Parametric resonances in a gas of two-level atoms in an external field

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When a gas of two-level atoms in a biharmonic field is sounded by a probe wave parametric resonances lead to the appearance of an additional complicated and subtle structure in the constant component of the probe-wave absorption coefficient. The absorption coefficient (in the steady-state regime) acquires an oscillating component, the amplitude of which also possesses a complicated behavior in a narrow neighborhood of the parametric resonance.

1. INTRODUCTION

Various aspects of the dynamics of a two-level atom placed in an external quasiresonance field consisting of several harmonics have recently been intensively discussed in the literature (see, e.g., Refs. 1–12). The greatest interest in this regard has been stimulated by the discovery and study of situations in which sharp singularities (peaks or dips) appear in the absorption spectrum of such a system. This interest is due to the possible use of such phenomena for purposes of frequency stabilization and frequency transfer over a band, and in spectroscopic applications.[13,14] An attractive feature of the systems under consideration is the possibility of purposeful controlled variation or choice of the external-field parameters, and thereby of the position and shape of the irregularities in the absorption spectrum. In this paper we shall discuss from this point of view the parametric resonances that arise in the absorption spectrum of a two-level atom under the action of external radiation.

We shall consider a gas of atoms under the action of an external field containing two components—primary and secondary; the amplitude of the primary wave is assumed to be considerably greater than that of the secondary. We use a linear approximation in the probe wave, i.e., its amplitude is assumed to be much smaller than the amplitude of the secondary wave. It is well known that under the action of monochromatic quasiresonance radiation the populations of the levels of an atom oscillate. We are interested in the question as to how the probe-wave-absorption coefficient changes if the secondary wave falls into resonance with the Rabi oscillations induced by the primary wave. As shown in Ref. 7, with neglect of relaxation and pumping for a single two-level atom placed in a bichromatic external field, parametric resonance leads to splitting of the Rabi spectrum of the system. For an analogous phenomenon to be possible for a gas of two-level atoms it is necessary that certain conditions on the parameters of the system be fulfilled. First, the relaxation constants should be sufficiently small in comparison with the frequency difference D between the components of the bichromatic field. Second, the amplitude of the primary harmonic of the bichromatic field should be a quantity of order D. Below we shall make the meaning of these restrictions more precise.

2. FORMULATION OF THE PROBLEM

Let the primary wave, secondary wave, and probe wave have frequencies Ω_1, Ω_2, Ω_3 amplitudes A_1, A_2, A_3 (which are assumed to be positive—an unimportant restriction adopted for simplification of the subsequent formulas), and wave vectors k_1, k_2, k_3, respectively. We shall discuss the situation when the primary and secondary waves propagate in one direction, and the probe wave propagates in the opposite direction. We shall assume that the thermal velocity u_T of the atoms, the frequency detuning $D = \Omega_2 - \Omega_1$, and the effective relaxation constant γ_0 of the atom satisfy the restriction $|D|u_T/c \ll \gamma_0$, where c is the velocity of light. Then, with sufficient accuracy, we can assume that $k_1 = k_2 = -k_3 = k$ (Ref. 8). Next, to avoid unnecessary complication, we assume that the longitudinal-relaxation constants of the two levels are equal: $\gamma_1 = \gamma_2 = \gamma$. We write out the system of Bloch equations that describes the dynamics of an atom in terms of the density matrix $\rho = \rho(v,z)$ (Refs. 13 and 14):

$$\begin{aligned}\frac{d}{dt}(\rho_{22}-\rho_{11}) &= -\gamma(\rho_{22}-\rho_{11})+4i[A_1\cos(\Omega_1 t-kz)\\&\quad+A_2\cos(\Omega_2 t-kz)\\&\quad+A_3\cos(\Omega_3 t+kz)](\rho_{21}-\rho_{12})+\Lambda,\\\frac{d}{dt}\rho_{12} &= -(\gamma_{12}+i\omega)\rho_{12}-2i[A_1\cos(\Omega_1 t-kz)+A_2\cos(\Omega_2 t-kz)\\&\quad+A_3\cos(\Omega_3 t+kz)](\rho_{22}-\rho_{11}),\\&\rho_{21}=\overline{\rho_{12}},\ d/dt=\partial/\partial t+v\partial/\partial z,\end{aligned}\tag{1}$$

where v is the velocity of the atom, z is the spatial (one-dimensional) coordinate of the atom, ω is the transition frequency, Λ is the pumping parameter, γ_{12} is the transverse-relaxation constant, and the bar denotes complex conjugation. We make the replacements

$$\begin{gathered}R=2^{-1/2}(\rho_{22}-\rho_{11}),\\\rho_{12}=R_{12}^{+}\exp[-i(\Omega_1 t-kz)]+R_{12}^{-}\exp[-i(\Omega_3 t+kz)],\\\rho_{21}=R_{21}^{+}\exp[i(\Omega_1 t-kz)]+R_{21}^{-}\exp[i(\Omega_3 t+kz)].\end{gathered}\tag{2}$$

Below we shall use the notation

$$\begin{gathered}\gamma D^{-1}=\varepsilon,\ \gamma_{12}D^{-1}=\Gamma\varepsilon,\ (\omega-\Omega_1)D^{-1}=\Delta_1 2^{1/2}A_3D^{-1}=a_p,\\2^{1/2}A_1D^{-1}=a_1,\ 2^{1/2}A_2D^{-1}=\varepsilon a_2,\ (\omega-\Omega_3)D^{-1}=\Delta_3,\\kD^{-1}=\varkappa,\ \Lambda D^{-1}=\lambda.\end{gathered}\tag{3}$$

Here, we assume that

$$0<\varepsilon\ll 1,\ \Gamma,\ \Delta_1,\ \Delta_3,\ a_1,\ a_2,\ \varkappa,\ \lambda=O(1)\tag{4}$$

(we note here that relative quantities do not depend on the choice of the order of the parameter λ). These relations make specific our choice of the range of parameters of the system. The choice of the relative orders of ε and the thermal velocity will be discussed below.

We shall calculate volume-averaged characteristics of the medium. As shown in Ref. 2, more precise allowance for the spatial structure of the solutions does not lead to appreciable distortion of the results. Using (1)–(4), the rotating-wave approximation, and the dilatation $\tau = Dt$, we arrive at the system of equations

$$\frac{\partial}{\partial\tau}R=-\varepsilon R+i\{[a_1+\varepsilon a_2\exp(-i\tau)]R_{21}^{+}+a_pR_{21}^{-}-[a_1+\varepsilon a_2\exp(i\tau)]R_{12}^{+}-a_pR_{12}^{-}\}+\lambda,$$

$$\frac{\partial}{\partial\tau}R_{12}^{+}=-[\Gamma\varepsilon+i(\Delta_1+\varkappa v)]R_{12}^{+}-i[a_1+\varepsilon a_2\exp(-i\tau)]R,$$

$$\frac{\partial}{\partial\tau}R_{21}^{+}=-[\Gamma\varepsilon-i(\Delta_1+\varkappa v)]R_{21}^{+}+i[a_1+\varepsilon a_2\exp(i\tau)]R.$$

$$\frac{\partial}{\partial\tau}R_{12}^{-}=-[\Gamma\varepsilon+i(\Delta_3-\varkappa v)]R_{12}^{-}-ia_pR,$$

$$R_{21}^{-}=\overline{R_{12}^{-}}. \qquad (5)$$

We assume that $a_p \ll \varepsilon$, i.e., we consider an approximation linear in the probe wave. In the framework of this approximation the last equation of the system (5) [for the function $R_{12}^{-}(v)$] is decoupled. Below it will be shown that the function $R(v)$ in the steady-state regime has the structure

$$R(v)=B_1(v)+2B_2(v)\cos[\tau+\vartheta(v)],$$

where the quantities $B_1(v)$, $B_2(v)$, and $\vartheta(v)$ are real, and $B_2(v)$ is nonzero in the case of parametric resonance. The experimentally measured quantity is the probe-wave-absorption coefficient

$$\chi(\Omega_3)=\int dv\exp[-v^2/u_T^2]\operatorname{Im}R_{21}^{-}(v)$$

(the integration here, and below when the limits are not indicated, is from $-\infty$ to ∞). Using the equation for the function $R_{21}^{-}(v)$, we obtain

$$\chi(\Omega_3)=\int dv\exp\left(-\frac{v^2}{u_T^2}\right)\left\{\frac{\Gamma\varepsilon B_1}{\Gamma^2\varepsilon^2+y^2}+B_2\left[\frac{\Gamma\varepsilon\cos(\tau+\vartheta)-(y+1)\sin(\tau+\vartheta)}{\Gamma^2\varepsilon^2+(y+1)^2}+\frac{\Gamma\varepsilon\cos(\tau+\vartheta)-(y-1)\sin(\tau+\vartheta)}{\Gamma^2\varepsilon^2+(y-1)^2}\right]\right\}, \qquad (6)$$

where

$$y=\Delta_3-\varkappa v.$$

Thus, to determine the probe-wave-absorption spectrum it is sufficient to construct the steady-state solution of the first three equations (5) under the assumption that $a_p = 0$. We write these equations in a form that explicitly includes the small parameter ε:

$$\frac{\partial}{\partial\tau}\mathbf{W}=[Q_0+\varepsilon Q_1(\tau)]\mathbf{W}+\mathbf{C}, \qquad (7)$$

where

$$\mathbf{W}=\begin{pmatrix}R\\R_{12}^{+}\\R_{21}^{+}\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}\lambda\\0\\0\end{pmatrix},\quad Q_0=i\begin{pmatrix}0&-a_1&a_1\\-a_1&-\sigma&0\\a_1&0&\sigma\end{pmatrix},$$

$$\sigma=\Delta_1+\varkappa v,$$

$$Q_1(\tau)=\begin{pmatrix}1&-ia_2\exp(i\tau)&ia_2\exp(-i\tau)\\-ia_2\exp(-i\tau)&-\Gamma&0\\ia_2\exp(i\tau)&0&-\Gamma\end{pmatrix}.$$

General perturbation-theory methods for constructing solutions of systems with periodic coefficients and a small parameter are described in Refs. 15 and 16. As follows from the results described there, the presence or absence of parametric resonances has an important bearing on the construction of the asymptotic forms of the solutions of such systems. The presence of parametric resonances considerably distorts the form of the solution. Below we shall construct only the leading term of the asymptotic expansion of the steady-state solution of the system (7), i.e., we shall study first-order parametric resonances.

Remark 1. In Eqs. (3) and henceforth it is assumed that $D > 0$. If, however, $\Omega_2 < \Omega_1$, the formulas must be modified appropriately.

Remark 2. In our formulation of the problem there are two quantities that can be used to introduce a time (or frequency) scale—the relaxation time of the atom, and the frequency difference between the primary and secondary waves. In the framework of the approach being developed here the second possibility appears to be the more natural, and we shall use it. In papers in which the absorption spectrum of an atom in a monochromatic field has been studied (see, e.g., Ref. 17; further references can be found in Refs. 13 and 14) the first quantity has usually been used to introduce the time scale, and the second has simply been absent. In the terms used in these papers, the assumptions adopted here are equivalent to assuming that the primary component of the external field is strong while the amplitude of the secondary component takes average values.

3. THE PARAMETRIC-RESONANCE CONDITION

We shall describe briefly the approach to the construction of the solution of the system (7). First we shall solve the homogeneous system, i.e., we shall seek the matrix $N(\tau)$ that is the solution of the problem

$$\frac{\partial}{\partial\tau}N(\tau)=[Q_0+\varepsilon Q_1(\tau)]N(\tau),\quad N(0)=I, \qquad (8)$$

where I is the unit matrix. We introduce the matrix U:

$$U=\frac{1}{F}\begin{pmatrix}\sigma&a_1&a_1\\-a_1&-a_1^2/(\sigma+F)&a_1^2/(F-\sigma)\\-a_1&a_1^2/(F-\sigma)&-a_1^2/(F+\sigma)\end{pmatrix},$$

$$F=(\sigma^2+2a_1^2)^{1/2}.$$

Then $U^{-1}Q_0U=E=\operatorname{diag}\{e_1,e_2,e_3\}$, where $e_1=0$, $e_2=iF$, and $e_3=-iF$, and for the matrix $L=U^{-1}N$ we obtain the system

$$\frac{\partial}{\partial \tau}L=[E+\varepsilon M(\tau)]L, \quad M(\tau)=U^{-1}Q_1(\tau)U. \tag{9}$$

To construct the solution of the system (9) we apply the many-scales method.[15] We introduce the "slow" time $\tau_1 = \varepsilon\tau$ and seek the solution of (9) in the form

$$L(\tau, \tau_1)=\exp(E\tau)[L_0(\tau, \tau_1)+\varepsilon L_1(\tau, \tau_1)+\ldots]. \tag{10}$$

Substituting (10) into (9), we find a recursive system of equations for successive terms of the expansion (10). We write out the first two equations:

$$\frac{\partial}{\partial \tau}L_0=0,$$

$$\frac{\partial}{\partial \tau}L_1=G(\tau)L_0-\frac{\partial}{\partial \tau_1}L_0,$$

$$G(\tau)=\exp(-E\tau)M(\tau)\exp(E\tau).$$

From the first equation it follows that $L_0(\tau,\tau_1) = L_0(\tau_1)$. We now consider the second equation. Its right-hand side should not contain secular terms. From this condition we obtain an equation for the matrix $L_0(\tau_1)$:

$$\frac{\partial}{\partial \tau_1}L_0=\langle G(\tau)\rangle L_0. \tag{11}$$

Here, $\langle\ldots\rangle$ denotes the discarding of terms oscillating with frequencies of order unity, i.e., the application of the procedure for averaging over τ. In the general case this procedure leads to a diagonal matrix. However, there exist relations between the parameters (parametric resonances) that lead to the presence of off-diagonal terms. The elements of the matrix M contain exponentials with exponents 0, $i\tau$, and $-i\tau$. Also, $G_{sr} = M_{sr}\exp[i\tau(e_r - e_s)]$, s, $r = 1,2,3$. Therefore, generally speaking, the following parametric resonances are possible:

$$F(v)=1+\varepsilon\nu, \quad \nu=O(1) \tag{12a}$$

which is the condition under which (for atoms with a given velocity) the Rabi frequency coincides asymptotically with the frequency difference between the primary and the secondary component of the external field, and

$$F(v)=0{,}5+\varepsilon\mu, \quad \mu=O(1), \tag{12b}$$

which corresponds to the situation in which the frequency difference between the primary and secondary components agrees asymptotically with the frequency difference between the Rabi harmonics. However, as follows from the structure of the matrix $M(\tau)$, in the leading term of the asymptotic expansion of the solution the resonance (12b) is absent.

Remark 3. In Ref. 7 a description was given (in somewhat different notation) of the parametric resonance (12a). In Ref. 7 the velocity distribution of the atoms was not taken into account, and therefore the formulas given here for the position of the parametric resonances differ somewhat from those obtained in Ref. 7. In the framework of the procedure based on the Schrödinger equation[7] the parametric resonances (12b) were again absent.

The argument of the left-hand side of the relation (12a) is the velocity of the atom, which can be arbitrary. Depending on whether or not there exist groups of atoms whose velocities satisfy the relation (12a), three different situations are possible.

1) $1 - 2^{1/2}a_1 = q < 0, q = O(1)$. In this case there is no group of atoms that satisfy the parametric-resonance conditions. The matrix $\langle G\rangle$ is diagonal for all velocities, and the calculation of $L_0(\tau_1)$ is trivial. In this case the solution of the system (8) for any velocity of the atom in the leading term does not differ from the solution in the absence of the secondary field, i.e., the secondary field A_2 has an influence only in the nonleading terms of the asymptotic form of the solution. For the principal term of the expansion in ε of the function $R(v)$ we obtain

$$R(v)=\lambda[1-2a_1^2\Gamma/(\sigma^2+2a_1^2\Gamma)]/\varepsilon. \tag{13}$$

This result coincides (after relabelings and discarding of the nonleading terms) with Eq. (2.105) of the book by Nayfeh.[13] The next terms of the expansion of $R(v)$ in the small parameter contain oscillating terms. We note here the fact that the field broadening leads to the result that the Bennet dip (the decrease of $R(v)$ in the neighborhood of the point $\sigma = 0$) has a width of order $O(1)$.

2) $2^{1/2}a_1 = 1 + \delta\varepsilon, \delta = O(1)$. There exists one group of atoms whose velocity satisfies the condition (12a). These atoms are parametrized by the expression

$$v=\{-\Delta_1\pm[2\varepsilon(\nu-\delta)]^{1/2}\}/\varkappa+O(\varepsilon), \tag{14}$$

where $\nu\geqslant\delta$. The number of such atoms is then proportional to $\varepsilon^{1/2}$. We shall call this situation a parametric resonance of type I.

3) $1-2^{1/2}a_1=q>0, \ q=O(1)$. There exist two groups of atoms whose velocities satisfy the condition (12a). Their velocities are parametrized by the relation

$$v=[q_1-\Delta_1+\varepsilon\nu/q_1]/\varkappa+O(\varepsilon^2), \quad q_1=\pm[q(2-q)]^{1/2}. \tag{15}$$

The plus sign in the expression for q_1 corresponds to one group of atoms, and the minus sign to the other. The number of atoms in both groups is of order ε. We shall call this case a parametric resonance of type II.

As follows from Eq. (6), to investigate the probe-wave-absorption spectrum it is necessary to construct the solution of Eqs. (7) for all values of the velocity v. Then, for a given set of parameters of the system, one of the situations described above is realized. We should have the possibility of taking into account the contribution to the integral (6) of each group of atoms, bearing in mind that we can select the parameters of the probe wave in such a way as to probe either group of atoms. In fact, the principal contribution to the time-independent part of the integral (6) is given by the group of atoms with velocities in an interval with width of order ε and with center at the point $\Delta_3\varkappa^{-1}$. As follows from what has been said above, those groups of atoms that satisfy the parametric-resonance conditions have sizes of order $O(\varepsilon)$ or $O(\varepsilon^{1/2})$. When the parameters of the probe field are tuned to a group of atoms satisfying the parametric-resonance condition these atoms make the principal contribution to the time-independent part of the velocity integral (6). An analogous assertion is also valid for the part of the integral (6) that oscillates with time. Thus, a parametric resonance (if it exists for the given choice of parameters) can appreciably distort the probe-wave-absorption spectrum.

4. ATOMS AT PARAMETRIC RESONANCE

We shall consider the dynamics of atoms whose velocities satisfy the parametric-resonance condition (12a). We need to construct the solution of the system (11) [and, with the aid of the latter, the solution of the system (8) and, finally, (7)] for such groups of atoms. Here we shall follow the standard prescriptions: In the matrix $G(\tau)$ we replace the expressions $\exp[\pm i(F(v)-1)\tau]$ by $\exp[\pm i\nu\tau_1]$. After this it is easy to obtain

$$\langle G(\tau,\tau_1)\rangle = G_1(\tau_1)$$

$$= -\Gamma + F^{-2}\begin{pmatrix} (\Gamma-1)\sigma^2 & i\xi\exp(i\nu\tau_1) & -i\xi\exp(-i\nu\tau_1) \\ i\xi\exp(-i\nu\tau_1) & (\Gamma-1)a_1^2 & 0 \\ -i\xi\exp(i\nu\tau_1) & 0 & (\Gamma-1)a_1^2 \end{pmatrix}.$$

$$\xi = Fa_2a_1^2(F-\sigma)^{-1}.$$

We can reduce the system of equations (11) with the matrix G_1 to a system of equations with constant coefficients, and construct its solution. Then, for the leading term of the expansion in ε of the matrix $N(\tau)$, we obtain

$$N(\tau) = U\exp(E\tau)Z(\tau)U_1K(\tau)U_1^{-1}U^{-1},$$

$$Z(\tau) = \operatorname{diag}\{1,\ \exp(-i\nu\varepsilon\tau),\ \exp(i\nu\varepsilon\tau)\}. \qquad (16)$$

We shall give the expressions for U_1 and K for two particular cases: a) for $\Gamma = 1$, and b) for $\sigma^2 = a_1^2$. In these situations the diagonal of the matrix $G_1(\tau_1)$ is proportional to the unit matrix, the solution of the problem is simplified (in the general case the corresponding formulas are too unwieldy, although all the qualitative conclusions remain valid), and we obtain

$$K(\tau) = \operatorname{diag}\{\exp(-\Gamma_1\varepsilon\tau),\ \exp[\varepsilon\tau(-\Gamma_1+iH)],$$

$$\exp[\varepsilon\tau(-\Gamma_1-iH)]\},$$

$$U_1 = H^{-1}\begin{bmatrix} \nu & -\xi & -\xi \\ \xi & -\xi^2(H-\nu)^{-1} & \xi^2(H+\nu)^{-1} \\ \xi & \xi^2(H+\nu)^{-1} & \xi^2(H-\nu)^{-1} \end{bmatrix},$$

$$H = (\nu^2+2\xi^2)^{1/2},$$

where $\Gamma_1 = 1$ for case a), and $\Gamma_1 = (1+2\Gamma)/3$ for case b). Here, $N(0) = 1$. We note that $Z^{-1}(\tau) = Z(-\tau)$ and $K^{-1}(\tau) = K(-\tau)$.

It follows from (7) that

$$\mathbf{W}(\tau) = N(\tau)\mathbf{W}(0) + N(\tau)\int_0^\tau N(-x)\,dx\,\mathbf{C}.$$

We are interested in the steady-state solution $\mathbf{W}_c$; therefore, the damping term outside the integral can be discarded, and in the integral we need be interested only in the value at the upper limit. Substituting (16), we find

$$\mathbf{W}_c = U\theta(\tau)U^{-1}\mathbf{C},$$

$$\theta(\tau) = \exp(E\tau)Z(\tau)U_1K(\tau)\int^\tau K(-x)U_1^{-1}Z(-x)\exp(-Ex)\,dx.$$

It is not difficult to convince oneself that the matrix $\theta(\tau)$ contains harmonics with exponents 0, $\pm i\tau$, and $\pm 2i\tau$. However, the harmonics with exponents $\pm 2i\tau$ have amplitudes of lower order than those with exponents $\pm i\tau$. Therefore, they can be assigned to those terms of the expansion of the solution of the next order of smallness, and omitted. Finally, we obtain the result for the function $R(v)$—the first component of the vector $\mathbf{W}_c$:

$$R(v) = S(v) + T(v),$$

where

$$T(v) = \lambda F^{-2}H^{-2}\varepsilon^{-1}\sigma^2[\nu^2\Gamma_1^{-1} + 2\xi^2/(\Gamma_1^2+H^2)],$$

$$S(v) = -2\lambda\sigma a_1\xi\varepsilon^{-1}F^{-2}\Gamma_1^{-1}[\nu\cos\tau + \Gamma_1\sin\tau]/(\Gamma_1^2+H^2). \qquad (17)$$

We note some consequences of this relation. First, irrespective of the velocity of the atom, the steady-state value oscillates with a frequency equal to unity (in the original quantities, this frequency is equal to D), whereas the Rabi frequency of each atom depends on the velocity. Similar facts are familiar (in another context) in the theory of parametric resonance; see, e.g., Ref. 15, in which it is shown that near a parametric resonance the solution has only harmonics with frequencies that are multiples of the frequency of the periodic perturbation. Second, the phase of the oscillating component of the function $R(v)$ depends on the velocity of the atom. Here it is necessary to specify precisely relative to what the phase shift is measured. As follows from our considerations, the reference origin is the moment at which the phases of the primary and secondary waves of the external radiation coincide. Furthermore, these relations contain the parameter ν that describes the fine tuning of the velocity of the atom to the parametric-resonance condition.

It is not difficult to verify that as $\nu \to \pm\infty$ (i.e., as we move away from the parametric-resonance condition) the constant term of this formula goes over into (13) and the oscillating term tends to zero. This corresponds to the above-mentioned absence of oscillating terms in the leading term of the expansion of $R(v)$ in ε away from the parametric resonance.

Remark 4. In the case of a resonance of type I, as follows from (14), the quantity σ is of order $O(\varepsilon^{1/2})$. Therefore, the constant term in (17) is of order $O(1)$. However, in the course of the calculations we discarded terms that could give a contribution of order $O(1)$ to the function $R(v)$. Thus, in this case the constant term in (17) can only be by way of an estimate.

5. PROBE-WAVE-ABSORPTION SPECTRUM

The study of the probe-wave-absorption spectrum rests on the relation (6). Here, for a qualitative analysis, it is sufficient to bear the following facts in mind. First, we consider the leading term of the expansion in ε of the asymptotic form of the function $\chi(\Omega_3)$. Second, the terms that appear in the curly brackets in the integrand in (6) contain as factors narrow (with width $\Gamma\varepsilon\varkappa^{-1}$) Lorentzian distributions with centers at the points $v_p = \Delta_3\varkappa^{-1}$ and $v_p \pm \varkappa^{-1}$, respectively. The neighborhoods of these points give the leading contribution of the integral (6). When v_p is varied, i.e., when the frequency Ω_3 of the probe wave is detuned away from the transition frequency, we probe different groups of atoms, since the leading contribution to the integral is given only by atoms in the ε-neighborhood of the points v_p and $v_p \pm \varkappa^{-1}$, respectively, for the constant and the oscillating component of the probe-wave-absorption coefficient. The numerators of the expressions in the curly brackets in (6) are described by the expression (13) outside the neighborhood of the parametric resonance, while in the $\varepsilon^{1/2}$-neighborhood of a resonance of type I, or in the ε-neighborhood of a resonance of type II, they are described by the relation (17). These neighborhoods have their center at the point $v_r = (q_1 - \Delta_1)/\varkappa$ (we set $q_1 = 0$ for a resonance of type I). The first factor in the integrand in (6) is the Doppler contour; for applicability of the Doppler limit (i.e., to permit the exponential evaluated at the point v_p to be taken outside the integral) it suffices to satisfy either the condition $u_T = O(\varepsilon^{1/2})$ or the condition $u_T = O(1)$.

If the parameters of the system are chosen in such a way that there are no atoms satisfying the parametric-resonance condition (12), the function $R(v)$ is specified by the relation (13). We then obtain a well known result: The function $\chi(\Omega_3)$ describing the absorption spectrum is a Doppler contour with a Bennet dip in it.[13,14] As already noted above, the width of the dip is a quantity of order $O(1)$. The oscillating component in the leading term of the expansion in ε of the absorption coefficient is absent.

If, however, for the chosen parameters of the system, there are atoms satisfying the condition (12a), i.e., $2^{1/2}a_1 < 1$, the $\chi(\Omega_3)$ dependence becomes more complicated. When v_p and $v_p \pm \varkappa^{-1}$ are varying outside the velocity intervals that satisfy the parametric-resonance condition, the function $\chi(\Omega_3)$ is again a Doppler contour with a Bennet dip in it, and there is no oscillating component in the leading term. We shall discuss the behavior of $\chi(\Omega_3)$ in the situation when v_p or $v_p \pm \varkappa^{-1}$ approaches v_r.

Suppose that the parameters of the system are such that there is a group of atoms satisfying the condition for a resonance of type I. Of course, for such a resonance it makes sense to consider only the particular case (a), i.e., $\Gamma_1 = \Gamma = 1$, since for case (b) $a_1^2 = \sigma^2 = O(\varepsilon)$. It was noted above that, using our results, for the constant component $T(v)$ we obtain $T(v) = O(1)$. At the same time, if we probe atoms off resonance, $T(v) = O(\varepsilon^{-1})$. Thus, a resonance of type I leads to a change of the order of magnitude of the constant component [the function $t(\Omega_3)$] of the probe-wave-absorption coefficient.

As regards the oscillating component $S(v)$, as follows from Eqs. (13) and (17) it changes in order of magnitude as the quantities $v_p \pm \varkappa^{-1}$ characterizing the probe-wave frequency approach the velocities of the group of atoms that are in a type-I resonance: Outside this group of atoms $S(v)$ is of order $O(1)$, while for these atoms it is of order $O(\varepsilon^{-1/2})$. Using simple arguments, we can calculate in leading order the oscillating component of the function $\chi(\Omega_3)$—the function $s(\Omega_3)$. Here we take into account that, as follows from (14), in the given case the function $S(v)$ is slowly varying in comparison with the Lorentzian contour: $S(v)$ varies over intervals of order $O(\varepsilon^{1/2})$, whereas the Lorentzian contour varies over intervals of order $O(\varepsilon)$, near the parametric-resonance point. The closeness of $v_p \pm \varkappa^{-1}$ to v_r here implies

$$v_p \pm \varkappa^{-1} - v_r = \zeta \varepsilon^{1/2} \varkappa^{-1}, \quad \zeta = O(1).$$

In the original notation this relation can be written in the form

$$\Omega_3 = 2\omega - \Omega_1 \pm D - \zeta D \varepsilon^{1/2}.$$

It follows from this that in the integration over v near $v_p \pm \varkappa^{-1}$ in the leading term we can assume the following relations to be valid:

$$F=1, \ \xi = a_1^2 a_2, \ \nu = \delta + \zeta^2/2,$$
$$H = [2a_1^4 a_2^2 + (\delta + \zeta^2/2)^2]^{1/2}, \ \sigma = \zeta \varepsilon^{1/2}.$$

Taking this into account, we obtain

$$s(\Omega_3) = -2\pi a_p \lambda \zeta \varepsilon^{-1/2} \varkappa^{-1} a_1^3 a_2 [(\varepsilon + \zeta^2/2)\cos\tau + \sin\tau]$$
$$\times [1 + (\varepsilon + \zeta^2/2)^2 + 2a_1^4 a_1^2]^{-1} \exp[-v_r^2/u_T^2].$$

Thus, the oscillating component of the function $\chi(\Omega_3)$ in the case of a parametric resonance of type I is of order $O(\varepsilon^{-1/2})$. Its amplitude has a zero at $\zeta = 0$ (i.e., for exact tuning to the parametric resonance) and two humps at nonzero values of ζ. Here the parameters δ and ζ [which are quantities of order $O(1)$] characterize the tunings of the external radiation and of the frequency of the probe radiation, respectively, to the parametric resonance.

We now consider the case of a parametric resonance of type II. When the corresponding conditions are fulfilled the function $\chi(\Omega_3)$ possesses complicated behavior when the quantities v_p and $v_p \pm \varkappa^{-1}$ pass through values in the ε-neighborhood of the point v_r. In this situation we can assume that the following relations are valid:

$$F=1, \ \sigma = q_1, \ \xi = a_1^2 a_2/(1-q_1).$$

To determine the behavior of the constant component $t(\Omega_3)$ of the absorption coefficient near the parametric resonance we assume that $v_p = v_r + \varepsilon\beta\varkappa^{-1}$, with $\beta = O(1)$. In the original quantities this implies that $\Omega_3 = 2\omega - \Omega_1 - q_1 D - \varepsilon\beta D$. From (6) and (17) it follows that

$$t(\Omega_3) = \lambda a_p q_1^2 \varepsilon^{-1} \varkappa^{-1} \exp(-v_p^2/u_T^2)$$
$$\times \int d\nu \, \Gamma q_1 [\Gamma^2 q_1^2 + (\nu - \beta q_1)^2]^{-1} V(\nu).$$
$$V(\nu) = [\nu^2 \Gamma_1^{-1} + 2\xi^2/(\Gamma_1^2 + \nu^2 + 2\xi^2)](\nu^2 + 2\xi^2)^{-1}.$$

The oscillating component of the absorption coefficient has singularities, associated with the presence of the parametric resonance, at $\Omega_3 = 2\omega - \Omega_1 - q_1 D \pm D - \varepsilon\beta D$, with $\beta = O(1)$. Using (6) and recognizing that in the given case $\sigma = q_1$, we find an expression for $s(\Omega_3)$:

$$s(\Omega_3) = a_p \frac{\exp(-v_r^2/u_T^2)}{q_1 \varkappa}$$
$$\times \int d\nu \, B_2 \frac{\Gamma\cos(\tau + \vartheta) \mp (\beta - \nu/q_1)\sin(\tau + \vartheta)}{\Gamma^2 + (\beta - \nu/q_1)^2}$$

$$B_2(\nu) = B_2(v(\nu)) = \frac{\lambda q_1 a_1 \xi}{\varepsilon \Gamma_1 (\Gamma_1^2 + H^2)(\nu^2 + \Gamma_1^2)^{1/2}},$$

$$\vartheta(\nu) = \pi + \arcsin[\nu(\nu^2 + \Gamma_1^2)^{-1/2}].$$

Here the choice of the signs in the integrand is opposite to the choice of the signs in Ω_3, and the parameters q_1 and β describe the tuning of the external radiation and the tuning of the probe wave to the parametric resonance. We note that the constant component and the amplitude of the oscillating component in the neighborhood of the parametric resonance are quantities of the same order. For specified primary-wave parameters and secondary-wave frequency, the parameter ξ depends linearly on the relative amplitude a_2 of the secondary wave. By varying a_2 (and, correspondingly, ξ), we can change the integrands, and thereby change the behavior of

$t(\Omega_3)$ and $s(\Omega_3)$. For example, for relatively large values of ξ the function $V(\nu)$ has a minimum at $\nu = 0$. Fixing the parameters of the primary and secondary waves, and measuring the probe-wave frequency Ω_3 (i.e., the parameter β), we find the minimum of $s(\Omega_3)$ at $\beta = 0$. Here, the relative magnitude of the minimum, i.e., its contrast, will be of order $O(1)$.

From the relations given above it is possible to find in explicit form the amplitude and phase of the oscillating component $s(\Omega_3)$ of the absorption coefficient. However, even without this, it is possible to conclude that they will also be rather complicated functions both of the primary-wave and secondary-wave parameters and of the probe-wave frequency Ω_3 (i.e., of the parameter β) and will possess minima (maxima) with contrast of order $O(1)$. This supplementary structure, associated with the presence of parametric resonances both for the constant component and for the oscillating component, has a width of order $O(\varepsilon^{1/2})$ for resonances of type I and $O(\varepsilon)$ for resonances of type II. Thus, parametric resonances lead to the presence of sharp singularities in the probe-wave-absorption spectrum. We emphasize once again that away from the parametric resonance there is no oscillating component in the leading term in the expansion in ε of the absorption coefficient, i.e., in the neighborhood of the parametric resonance a change (increase) is observed in the order of magnitude of the amplitude of the oscillating component.

In order that the effects described above be manifested in experiment, it is necessary, as already mentioned, that certain conditions on the parameters of the atom and external field be fulfilled. Namely, the Rabi parameter A_1 of the primary wave and the detuning D between the frequencies of the primary and secondary waves should be considerably greater than the widths of the levels of the atom, while the Rabi parameter A_2 of the secondary wave should be of the same order as the widths of the levels. In addition, the condition $2A_1 < D$ should be fulfilled. The choice of these parameters (the intensities and frequencies of the components of the external field) is in the hands of the experimenter, and can be made in such a way that these conditions are fulfilled. Following Chapter 2 of Ref. 14, we shall consider as an example the 4^1S_0–4^3P_1 transition of the ^{40}Ca atom. The radiative width of the transition is equal to 410 Hz, which is much smaller than the Rabi parameter $A_1 = 1 \times 10^6$ Hz of the primary wave for an intensity of the latter of 1 W/cm^2. If the intensity of the secondary wave is of the order of 10^{-6} W/cm^2, the value of its Rabi parameter is of the order of the width of the levels. By taking a detuning between the frequencies of the primary and secondary components that is greater than 2×10^6 Hz we fulfill the conditions necessary for the appearance of parametric resonances.

We note here that in the derivation of our formulas we assumed that a number of restrictive conditions on the parameters of the system are fulfilled, e.g., that the widths of the two levels of the atom coincide. However, there are grounds to suppose that the effects described will also occur in a more general situation.

6. CONCLUSION

We shall summarize the results. For a gas of two-level atoms placed in an external field (consisting of a primary and a secondary wave) and probed by a probe wave we have written out the conditions for the existence of parametric resonances, which can be of two types. Parametric resonances lead to the appearance of a complicated and subtle structure for the constant component of the probe-wave-absorption spectrum. This structure is observed upon variation of the probe-wave frequency within a range of order $O[(\gamma D)^{1/2}]$ about the value $2\omega - \Omega_1$ (for a resonance of type I) or within a range of order $O(\gamma)$ about the value $2\omega - \Omega_1 - q_1 D$ (for a resonance of type II); here, γ is the natural line width. We recall that the principal initial assumption was the condition $D \gg \gamma$, which can be realized easily in experiment. In addition, the probe-wave-absorption coefficient in the steady-state regime contains a component oscillating with frequency D. Away from the parametric resonance its relative amplitude is a quantity of order $O(1)$. As the probe-wave frequency passes through the neighborhoods of the values $2\omega - \Omega_1 - q_1 D \pm D$ the relative amplitude of the oscillating component of the absorption coefficient is a quantity of order $O[(D/\gamma)^{1/2}]$ for a resonance of type I, and $O(D/\gamma)$ for a resonance of type II. This amplitude also possesses complicated behavior in an asymptotically small neighborhood of the parametric resonance. The phase of the oscillating component is related to the velocity of the group of atoms being probed. Field broadening of these structures (the principal wave is not assumed to be small) is absent. By choosing the parameters of the system—in particular, the parameters of the primary and the secondary wave, it is possible to change the dependence of the measured quantities on the probe-wave frequency. These circumstances make such systems promising in spectroscopic applications, and also in the creation of optical reference frequencies and in laser-frequency stabilization.

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