Does the statistical attractor of the nonlinear Schrödinger equation display a dark soliton?

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Particular solutions in the shape of solitons play an important role in integrable wave systems and an even more important one in nonintegrable systems. Nonintegrable systems with wave repulsion are able to form dark solitons—local reductions in the wave intensity. In the present paper we use a thermodynamic analysis and the stability of the dark solitons of the nonlinear Schrödinger equation (NLS) to formulate a hypothesis about the possibility that their modulational amplitude increases and that they display attractor properties. We state the initial conditions for the NLS which make it possible to check experimentally and numerically the predicted manifestations of attractor properties of dark solitons.

1. INTRODUCTION

Particular soliton solutions are able to emerge as the result of the evolution of a broad class of initial conditions. For integrable systems the solitons appear in the decay of localized perturbations and the amplitude of the solitons is in that case unchanged when they collide with one another.[1,2] One can say that the solitons were superimposed on one another in the initial condition. The evolution merely made it possible for them to disperse and to be made visible. In the more general nonintegrable systems, solitons can grow by absorption of waves and this makes them even more important. Initially only solutions of integrable systems were called solitons and the term "solitary wave" was used for nonintegrable systems. Later, however, all localized solutions were named solitons and we shall follow this more recent tendency.

The statement about the condensation of waves into solitons for nonintegrable systems appeared as an analytical prediction based on statistical considerations and the maximum-entropy principle.[3,4] The main result is the domination of the fusion of solitons over splitting the the possibility of an unbounded growth of strong solitons, and this made it possible to call them statistical attractors.[3–5] The analytical theory is complicated and contains unclear assumptions, so that it was of great value to have a straightforward numerical verification of the hypothesis of the growth of solitons for the NLS with an attractive kind of nonintegrable potential[5] and without a prohibition of a decrease in the number of solitons during the evolution. A similar result was obtained independently by another group of authors[6] who used another thermodynamic-equilibrium construction which required in the last stage computer calculations.

The thermodynamic predictions seem completely general but encountered difficulties when being applied to NLS with wave repulsion,

$$i\dot{\Psi}_t+\Psi_{xx}''-U(|\Psi|^2)\Psi=0, \quad (1)$$

which have solutions of the type of local lowering of the intensity $|\Psi|^2$ (dark solitons[7,8]). The "depth" of such "holes" is clearly restricted by the intensity of the background, and an unbounded increase in the amplitude of the modulation of dark solitons is impossible. Moreover, there appear additional boundary conditions connected with the fact that the change in the phase of the scalar field Ψ is constant and that it is impossible to change the advance in phase. This makes dark solitons, in contrast to the usual solitons of the NLS with an attractive type potential, single-parameter solitons. The velocity of a dark soliton is strictly related to the amplitude of its modulation and this must be taken into account in a thermodynamic analysis. An important feature of the waves of the NLS with a repulsive type of potential is that they are able to interact with the background, and this leads to a more restricted equilibrium distribution than in Refs. 3 to 6.

In the present paper we give a thermodynamic analysis of the "dark soliton + waves" system. We show that the condition for an increase in the modulational amplitude of dark solitons to be thermodynamically favored is connected with the condition that they be Lyapunov stable. Such a stability is proved for dark solitons of the NLS with an integrable repulsive potential and a modulational amplitude not larger than some limit; this possibly enables us to explain the experimental results of Ref. 9. The authors propose a straightforward numerical and experimental study of the dark solitons of the NLS, which would make it possible to check their attractor properties.

2. PROPERTIES OF THE DARK SOLITONS OF THE NLS WITH A REPULSIVE TYPE POTENTIAL

The NLS with a repulsive type potential describes the propagation of modulated ion-sound waves $[U\simeq|\Psi|^2$ (Ref. 9)] and of nonlinear waves in light conductors with a "normal" dependence of the refractive index on the lightguides $(U\simeq|\Psi|^2+\alpha|\Psi|^4)$ (Ref. 10), $U\simeq|\Psi|^2$ (Refs. 7 and 11 to 15), as well as the spatial picture of diffraction of a laser beam passing through a diffraction lattice and scattering material $[U\simeq|\Psi|^2$ (Ref. 16)]. In the last case x is the coordinate in the cross section of the beam and t is the coordinate along the beam.

A study of the structure of laser pulses propagating along lightguides is important from the point of view of applications which may arise.[17] In many cases[14,15] the laser pulse can be assumed to be much longer than the minima in the wave intensity which occur on its background. This makes it possible to consider an NLS with nonvanishing boundary conditions (for $x\to\pm\infty$).

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The first analytical results about the properties of the solutions of an NLS with non-vanishing boundary conditions were obtained in Refs. 2, 7, 8, and 18, where an integrable quadratic potential was considered ($U \simeq |\Psi|^2$). It was shown in Ref. 18 that small perturbations of the background solution of the NLS equation can in the first approximation in the nonlinearity and dispersion parameters be described by a sound-wave type equation close to the KdV equation, which has soliton solutions.[2] It was shown[8] by using a modified inverse scattering method that the evolution of any sufficiently strongly localized initial solution leads to the appearance of a set of dark solitons and waves.

The analytical form of the dark solitons of an NLS with integrable (quadratic) potential $U \simeq |\Psi|^2$,

$$i\dot{\Psi}_t + \Psi_{xx}'' - |\Psi|^2\Psi = 0, \tag{2}$$

and the boundary conditions: $|\Psi|^2 \to \omega_0$ (1) and $\Psi_x' \to 0$ (2) as $x \to \pm\infty$, was given in Refs. 7 and 8.

The solution of the NLS was found in the form[7]

$$\Psi^0 = \Psi_0(x, t)\exp[-i\omega_0 t], \quad \Psi_0(x, t) = \Psi_0(\zeta), \quad \zeta = x - Vt,$$
$$\Psi_0 = |\Psi_0|\exp[i\delta\theta(\zeta)], \tag{3}$$
$$|\Psi_0|^2 = V^2/2 + (\omega_0 - V^2/2)\,\mathrm{th}^2((\omega_0/2 - V^2/4)^{1/2}\zeta), \tag{4}$$

$$\delta\theta(\zeta) = \frac{V}{2}\int^{\zeta}\left(\frac{\omega_0}{|\Psi_0|^2} - 1\right)d\zeta. \tag{5}$$

The total advance in phase, $|\delta\theta| = 2\cdot\arctan \times (\sqrt{2\omega_0/V^2 - 1})$, can vary between the limits 0 and π. The velocity V of the dark soliton is, in contrast to the usual soliton, completely determined by the modulation amplitude, since in a dark soliton the constant amplitude $|\Psi_0|$ is also fixed, as well as the rate of change in phase $\Psi_{0_x}' = 0$ as $x \to \pm\infty$.

The collision of N dark solitons of the integrable NLS was considered analytically in Ref. 8 and it was shown that their modulation amplitude is unchanged after collisions. Numerical experiments have in practice confirmed this conclusion for dark solitons, both those moving in opposite directions and those moving in the same direction,[11,15] which is characteristic for an integrable NLS which has a denumerable set of integrals of motion.[19]

The aim of our paper is a consideration of the behavior of the solutions of an NLS with a repulsive type potential in the general, nonintegrable case. This means that we consider an equation of the form

$$i\dot{\Psi}_t + \Psi_{xx}'' - |\Psi|^2\Psi = H_{int}. \tag{6}$$

The perturbation H_{int} can here contain both nonlinear terms as well as corrections to the dispersion. If H_{int} is small the functional form of the dark solitons will be close to the integrable case (5) which we considered. The role of H_{int} reduces to switching on interactions and the possibility to change the modulational amplitude and the number of dark solitons. In that sense the situation is similar to constructing a model of the thermodynamics of a perfect gas—one neglects the interactions when considering equilibrium, but one understands them to be necessary for the establishing of the equilibrium.

3. STABILITY OF THE DARK SOLITONS OF THE NLS

If a particular solution claims to play an important role in the asymptotic behavior it must at least be stable. The stability of a soliton of an NLS with an attractive power-law potential is independent of its amplitude,[20] this does not contradict the possibility of the unbounded growth of strong solitons. Small-amplitude dark solitons of an NLS are described by a sound-wave type equation which is close to the KdV equation which has solitons which are stable[18] and show attractor properties.[4] In the present section we show the stability of dark solitons with a finite modulational amplitude which are not described by the KdV equation.

The stability of dark solitons of an NLS with an integrable repulsive type potential can be proved by the Lyapunov integral method. A dark soliton Ψ^0 is an extremal of the functional L composed of the usual integrals (characteristic for the nonintegrable NLS):

$$L = E - \omega_0 N - \frac{1}{2}VP = \int \left|\Psi_x' - \frac{i}{2}V\Psi\right|^2 + \frac{1}{2}|\Psi|^4 - \Omega|\Psi|^2\,dx. \tag{7}$$

$$\Omega = \omega_0 + V^2/4,$$

$$E = \int |\Psi_x'|^2 + \frac{1}{2}|\Psi|^4\,dx, \qquad N = \int |\Psi|^2\,dx,$$
$$P = i\int \Psi\Psi_x^{*\prime} - \Psi^*\Psi_x'\,dx, \tag{8}$$

where E, N, and P are, respectively, the energy, number of waves, and momentum integrals. The second variation has the form

$$\frac{1}{2}\delta^2 L = \int \left|\delta\Psi_x' - \frac{i}{2}V\delta\Psi\right|^2 + \frac{1}{2}(\delta|\Psi|^2)^2 - (|\Psi_0|^2 - \Omega)|\delta\Psi|^2\,dx. \tag{9}$$

Solutions for which $\delta^2 L$ has a fixed sign are Lyapunov-stable. It was made clear numerically (by an expansion of an arbitrary test function $\delta\Psi$ in Fourier harmonics) that $\delta^2 L > 0$ if the advance in phase $|\theta| < \pi/2$. The dark solitons with a modulational amplitude less than $\sqrt{1/2}$ are thus Lyapunov-stable. In proving the stability we did not use the integrability of the NLS. This possibly enables us to extend the conclusion about the stability of a dark soliton with a finite (though smaller than critical) modulation amplitude to arbitrary potentials which are sufficiently close to the integrable one considered, since the nonintegrable NLS conserves all three integrals E, N, and P.

4. THERMODYNAMICS OF THE "DARK SOLITON + WAVES" SYSTEM

Thermodynamically an equilibrium distribution of waves in a dark soliton is possible only in the case where the dark soliton is not only a stationary point of some integral of motion, but also realizes its maximum (minimum).

An integrable NLS has a denumerable set of polynomial type integrals of motion.[19] The establishment of a thermodynamic equilibrium distribution is therefore impossible on a hypersurface of the function L considered above which does not take into account the limitations imposed by the integrability.

A change in the potential or in the dispersion of the integrable NLS leads to the loss of additional integrals. It becomes possible to establish a thermodynamic "dark soliton + waves" equilibrium distribution on a hypersurface given by the conservation of the main integrals of the NLS (energy, mometum, and number of waves) similar to the thermodynamic "soliton + waves" thermodynamic equilibrium considered before.[3,4] If the changes in the integrable NLS are small the hypersurface of the fixed-sign functional L considered above changes little and the thermodynamic equilibrium distribution is established on a surface close to an L hypersurface. We shall therefore, as we have already said earlier, use for our construction of an equilibrium model integrals of motion which are analytical expressions for dark solitons and waves and were obtained for the integrable NLS. If there is no complete equilibrium the thermodynamics shows a tendency for the evolution of the process.

On the basis of the maximum entropy principle the tendency of the "soliton + waves" system of an NLS with an attractive type potential to increase the amplitude of the solitons was justified in Refs. 3 and 4. The thermodynamic equilibrium distribution of any system contains only thermodynamic constants however many integrals of motion there are.[21] The distribution of waves considered which is in equilibrium with the dark soliton has the following form

$$\ln f_{\text{equil}} \simeq -(E_w + aN_w + bP_w)/T, \tag{10}$$

where E_w, P_w, and N_w are the energy, momentum, and number of waves; the T, a, and b are thermodynamic parameters.

The integrals E_w^0, P_w^0, and N_w^0 of free waves on a constant background $|\Psi_0|^2 \equiv \omega_0$ can be calculated analytically. Indeed, perturbations $\delta\Psi_k$ of the background of the NLS with an integrable attractive type potential are given up to second order in the small amplitude, by the expressions[22]

$$\begin{gathered}\delta\Psi_k = \{\phi_k^{(1)} + \phi_k^{(2)}\}\exp[-i\omega_0 t],\\ \phi_k^{(1)} = \phi_k^{\pm}\left[\cos(\omega_k^{\pm}t - kx) - i\frac{\omega_k^{\pm}}{k^2}\sin(\omega_k^{\pm}t - kx)\right],\\ \omega_k^{\pm} = \pm k(2\omega_0 + k^2)^{1/2},\\ \phi_k^{(2)} = -|\phi_k^{\pm}|^2\left(1 + \frac{\omega_0}{2k^2}\right)\Big/\,\omega_0^{1/2} + iA + B\cos[2(\omega_k^{\pm}t - kx)]\\ + iC\sin[2(\omega_k^{\pm}t - kx)],\end{gathered} \tag{11}$$

where A, B, and C are real parameters which are inessential for evaluating N_w^0, P_w^0, and E_w^0:

$$\begin{gathered}N_w^0 = \int |\phi_k^{(1)}|^2 + 2\omega_0^{1/2}\,\mathrm{Re}\,\phi_k^{(2)}\,dx = |\phi_k^{\pm}|^2\frac{\omega_0}{2k^2},\\ P_w^0 = 2\,\mathrm{Im}\int \phi_k^{(1)*}\,\phi_{kx}^{(1)\prime}\,dx = 4\frac{k}{\omega_0}\omega_k^{\pm}N_w^0,\\ E_w^0 = \int |\phi_{kx}^{(1)\prime}|^2 + \omega_0\{2\omega_0^{1/2}\,\mathrm{Re}\,\phi_k^{(2)} + |\phi_k^{(1)}|^2 + 2(\mathrm{Re}\,\phi_k^{(1)})^2\}dx\\ = \omega_0\left(1 + \frac{2\omega_k^{\pm 2}}{\omega_0^2}\right)N_w^0.\end{gathered} \tag{12}$$

The thermodynamic parameters a and b in Eq. (10) are determined by the condition that the chemical potentials of the waves and of the dark soliton which is in equilibrium with them are the same. This condition of thermodynamic equilibrium is equivalent to regarding the functional L considered in the preceding section as extremal. Therefore we have

$$a = -\omega_0,\ b = -V/2;\ E_w + aN_w + bP_w = L - L_0(V) = {}^1/_2\delta^2 L, \tag{13}$$

where $L_0(V)$ is the value of L in a dark soliton which moves with velocity V.

The eigenfunctions of the NLS differ in the particular case of a small-amplitude dark soliton (almost constant background, $|\Psi_0|^2 \equiv \omega_0$) little from free waves and their thermodynamic equilibrium distribution can be found analytically. The potentials a and b of the free waves are not arbitrary, owing to the interaction with the background: $a = -\omega_0$, $b = -\sqrt{\omega_0/2}$ (the sign of b is determined by the direction in which the dark soliton moves, in the present case in the positive direction). Another difference between the free waves and the waves considered in Refs. 3 and 4 is that the eigenmodes (11) are each a sum of two functions which depend exponentially on the time. The equilibrium distribution of the waves is therefore not determined by a modified Rayleigh-Jeans law, as in Refs. 3 and 4. Indeed, at thermodynamic equilibrium we have $\frac{1}{2}\delta^2 L = T$ and from (12) and (13) we get the following equilibrium distribution for the free waves:

$$|\phi_k^{\pm}|^2 \simeq \frac{T}{\omega_0}\left(1 + \frac{k^2}{2\omega_0} \mp \left[1 + \frac{k^2}{2\omega_0}\right]^{1/2}\right)^{-1}. \tag{14}$$

Although there were, as in Refs. 3 and 4, originally three integrals of motion, the distribution (14) depends only on two parameters (T and ω_0). The dependence on the third parameter—the momentum—has been removed by the special choice of the coordinate system (the boundary condition $\Psi_x' \to 0$ as $x \to \pm\infty$. The branches have for $k \ll \sqrt{\omega_0}$ different asymptotic forms: $|\phi_k^-|^2 \simeq T/2\omega_0$, $|\phi_k^+|^2 \simeq 4T/k^2$. The distribution $|\phi_k^+|^2$ describes equilibrium waves moving in a direction which is the same as the direction in which the small-amplitude dark soliton moves. Indeed, the equilibrium distribution of the waves in the KdV is the same for $k \to 0$ as the $|\phi_k^+|^2$ distribution. The $|\phi_k^-|^2$ distribution describes equilibrium waves moving in opposite directions and it cannot be obtained from the KdV equation.

The phase volume ΔF of the waves in equilibrium with a dark soliton moving with a velocity V is

$$\Delta F \simeq f_{\text{equil}}^{-1} \simeq \exp\left[\left(E_w - \omega_0 N_w - \frac{V}{2}P_w\right)\Big/\,T\right] \tag{15}$$

and, hence, the entropy of the waves will increase if the quantity $E_w - \omega_0 N_w - \frac{1}{2}VP_w$ increases during an increase in the modulational amplitude of the dark soliton. According to (13) we have $E_w - \omega_0 N_w - \frac{1}{2}VP_w = \frac{1}{2}\delta^2 L$. We have shown in the preceding section that the quantity $\delta^2 L$ is positive if the modulation amplitude of the dark soliton is not more than some finite magnitude ($\sqrt{1/2}$ for a dark soliton of a NLS with a nearly integrable potential). When the modulation amplitude of the dark soliton increases, the number of waves in it decreases and N_w correspondingly increases. The increase in N_w causes an increase in the positive definite $\delta^2 L$ so that up to a well defined finite modulation amplitude the dark soliton can show attractor properties.

There exist circumstantial experimental data which make it possible to confirm the suggestion made above that the increase in the modulation amplitude of a dark soliton saturates. For instance, in Ref. 9 the experimental evolution of an initially maximally modulated ion-sound wave led to a diminution in the modulation amplitude.

We now consider a thermodynamic system consisting of several dark solitons. If there are no direct collisions, each dark soliton interacts only with waves. Such an interaction was considered above and leads to the growth of the modulation amplitude of each not too strongly modulated soliton. There are no explicit integral restraints on such a process—the conservation of merely the three integrals of motion (8) is not difficult as one can dump energy, momentum, and numbers of waves from the dark solitons into the functionally rich nonequilibrium distribution of the waves.

The possibility of the simultaneous growth of the modulation amplitudes of a system of dark solitons distinguishes them from the system of KdV solitons and solitons of the NLS with an attractive type potential considered in Refs. 3 and 4, where the simultaneous growth of the soliton amplitudes was impossible. The reason is that a growth of the solitons is accompanied with absorption of waves and conservation of the total number of waves is possible only by reducing the amplitudes of some of the solitons.

The growth of small-amplitude dark solitons must have interesting features. In fact, it has been shown earlier[18] that in the case the solution of the NLS with a repulsive type potential is described by an equation close to the KdV the solitons of which cannot grow simultaneously. The contradiction appearing here is connected with the fact that the above-mentioned reduction of the NLS to the KdV contains an assumption that the interaction between waves moving in different directions is small. This leads to the appearance of "weak" integrals—the momenta of waves moving to the right and moving to the left are conserved separately; a restraint emerges on the use of the background as a wave reservoir. This all probably leads to a slowing down of the simultaneous growth of dark solitons with a small modulational amplitude or even to the disappearance of some of them—until the increase in the moudlational amplitude of the remaining dark solitons lifts the prohibitions imposed by the "weak" integrals.

A direct numerical and experimental simulation of the evolution of dark solitons of the nonintegrable NLS might give more rigorous results. Experiments with nonlinear waves in light conductors make it possible to study the interaction between dark solitons of the NLS with a controlled nonintegrable correction $\alpha|\Psi|^4$ to the quadratic potential $U \simeq |\Psi|^2$.[10] The possibilities for numerical simulations are greater. Depending on the scheme chosen it is easy to add to the integrable NLS either dispersion corrections or one can change the potential.

As initial conditions one can choose:

1) either a weakly modified background—to confirm the attractor properties of dark solitons with a small modulational amplitude and to clarify the features of their simultaneous growth;

2) or strongly modulated waves—to verify the hypothesis about the saturation of the attractor properties of dark solitons with a modulational amplitude above some critical value.

5. CONCLUSION

We have given for the nonintegrable nonlinear Schrödinger equation (NLS) with a repulsive type potential an analytical description of the "dark solitons + waves" equilibrium showing the thermodynamic advantage of a growth in the modulational amplitudes of the dark solitons up to some finite value.

The growth of the modulation amplitude of a dark soliton, in contrast to the growth of the usual soliton, is accompanied by the emission of waves and not by their absorption. The simultaneous growth of all dark solitons is therefore not prohibited by the necessity to conserve the standard integrals (8) of the NLS. At the same time the simultaneous growth of dark solitons with a small modulation amplitude is apparently made difficult by the existence of "weak" integrals of motion of the NLS operating only when the modulation amplitude of the background is small. These integrals prohibit the use of the background as a wave reservoir and can lead to the annihilation of some of the dark solitons until the growth in the modulation amplitude of the remaining dark solitons lifts the restraints imposed by those "weak" integrals.

The situation remains unclear for dark solitons with a modulation amplitude above a critical value ($\sqrt{1/2}$ for the NLS with a nearly integrable potential). In that case there exists both a group of waves with $\delta^2 L < 0$ and a group of waves with $\delta^2 L > 0$ and it is thermodynamically advantageous to transfer waves by means of a dark soliton. Of course, the growth of the waves will be restricted by nonlinear corrections and it also unclear what will happen to the modulation amplitude of the dark soliton. It is possible that such a dark soliton with a modulation amplitude close to the maximum one will split up during the evolution of an NLS with a repulsive potential close to a collapsing one ($U = |\Psi|^4$). It is advisable to study its behavior by numerical or experimental methods. Such a study might involve determining the relative growth rates of dark solitons with different modulational amplitudes which is essentially a nonlinear problem.

The considerations presented in this paper might in principle be applied also to the two-dimensional case as was done in Ref. 23 for the case of two-dimensional solitons of the NLS with an attractive type of potential.

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[1] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).
[2] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).
[3] V. I. Petviashvili and V. V. Yan'kov, in *Problems of Plasma Theory* (Ed. B. B. Kadomtsev) Vol. 14 (1985) [English translation published by Consultants Bureau].
[4] S. F. Krylov and V. V. Yan'kov, Zh. Eksp. Teor. Fiz. **79**, 82 (1980) [Sov. Phys. JETP **52**, 41 (1980)].
[5] V. E. Zakharov, A. N. Pushkarev, V. F. Shvets, and V. V. Yan'kov, Pis'ma Zh. Eksp. Teor. Fiz. **48**, 79 (1988) [JETP Lett. **48**, 83 (1988)].
[6] J. L. Lebowitz, H. A. Rose, and E. R. Speer, J. Stat. Phys. **50**, No. 3/4 (1988); **54**, No. 1/2 (1989).
[7] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, No. 3, 142 (1973).
[8] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **64**, 1627 (1973) [Sov. Phys. JETP **37**, 823 (1973)].

[9]H. Sugai, R. Hatakeyama, K. Saeki, and M. Inutake, Phys. Fluids **19**, 1753 (1976).
[10]H. Puell, K. Spanner, W. Falkenstein, W. Kaiser, and C. R. Vidal, Phys. Rev. **14**, 2240 (1976).
[11]L. J. Mulder and R. H. Enns, IEEE **25**, 2205 (1989).
[12]D. Krokel, N. J. Halas, G. Giuliani, and D. Grischowsky, Phys. Rev. Lett. **60**, 29 (1988).
[13]A. M. Weiner, J. P. Heritage, R. J. Hawkins, *et al.*, Phys. Rev. Lett. **61**, 2445 (1988).
[14]W. J. Tomlinson, R. J. Hawkins, A. M. Weiner, J. P. Heritage, and R. N. Trurston, J. Opt. Soc. Am. B, Opt. Phys. **6**, No. 3, 329 (1980).
[15]R. N. Trurston and J. P. Heritage, J. Opt. Soc. Am. B, Opt. Phys. **8**, No. 2, 471 (1989).
[16]G. A. Swartzlander, D. R. Andersen, J. J. Regan, H. Yin, and A. E. Kaplan, Phys. Rev. Lett. **66**, 1583 (1991).
[17]A. Hasegawa, Optical Solitons in Fibers, Springer (1989).
[18]V. I. Karpman and E. M. Kruskal, Zh. Eksp. Teor. Fiz. **55**, 530 (1968) [Sov. Phys. JETP **28**, 277 (1969)].
[19]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].
[20]N. G. Vakhitov and A. A. Kolokolov, Izv. Vysshh. Uchebn. Zaved. Radiofiz. **16**, 1020 (1973) [Radiophys. Qu. Electron. **16**, 783 (1975)].
[21]L. D. Landau and E. M. Lifshitz, Statistical Physics, Vol. 1, Nauka, Moscow (1976) [English translation published by Pergamon Press, Oxford].
[22]H. C. Yuen and B. M. Lake, *Theory of Nonlinear Waves on Deep Water, in Solitons in Action* (Eds. K. Lonngren and A. Scott), Academic Press, New York (1978).
[23]A. I. D'yachenko, V. E. Zakharov, A. N. Pushkarev, V. F. Shvets, and V. V. Yan'kov, Zh. Eksp. Teor. Fiz. **96**, 2026 (1989) [Sov. Phys. JETP **69**, 1144 (1989)].

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