Self-focusing and relativistic waveguiding of an ultrashort laser pulse in a plasma

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We have studied the self-interaction that takes place in an ultrashort pulse of electromagnetic radiation as it generates plasma waves, resulting from competition between ponderomotive and relativistic nonlinearities. We have found self-consistent self-similar solutions that describe the dynamics of a square-wave pulse when these nonlinearities are weak, along with the (exponential) amplification of the plasma wave amplitude in the nonlinear regime associated with increased fields in the "trailing-edge" portion of the pulse. In cases of practical interest (e.g., for particle acceleration, frequency conversion, etc.), in which the fields are relativistically strong, we have integrated these equations numerically both in "exact" form and in the approximation of paraxial optics (i.e., for beams of quasi-Gaussian shape). We find that although the leading-edge portion of the pulse propagates as in the linear case and undergoes diffractive spreading, its central region can be trapped in a waveguiding regime. We show that there is a considerable increase (compared with the linear case) in the intensity of the plasma waves and an increase of the longitudinal size of the region in which this effect is observed.

A promising way to generate strong electromagnetic fields in a plasma and thereby accelerate high-energy particles is to use the plasma waves created by ultrashort laser pulses.

This idea was formulated for the first time in Ref. 1 with regard to accelerator applications; however, the corresponding nonlinear processes in the plasma were investigated in detail only within the last year,[2–6] when the possibility of doing the appropriate experiments became realistic. The plasma waves are excited by short (in comparison with the plasma wavelength laser pulses through the action of the ponderomotive force; from a physical point of view, this excitation is analogous to the radiation of plasma waves by a moving charge. For fixed laser pulse energy, the amplitude of the resulting plasma waves increases as their wavelength decreases, and is in order of magnitude equal to the ponderomotive potential.[4] From this it follows that in order to obtain strong electric fields in a plasma it is necessary to use short laser pulses that are relativistically strong, in which case the propagation characteristics of the pulse through the plasma are determined by the competition between electrostrictive and relativistic nonlinearities. Under these conditions, relativistic self-focusing effects and optical waveguiding of the laser beam can occur, which in turn lead to cumulative generation of a plasma wake and a considerable increase in the longitudinal size of the latter.

The effect of steady-state self-focusing of beams of electromagnetic waves in a plasma associated with the relativistic dependence of the electron mass on its oscillatory energy was studied for the first time in Ref. 7, and later discussed in Refs. 8 and 9. In these and in subsequent and often-cited articles (see, e.g., Refs. 10–13), it was shown that if the power of the wave beam P exceeds the so-called critical self-focusing power $P_{\rm cr}$, which for the case of a relativistic nonlinearity is given by the expression $P_{\rm cr} = 15\ (\omega^2/\omega_p^2)$ GW, then focusing of the beam occurs down to dimensions determined by nonlinearity saturation effects, and as a result power on the order of the critical power is trapped in nonlinear optical waveguiding. However, as was shown in Ref. 4, this picture is valid only for relatively long pulses ($L \gg \lambda_p$), whereas for ultrashort pulses the primary role is played by competition between the electrostrictive and relativistic nonlinearities. In this case, excitation of plasma oscillations at the front of the pulse leads to weakening of the overall nonlinearity of the plasma. The qualitative nature of the self-focusing for this case was discussed in Ref. 4, where it was shown that the leading edge of a packet should propagate as in the linear case, undergoing diffractive spreading, while the main portion should be trapped in the optical waveguiding regime. However, a quantitative theory of this process has yet to be developed. In this paper we investigate in detail the process of dynamic self-focusing of an ultrashort laser pulse as it propagates through a low-density plasma

1. FUNDAMENTAL EQUATIONS

Excitation of a plasma wake by a short laser pulse, including nonlinear distortion of its transverse and longitudinal spatial structure, can be investigated by using the following system of equations:

$$-2ik\frac{\partial A}{\partial z}+\frac{\omega_p^2}{u^2\omega^2}\frac{\partial^2 A}{\partial\tau^2}+\Delta_\perp A+\frac{\omega_p^2}{c^2}\frac{\Phi}{1+\Phi}A=0, \tag{1}$$

$$\frac{\partial^2\Phi}{\partial\tau^2}=\omega_p^2\frac{1+|A|^2-(1+\Phi)^2}{2(1+\Phi)^2}, \tag{2}$$

which are elementary generalizations of the equations used in Refs. 4 and 5 for the one-dimensional case. In these equations the following notation has been used: $\Phi = e\varphi/m_0c^2$ is the scalar potential of the plasma oscillations normalized by the relativistic quantity m_0c^2/e, A is the slowly varying complex amplitude of the vector potential of a circularly polarized electromagnetic wave normalized by the same quantity, $\tau = t - z/u$, where u is the local group velocity of the packet, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian, $\omega_p = (4\pi e^2N_0/m_0)^{1/2}$ is the Langmuir frequency, N_0 is the electron density in the unperturbed plasma, and ω and k are the frequency and wave number of the electromagnetic wave.

In what follows we investigate propagation of a short laser pulse (short compared to the wavelength $\lambda_p = 2\pi c/\omega_p$

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of the excited plasma waves) in an underdense ($\omega_p \ll \omega$) plasma. Propagation takes place over a finite distance $z \ll c\omega^4/\omega_p^3$, in which case we may neglect the effects of dispersive spreading, i.e., omit the terms in Eq. 1 with $\partial^2 A/\partial\tau^2$.

2. THE CASE OF WEAK NONLINEARITY

In the simplest case of a weak nonlinearity ($\Phi \ll 1$), the amplitude of the plasma wave generated by this short pulse is determined by the expression

$$\Phi_m = \omega_p \int_{-\infty}^{\infty} |A|^2 \, d\tau. \tag{3}$$

Then the self-consistent field distribution A is determined by the following system of equations:

$$-i\partial A/\partial z + \Delta_\perp A + \Phi A = 0, \tag{4}$$

$$\partial^2\Phi/\partial\tau^2 = +|A|^2, \tag{5}$$

where we have introduced new dimensionless variables

$$z_{(n)} = \frac{\omega_p}{2\omega} k_p z, \quad \mathbf{r}_{(n)} = k_p \mathbf{r}_\perp, \quad \tau_{(n)} = \omega_p \tau, \quad k_p = \omega_p/c,$$

the label "n" will be omiteed in what follows.

Equations (4) and (5) have the usual integral

$$\int_{-\infty}^{\infty} |A(\mathbf{r}, z, \tau)|^2 \, d\mathbf{r} = W(\tau), \tag{6}$$

corresponding to conservation of the total energy flux in the beam:

$$\tilde{P} = PW(\tau), \qquad P = \frac{m^2c^5}{4\pi e^2} \frac{\omega^2}{\omega_p^2}.$$

In order to analyze the characteristics of the solution we can make use of the procedures developed in Ref. 14 and elsewhere to investigate spatial wave collapse in media with time-dependent nonlinearity. For the case of a square-wave pulse ($W(\tau) =$ const where $0 \leqslant \tau \leqslant \tau_0$) the system (4), (5) has a set of self-similar solutions

$$A = \alpha u(\zeta) \exp(-\alpha\tau + i\gamma z e^{2\alpha\tau}),$$

$$\Phi = e^{2\alpha\tau} v(\zeta), \quad \zeta = r e^{\alpha\tau}, \tag{7}$$

where α and γ are positive constants. These solutions describe a quasistationary pulse in the shape of a horn which flares in the direction of motion. The self-similar functions $V(\zeta)$ and $U(\zeta)$ correspond to a localized solution to the system of equations

$$\Delta u + (v - \gamma) u = 0,$$

$$\zeta^2 \frac{\partial^2 v}{\partial \zeta^2} + 5 \frac{\partial v}{\partial \zeta} + 4v = u^2 \tag{8}$$

The structure of the fundamental self-similar mode, which is shown in Fig. 1, is characterized by an exponential decrease of the vector potential with respect to radius and a power-law decrease ($-\zeta^{-2}$) of the plasma-wave potential. The self-similarity parameter α is completely determined by the power in the transverse cross section of the pulse

$$\alpha = (P/K\tilde{p})^{1/2}, \tag{9}$$

where $K = 2\pi \int_0^\infty u^2 \zeta d\zeta = 79.9$. The parameter γ gives the

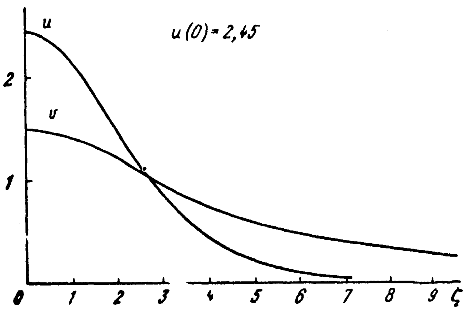

FIG. 1. Field distribution (u) and plasma wave potential (v) in the self-similar mode.

transverse scale of the distribution and can be found from the condition that the self-similar distribution be matched with the solution at the leading edge of the pulse, which is not subject to nonlinear distortion due to the smooth switching-on of the potential Φ [see (5)]. From dimensional considerations it follows that $\gamma = c/a_0^2$, where a_0 is the characteristic width of the profile at the leading edge, which changes its shape smoothly with z (on a scale much larger than the longitudinal size of the pulse) along the path according to the laws of linear diffraction; c is a constant of order unity which depends on the form of the pulse.

Using Eq. (3) and the solution we have obtained for the transverse structure of the mode u_M ($\zeta, \gamma = 1$), we derive an expression for the amplitude of the plasma oscillations in the Killwater wake:

$$\Phi_M = \frac{\alpha}{2r^2} \int_{\gamma r^2}^{\gamma r^2 \exp(2\alpha\tau_0)} u_M^2[\xi^{1/2}] \, d\xi. \tag{10}$$

In the near-axis region we have in dimensional variables

$$\Phi_m = \frac{mc^2}{2e} \frac{\omega^2}{\omega_p^2} \frac{c}{k_0^2 a_0^2} \frac{p^{1/2}}{e} \frac{u_0^2}{k^{1/2}} \Big[e^{2(p/k\tilde{p})^{1/2}\omega_p\tau_0} - 1 - \frac{(u_0^2 - 4)}{3^2} \frac{cr^2}{a_0^2} \left(e^{4(p/k\tilde{p})^{1/2}\omega_p\tau_0} - 1\right) \Big]. \tag{11}$$

It follows from (11) that when $(p/\tilde{p})^{1/2}\omega_p\tau_0 \ll 1$ we may neglect the effect of nonlinear pulse compression. This compression will play an important role only if the power exceeds the value $\tilde{p}_{cr} = 4\tilde{p}(\omega_p\tau_0)^{-2}$, which is in fact the critical power for self-focusing in the nonstationary case (an analogous estimate was obtained in Ref. 4 based on the one-dimensional model). For $p \gg \tilde{p}$ the amplitude of the wake wave increases exponentially $\sim \exp[(kp/\tilde{p})^{1/2}\omega_p\tau_0]$ and is simultaneously accompanied by a decrease in the radius of the wake region.

3. THE APPROXIMATION OF PARAXIAL OPTICS

The self-similar relations we have obtained for the case of weak nonlinearity cannot do more than familiarize us somewhat with the problem, because they do not take into account effects that saturate the nonlinearity, which play an important role in cases of practical interest, e.g., accelerators, where relativistically strong fields $A^2 \gg 1$ are involved. The question of whether the effect of nonlinear waveguiding of a beam can be used to increase the size of the plasma wake can be answered only by investigating more complex equations that can be written in the form

$$-i\frac{\partial A}{\partial z}+\Delta_\perp A+\frac{\Phi}{1+\Phi}A=0, \tag{12}$$

$$\frac{\partial^2\Phi}{\partial\tau^2}=\frac{1+|A|^2-(1+\Phi)^2}{2(1-\Phi)^2}. \tag{13}$$

Let us investigate this sytem in the approximation of paraxial optics (the so-called aberrationless approximation),[10] in which we assume that the wave beam has a quasi-Gaussian shape

$$A=\frac{W^{1/2}}{a_0 b}\exp\left(-\frac{r^2}{2a_0^2b^2}+i\varkappa r^2\right) \tag{14}$$

with a relative width b that depends on z and τ: $b=b(z,\tau)$. We will approximate the field distributions of the plasma index of refraction $n(r)$ and potential $\Phi(r)$ in the near-axis region as determined by Eq. (13) by the parabola

$$n(r)=n_0+n_2r^2=\frac{\Phi(r)}{1+\Phi(r)}, \tag{15}$$

$$\Phi(r)=\Phi_0+\Phi_2r^2.$$

where a_0 is the initial width of the Gaussian beam at the boundary of the plasma.

As a result we obtain the standard equations for the beam width used in the theory of self-focusing within the paraxial appoximation:

$$\frac{\partial^2 b}{\partial\zeta^2}=\frac{1}{b^3}+n_2 b,\qquad n_2=\frac{\Phi_2}{(1+\Phi_0)^2}, \tag{16}$$

in which we have introduced the variable $\zeta=z/a_0^2$; the quantities Φ_0 and Φ_2 are in turn solutions to the following equations:

$$\frac{\partial^2\Phi_0}{\partial\tau^2}=-\frac{\Phi_0(1+\Phi_0/2)}{(1+\Phi_0)^2}+\frac{W}{2a_0^2b^2(1+\varphi_0)^2}. \tag{17}$$

$$\partial^2\Phi_2/\partial\tau^2=-\frac{\Phi_2(1+W/a_0^2b^2)}{(1+\Phi_0)^3}-\frac{W}{2a_0^4b^4(1+\Phi_0)^2}. \tag{18}$$

The constant $\varkappa$ entering into Eq. (14) is determined by the expression $\varkappa=(1/2b)\partial b/\partial\xi$.

The system of equation (16) and (17) is simpler than the original system because it has fewer independent variables: in this system $b=b(\xi,\tau)$, i.e., we have eliminated the dependence on the transverse coordinate r.

To begin with, let us investigate the case of weak nonlinearity for a short excitation pulse within the framework of the paraxial approximation. For this we neglect Φ_0 compared to 1 and the first term on the right side of Eq. (18) compared to the second.

Then by comparing the diffractive and nonlinearity terms in (16) we obtain a condition for the time τ_0 at which an originally rectangular pulse begins to self-focus on its trailing edge:

$$\tau_0>2a_0^2/W^{1/2}. \tag{19}$$

Over distances $z_*\leqslant 2a_0^2b_0^2/\tau_0W^{1/2}$ (if it is smaller than the diffraction length) we should expect a horn-shaped pulse with self-similar structure that is "corrugated" by wavelike fine-scale perturbations of its envelope traveling in the direction of the leading edge of the pulse. The smooth structure of the horn is determined in the adiabatic approximation, where we can neglect the second derivatives with respect to radius in (16) and use the local relation $\Phi_2=-1/b^4$. The expression obtained in this case:

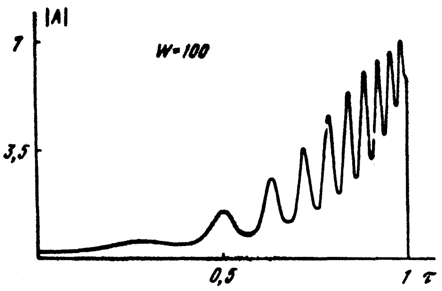

FIG. 2. On-axis field distribution ($v_\perp=0$) for a pulse of rectangular form when $z=2$.

$$b=\exp\left(-\frac{t(p/2)^{1/2}}{4}\right) \tag{20}$$

corresponds to Eq. (7), which we derived in terms of self-similar variables. This type of behavior, i.e., oscillating perturbations against a background, is characteristic of wave structures that are described by nonlinear equations of the type (16), and reflects a competition between diffractive and nonlinear effects as the system approaches the quasistationary distribution (20). Analogous oscillations also appear in the amplitude of the plasma waves generated. The structure shown in Fig. 2 is typical for such a corrugated horn. With increasing pulse length τ_0 or power W, the potential Φ_0 must inevitably increase to relativistic values $\Phi_0\approx 1$. In this case it is necessary to investigate the full system (16)–(18).

4. NUMERICAL INVESTIGATION OF WAVEGUIDING OF THE PULSE UNDER CONDITIONS WHERE THE RELATIVISTIC NONLINEARITY IS SATURATED

We integrated the "exact" equations (12) and (13) numerically for a converging Gaussian beam with width b_0 and convergence $\partial b/\partial\xi$ $(\xi=0)=b_0'$ specified at the plasma boundary. In Fig. 3 we show the dynamics of the vector potential envelope $A(z,\tau)$ in the plasma for an intensity profile that is close to rectangular:

$$p=p_0\exp\left[-\frac{(\tau-\tilde{\tau})^8}{\tau_0^8}\right]$$

and the parameters $a_0=10$, $b_0=2^{1/2}$, $b_0'=-1/2^{1/2}$, $\tau_0=1$, $\tilde{\tau}=3$, and $p_0=100$. The quantity $b_0'=-1/2^{1/2}$ corresponds to a beam whose linear focal distance from the plasma boundary is $z=1$; in the linear approximation, the longitudinal size of the caustic for this beam corresponds to $z=b_0^2=2$.

The qualitative evolution of the pulse is in agreement with the scenario described in Sec. 3, i.e., formation of a horn-like structure, although the field amplitude at the trailing edge increases more slowly on the average. Because the power flux through an arbitrary cross section of the pulse is conserved, the oscillations of the vector potential correspond exactly to the modulation depth (corrugation) of the spatial structure of the field. As it propagates in the plasma, the pulse breaks up in the longitudinal direction into a larger and larger number of bunches, although the average distribution (with respect to length) adiabatically traces out the

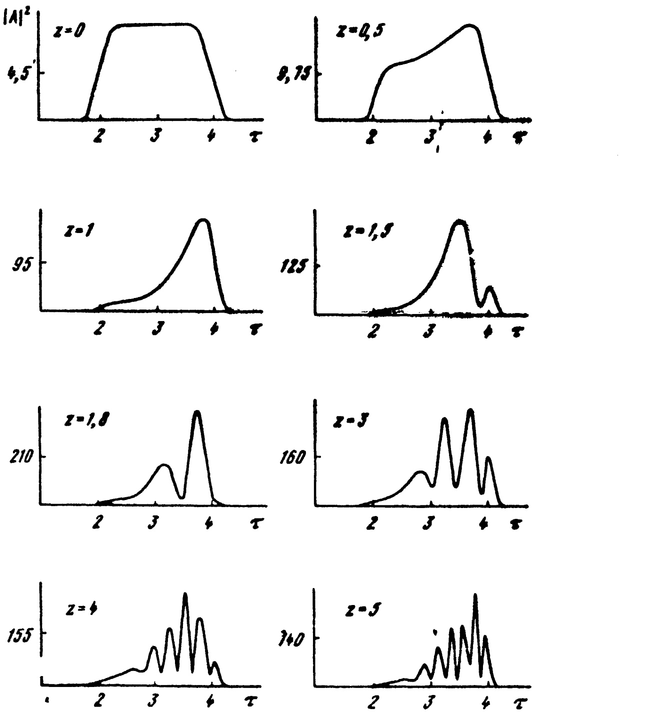

FIG. 3. On-axis field distribution ($v_\perp = 0$) for a pulse with a shape close to rectangular when $z = 0.0$, $z = 0.5$, $z = 1.0$, $z = 1.5$, $z = 1.8$, $z = 3.0$, $z = 4.0$, and $z = 5.0$.

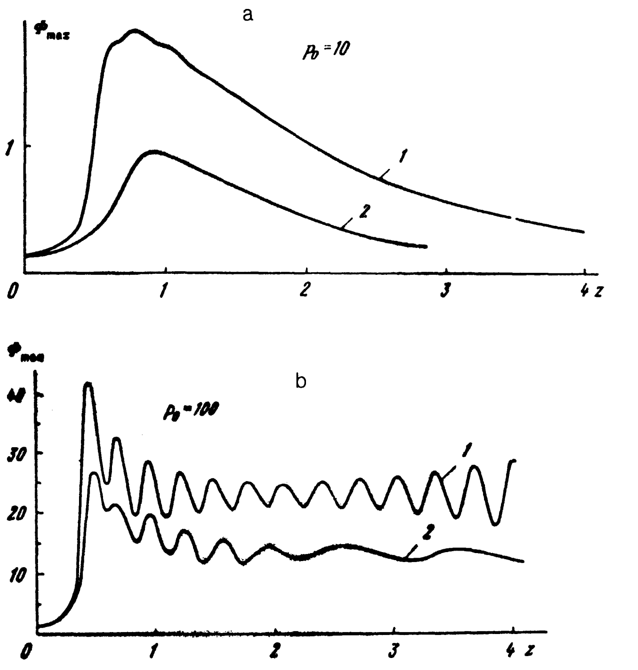

FIG. 4. Dependence of the maximum on-axis plasma wave amplitude ($v_\perp = 0$) on the location of the point where it is excited by a pulse of Gaussian shape.

diffractive distribution of the leading edge.

The effect of relativistic waveguiding of a short pulse is characterized by lengthening of the region of constriction compared to the linear (diffractive) propagation picture. Let us define the length scale of the region over which the field is amplified in terms of the coordinate z_{NL} at which the trailing edge of the pulse, which is broadened overall as it leaves the waveguiding zone, reaches a size corresponding to the dimensions of the linear focal spot. At this point, the region of saturation obviously terminates, so that we can use the relations derived for the self-similar pulse structure in the weakly relativistic approximation as an estimate. For the radius of the trailing edge we have

$$\Phi(z,\ \tau_0) \approx \Phi_0(z)\ \exp\ (-\alpha\tau_0),$$

where $\Phi_0(z)$ obeys the law of diffractive spreading $\Phi_0(z) \approx 4z/\Phi_{0\min}$. Thus,

$$z_{NL} \approx {}^1/_4 \Phi_{0\,min}^{2} \exp(\alpha\tau_0) = z_L \exp(\alpha\tau_0),$$

consequently the nonlinear constriction region turns out to be longer than the linear constriction region by a factor equal to the amount by which the self-similar structure is compressed from its leading to its trailing edge, i.e., it is uniquely determined by the intensity and length of the pulse. For the case we are investigating we have $\alpha = 10$, $\tau_0 = 1$, and $z_{NL}/z_L = \exp(10)$; clearly the length of the strong-field region is increased by a large factor.

Our numerical investigation showed that pulses with smoother (Gaussian) intensity profiles

$$p = p_0 \exp\left[-\frac{(\tau - \bar{\tau})^2}{\tau_0^2}\right]$$

also evolve in a way that is close to that described above, although in this case the self-focusing of the trailing edge of the pulse is found to be less sharp. As for the question of generation of wake waves, we note that due to the integration of the pulse over its length, the sharp space-time oscillations of the plasma waves excited by the source are smoothed out, and spatial modulation of the amplitude of the Langmuir wake is found to be insignificant. The region of strong wake waves obviously is clamped to the pulse axis, with an extent on the order of the nonlinear constriction length z_{NL}. In Fig. 4 we plot the on-axis Langmuir wave oscillation amplitude versus the position of its excitation point for pulses with Gaussian intensity profiles for the two values $p_0 = 10$ and $p_0 = 100$. In this figure, we show for comparison analogous curves calculated using the aberrationless approximation, which also demonstrates the qualitative agreement with the "exact" theory.

Thus, our investigations allow us to assert that ultrashort relativistically strong laser pulses undergo considerable structural distortion as they propagate through the plasma, acquiring a hornlike shape with strong longitudinal intensity modulation. Due to the inertia of the nonlinear response, the leading edge of the pulse undergoes linear diffractive spreading; however, its primary portion is found to be trapped in the nonlinear waveguiding regime, causing the region left behind by the pulse, which takes the form of an intense Langmuir wake, to be significantly lengthened. The self-similar solutions we have found for the dynamic self-focusing of the pulse allow us to determine the wake wave amplitude and the size of the region over which these waves are effectively excited.

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