \times\left\{2\left(\frac{m_R}{m}\right)^3\left[1+\left(\frac{m}{M}\right)^3\right] - \left(\frac{m_R}{m}\right)^2\left[1+\left(\frac{m}{M}\right)^2\right]+\frac{4m_R^2}{mM}\right\}\delta_{l0}, \qquad (23)$$

where $\nu_L(n)$ is the coefficient of the logarithm $\ln[1/(Z\alpha)^2]$ in the leading contribution to the Lamb shift:

$$h\nu_L(n) = \frac{4}{3}\frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\frac{m_R^3}{m^2}. \qquad (24)$$

This doubly logarithmic correction (23) to the Lamb shift is completely analogous to the correction (1) to the hyperfine splitting. In the same way it is easy to take account of the recoil [cf. (7)] and of the electrodynamic correction to the form factor of the nucleus [cf. (8)]. In the case of the Lamb shift there is no further, more complicated contribution,[2)] that does not reduce to graphs with a factorized block structure (cf. Fig. 1). It too can be calculated, although allowance for the ratio of the masses in the doubly logarithmic corrections in the Lamb shift is less urgent than in the hyperfine splitting. The point is that the shifts in muonium and positronium have been measured insufficiently accurately, while in hydrogen the corrections are too small because of the small mass ratio m_e/m_p. However, in the case of the Lamb shift there is a triply logarithmic contribution that is important for comparison with experiment. Thus, the graph depicted in Fig. 8, under the condition that both vertices are electric form factors, makes a logarithmic contribution to the Lamb shift. In contrast to (3), there is only one diagram, and the matrix elements coincide and are equal to (4). After straightforward calculations, for the contribution to the Lamb shift of the s-level with principal quantun number n we obtain

$$\delta\nu_{log}(n) = -\frac{16}{9}\nu_L(n)\frac{\alpha}{\pi}(Z\alpha)^2\ln^3\left(\frac{1}{Z\alpha}\right)\delta_{l0}. \qquad (25)$$

It can be shown that there are no triply logarithmic contributions analogous to the doubly logarithmic corrections induced by the graphs with kernels M_0, M_1, and M_2 (see Ref. 21).

The correction (25) found amounts numerically to $-0.49\alpha^2\delta\nu_L(n)$, or -3.6 kHz (for $n=2$), and is important for comparison of theory and experiment.

We note also the following circumstance. The standard structure of the logarithmic corections for Coulomb systems

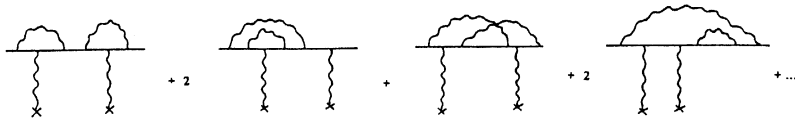

FIG. 9. Some graphs for contributions of order $\alpha^2(Z\alpha)\nu_F$.

involves expressions proportional to the quantity $[Z\alpha^2\ln(1/Z\alpha)]^n$. We could also consider here a series built from logarithmic corrections of the form

$$\nu_L(n)\ln\left(\frac{1}{Z\alpha}\right)\left[\alpha(Z\alpha)^2\ln^2\left(\frac{1}{Z\alpha}\right)\right]^n\delta_{l0},$$

the first term of which is the leading contribution to the Lamb shift, while the second was found above [see (25)]. These logarithmic corrections arise from graphs consisting of electron form-factor blocks linked by Coulomb Green's functions.

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APPENDIX: CONTRIBUTION OF DIAGRAMS WITH TWO-LOOP INSERTIONS IN AN ELECTRON LINE TO THE HYPERFINE SPLITTING IN MUONIUM

The contribution of diagrams with two-loop insertions can be expressed in terms of the product of the square of the Coulomb wave function $\phi_1(0)$ and matrix elements (between the large components of the spinors) of kernels corresponding to 30 diagrams for the two-loop virtual Compton amplitude for forward scattering of an electron with a spin flip (see Fig. 9). All the radiative corrections should be renormalized, and the contribution of the anomalous moment subtracted. The corresponding correction is equal to

$$\frac{\alpha^2(Z\alpha)}{\pi}\nu_F\frac{8}{\pi^2}m\int_0^\infty\frac{dk}{k^2}F(k), \tag{A1}$$

where F is the effective form factor, equal to unity in the absence of the radiative corrections, and k is the three-dimensional momentum of integration. About the quantity $F(k)$ it is known that at large momenta it behaves as a constant while at small momenta it contains a logarithm:

$$F(k)\approx-\frac{4}{3}\ln\left(\frac{4m^2}{k^2}\right)\frac{k^2}{4m^2}. \tag{A2}$$

The coefficient of the logarithm is easily found: It can be shown that in the Fried–Yennie gauge it arises from the contribution of the first two graphs of Fig. 9. The low-energy asymptotic form of the first graph is proportional to the electric form factor of the electron, while that of the second is proportional to the product of the Coulomb exhange and the form factor corresponding to the anomalous moment. We shall assume that the quantity $F(k)$ does not change sign and that the unknown constants have the same order of magnitude as the coefficient of the logarithm. In this case we can propose the following approximation for $F(k)$:

$$F(k)\approx-\frac{4}{3}\left[\frac{k^2}{k^2+4m^2}\ln\frac{k^2+4m^2}{k^2}\pm\frac{4m^2k^2}{(k^2+4m^2)^2}+\left(\frac{1}{2}\pm\frac{1}{2}\right)\frac{k^4}{(k^2+4m^2)^2}\right],$$

integration of which as in (A1) leads to the result $-1.47(48)[\alpha^2(Z\alpha)\pi]\nu_F$ used in the text. We note that the analogously determined estimate for the known[4] one-loop contribution of order $\alpha(Z\alpha)\nu_F$ leads to the coefficient $-2.94(95)$ instead of $-2.56...$.

It is possible to obtain a preliminary estimate close to this from considerations based on the analogy between the effective one-loop and two-loop form factors as functions of the square k^2 of the Euclidean momentum. These form factors possess similar analytical properties and have two identical branch points, at values of k^2 equal to zero and $-4m^2$. The asymptotic forms of the effective form factors are also analogous to each other in many respects: In both cases logarithmic terms are present at small momenta and absent at large momenta. The coefficients of the low-energy logarithm differ by the factor $\alpha/2\pi$. We shall assume that the characteristic behavior of these effective form factors is approximately the same, and that, therefore, the two-loop contribution is equal (to within 50%) to the one-loop form factor[4] with the factor $\alpha/2\pi$; this leads to a numerical coefficient $-1.28(64)$ instead of the coefficient $-1.47(48)$ obtained above.

The preliminary estimates obtained here have a somewhat speculative character. Nevertheless, they look rather plausible, and, not unimportantly, they reinforce each other.

[1)]In this paper we use relativisitc units with $\hbar=c=1$.
[2)]In the terminology of Ref. 21 it corresponds to matrix elements of the kernels M_0, M_1, and M_2.

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