Behavior of temporal correlation functions in the decay of a quasistationary state: the Fock–Krylov theorem

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We examine the time and frequency behavior of correlation functions in the decay of a quasistationary state and show that it strongly depends on the ratio of the decay width $1/\tau$ and the level separation in the system. We also discuss the possibility of studying via experiments the correlations functions in superconducting quantum interference devices (SQUIDs).

1. INTRODUCTION

In recent years there has been extensive analysis of the decay of metastable states in superconducting quantum interference devices (SQUIDs).[1-3] A notable achievement in this field has been the theoretical and experimental investigation of the interaction of a tunneling particle with the environment and the dissipative processes emerging in the particle motion. At low temperatures the particle may move in a nondissipative manner, and the problems associated with this mode of motion stem from the usual questions discussed in connection with the decay of a quasistationary state in nuclear physics,[4,5] one of which deals with the description of the transition from the mode occurring in the decay of a quasistationary state[4] to the Fock–Krylov oscillation mode[5] in a two-well potential.

An essential advantage of experiments conducted in studying the decay of a quasistationary state in SQUIDs over experiments involving other metastable systems (say, the decay of a radioactive nucleus) is that by simple variations of the electric parameters of the SQUIDs (resistance, capacitance, the time dependence of the electromagnetic field) we can achieve considerable variation in the extent of dissipation, the effective particle mass, the potential in which the tunneling state moves, and other characteristics of the transition. The simplicity of the system makes it possible to measure not merely the dependence of the potential on the current but the derivatives of these two quantities, which markedly improves the possibilities of observation. Measuring the first derivative yields the probability density rather than the transition probability and measuring the second derivative provides even subtler transition of characteristics. This is accompanied by striking physical phenomena[6,7] that still await theoretical analysis and interpretation. For instance, the authors of Ref. 6 report on the existence of a stochastic pattern in tunneling observations at certain parameters.

In this paper we consider the case of a highly asymmetric two-well potential (Fig. 1), where the level separation in the right well is considerably smaller than that in the left, analyze the decay probability amplitude for a state that initially was in the left well and in the course of time spread over both wells, and calculate the temporal correlations of the probability amplitude. This also enables one to observe the transition from the Fock–Krylov oscillation mode[5] to the Gamow mode of the decay of a quasistationary state.[4] In addition, we have calculated the behavior of the temporal correlation functions related to the dimensional quantization in the right well (Fig. 1). We discuss the application of the results to experiments with SQUIDs[6] when there is a transition from one experimental mode, the resonance mode, to another, the stochastic.

2. STATEMENT OF THE PROBLEM: THE TRANSITION PROBABILITY AMPLITUDE; TEMPORAL CORRELATION FUNCTIONS

Let us consider two potential wells separated by a barrier (Fig. 1). The left well *1* contains a stationary level E_0, provided that we ignore all interactions with the right well *2*. We assume that the right well is much larger than the left. As a result the level separation in the right well is much smaller than in the left. The wave function $\psi(x,t)$ satisfies the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=\left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x,t). \tag{1}$$

The initial state at $t=0$ corresponds to a state in the left well with a wave function $\psi_0(x)$ and energy E_0:

$$\psi(x,t)|_{t=0}=\psi_0(x). \tag{2}$$

We are interested in the amplitude $p(t)$ of the probability that this state at time t with the wave function

$$\psi_0(x,t)=\psi_0(x)\exp(-E_0t)$$

will go over to the state with a wave function $\psi(x,t)$ satisfying Eq. (1) with the initial condition (2):

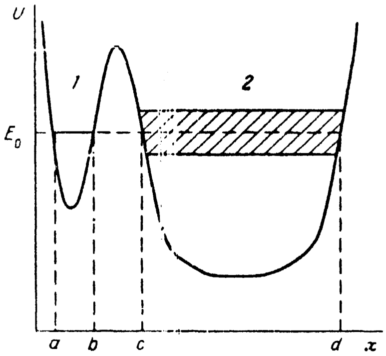

FIG. 1. The two-well potential of the system; E_0 is the energy of the metastable state.

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$$p(t)=\int dx\psi_0(x,t)\psi^*(x,t).$$

We expand $\psi(x,t)$ in the normalized eigenfunctions $\psi_E(x)$ of Eq. (1):

$$\psi(x,t)=\sum_E C_E\psi_E(x)\exp(-iEt).$$

The coefficients C_E are determined by the initial condition (2):

$$C_E=\int dx\psi_0(x)\psi_E^*(x). \quad (3)$$

As a result the probability amplitude $p(t)$ assumes the form

$$p(t)=\sum|C_E|^2\exp(it\delta E),\quad \delta E=E-E_0. \quad (4)$$

We calculate C_E, which is expressed in terms of $\psi_0(x)$ and $\psi_E(x)$ [see Eq. (3)]. To this end we confine ourselves to the quasiclassical approximation, which to within an unimportant phase factor completely describes the problem. If the wave functions $\psi(x)$ has the form

$$\psi(x)=A_1p^{-1/2}(x)\cos\left(\int_a^x p(x)dx-\frac{\pi}{4}\right)+B_1p^{-1/2}(x)\sin\left(\int_a^x p(x)dx-\frac{\pi}{4}\right),$$

in the classically accessible region $a<x<b$ (Fig. 1) and the form

$$\psi(x)=A_2p^{-1/2}(x)\cos\left(\int_c^x p(x)dx-\frac{\pi}{4}\right)+B_2p^{-1/2}(x)\sin\left(\int_c^x p(x)dx-\frac{\pi}{4}\right),$$

in the classically accessible region $c<x<d$, the coefficients A_2 and B_2 are related to A_1 and B_1 via the following matrix relation:

$$\begin{pmatrix}A_2\\B_2\end{pmatrix}=\begin{pmatrix}2e^D\cos S_0 & 2e^D\sin S_0\\ -{}^1/{}_2e^{-D}\sin S_0 & {}^1/{}_2e^{-D}\cos S_0\end{pmatrix}\begin{pmatrix}A_1\\B_1\end{pmatrix}. \quad (5)$$

An expression for the matrix elements can be obtained from the matching formulas, which are given in Ref. 8 in a form most convenient for applications (a rigorous treatment carried out in Ref. 9). In Eq. (5),

$$S_0=\int_a^b p(x)dx,\quad p(x)=[2m(E-U(x))]^{1/2},\quad D=\int_b^c p_E(x)dx,$$
$$p_E(x)=[2m(U(x)-E)]^{1/2}.$$

For $x<a$ the wave function decreases exponentially, so we have $B_1=0$. Similarly, the wave function decreases exponentially for $x>d$, which implies

$$A_2\cos S+B_2\sin S=0,\quad \text{where}\quad S=\int_c^d p(x)dx.$$

Substituting $A_2=2e^D\cos S_0$ and $B_2=-\frac{1}{2}e^{-D}\sin S_0$, we arrive at a dispersion equation determining the energy levels of the system:

$$\cos S_0(E)\cos S(E)=\Delta^2\sin S_0(E)\sin S(E),\quad \Delta={}^1/{}_2e^{-D}. \quad (6)$$

Here the wave function of the initial state is $\psi_0(x)$

$$\psi_0(x)=A_0p^{-1/2}(x)\cos\left(\int_a^x p(x)dx-\frac{\pi}{4}\right). \quad (7)$$

Its normalization is determined, as usual,[8,9] by

$$A_0^2=2m\pi/\omega_0(E_0), \quad (8)$$

where $\omega_0(E_0)$ is the frequency of motion of a particle with energy E_0 in the left well. The wave function $\psi_E(x)$ of the two-well system in the classically accessible regions is given by the following formula:

$$\psi_E(x)=A(E)p^{-1/2}\cos\left(\int_a^x p(x)dx-\frac{\pi}{4}\right),\quad a<x<b,$$
$$\psi_E(x)=A(E)Q(E)p^{-1/2}\cos\left(\int_x^d p(x)dx-\frac{\pi}{4}\right),\quad c<x<d, \quad (9)$$

where $Q(E)$ is determined by the matrix in (5),

$$Q(E)=\Delta^{-1}\cos S_0(E)\sin S(E)+\Delta\sin S_0(E)\cos S(E).$$

Like (8), the quantity $A^2(E)$ has the form

$$A^2(E)=\frac{2m}{\pi}\left[\frac{1}{\omega_0(E)}+\frac{Q^2(E)}{\omega(E)}\right]^{-1}. \quad (10)$$

Usually the matrix element between two wave functions is calculated as the Fourier coefficients of the respective classical quantity.[8,10] When calculating $C(E)$ in Eq. (3), there is no need to evaluate the Fourier coefficients since we are far from other levels in the left well and we can simply put $E=E_0$. As a result we can employ the same approximation as we did in (8)and (10):

$$C(E)=\pi A_0(E)A(E)/(2m\omega_0(E)). \quad (11)$$

Substituting (8) and (10) into (11) yields

$$C^2(E)=\omega_0^{-1}(E)[\omega_0^{-1}(E)+Q^2(E)\omega^{-1}(E)]^{-1}. \quad (12)$$

Using the dispersion equation (6), we can transform the expression for $Q^2(E)$ into

$$Q^2(E)=(\Delta^{-2}+\Delta^2)\cos^2 S_0(E_0)+\Delta^2.$$

Since $\cos S_0(E_0)=0$, we can write

$$Q^2(E)=(\Delta^{-2}+\Delta^2)[\pi\delta E/\omega_0(E_0)]^2+\Delta^2.$$

In view of its smallness, the second term inside the parentheses on the right-hand side can be ignored. As a result $Q(E)$ we obtain

$$Q^2(E)=\Delta^2[1+(\tau\delta E)^2],\quad \tau=\pi/(\omega_0\Delta^2)=4\pi e^{2D}/\omega_0.$$

The quantity $C(E)$ can be represented in the form

$$|C(E)|^2=\left[1+\frac{T}{\tau(1+(\tau\delta E)^2)}\right]^{-1}, \quad (13)$$

where $T=\pi/\omega(E_0)$ is the half-period of the motion of a particle in the right well with energy E_0. This is the final expression, and we will operate with it in calculating the probability amplitude $p(t)$ of Eq. (4).

In what follows we also consider the correlation function of two probability amplitudes as a function of the time delay between them:

$$K(t)\leq\lim_{t'\to\infty}\frac{1}{t'}\int_0^{t'}p(t'')p^*(t''+t)\,dt''.$$

Substituting into this formula (4) for probability amplitudes and employing the fact that

$$\lim_{t'\to\infty}\frac{\exp\{i(E-E')t'\}-1}{i(E-E')t'}=\delta(E-E'),$$

we arrive at a formula for a temporal correlation function $K(t)$ similar to (4):

$$K(t)=\sum|C_E|^4\exp\{it\delta E\},\quad \delta E=E-E_0. \tag{14}$$

3. THE FOCK-KRYLOV THEOREM AND THE DECAY OF A QUASISTATIONARY STATE

The correlation function $K(t)$ given by formula (14) with the coefficients $C(E)$, Eq. (13), of the energy spectrum (6) of the total system is evaluated by summing via Poisson's formula as was done in Ref. 11 for the probability amplitude $p(t)$ given by Eq. (4).

Bearing in mind that

$$\cos S_0(E)=-\sin[S_0(E_0)]\pi\delta E/\omega_0(E_0),$$

we can transform the dispersion equation (6) into $\tan S(E)=-\tau\delta E$, whose solution δE_n satisfies the condition

$$S(E_0)+\pi\delta E_n/\omega_0(E_0)=\pi n-\arctan(\tau\delta E_n). \tag{15}$$

Note that stepping outside the scope of the quasiclassical approximation only changes $S(E_0)$ by a constant of $\sim\pi/4$, which, as we will see shortly, leads only to an unimportant phase shift.

We represent $K(t)$ as

$$K(t)=\sum_{k=-\infty}^{\infty}Q_k,$$

$$Q_k=\int_{-\infty}^{\infty}dn\,|C_E|^4\exp(2\pi ikn+i\delta E_nt). \tag{16}$$

We now go from the integration variable n to δE. Here $dn/d(\delta E)$ is determined by condition (15):

$$\frac{dn}{d(\delta E)}=\frac{\tau}{\pi}\left\{1+\frac{T}{\tau(1+(\tau\delta E)^2)}\right\}\frac{1}{1+(\tau\delta E)^2}. \tag{17}$$

As a result, for Q_k we get

$$Q_k=\Phi_k(0)-\Phi_k(\sigma),\quad \sigma=\tau/T,$$

$$\Phi_k(\sigma)=\frac{\tau}{\pi}\int_{-\infty}^{\infty}\frac{\exp(2\pi ink+ixt)\,dx}{1+\sigma+(\tau x)^2}. \tag{18}$$

The probability amplitude $p(t)$ given by (4) is also expressed in terms of $\Phi_k(\sigma)$:

$$p(t)=\sum_{k=-\infty}^{\infty}p_k,\quad p_k=\Phi_k(0). \tag{19}$$

Substituting n specified by (15), we can represent $\Phi_k(\sigma)$ as follows:

$$\Phi_k(\sigma)=\frac{1}{\pi}\exp[2ikS(E_0)]\int_{-\infty}^{\infty}\exp(2ik\,\mathrm{arctg}\,x+ixt_k)\frac{dx}{1+\sigma+x^2}, \tag{20}$$

with $t_k=(t+2kT)/\tau$. Since

$$\exp(i\arctan x)=(1+ix)/(1+x^2)^{1/2},$$

the integral in (20) can be written as

$$\tilde\Phi_k(\sigma)=\int_{-\infty}^{\infty}\left(\frac{1+ix}{1-ix}\right)^k\exp(ixt_k)\frac{dx}{1+\sigma+x^2}. \tag{21}$$

This integral can easily be evaluated by the theory of residues since the integrand contains only three poles: $\pm i\sqrt{1+\sigma}$ and $i\,\mathrm{sgn}\,k$. We assume that t satisfies the inequalities $0<t<2T$. In this case the sign of t_k coincides with the sign of k. Hence, for $k\geqslant0$ the integration contour must be closed in the upper half-plane, with only one singular point, $i\sqrt{1+\sigma}$, lying inside the contour. As a result for $k\geqslant0$ and $t\geqslant0$ we have

$$\tilde\Phi_k(t,\sigma)=\frac{\pi}{\rho}\frac{(-\sigma)^k}{(1+\rho)^{2k}}\exp\left[\frac{-\rho(t+2kT)}{\tau}\right],\quad \rho=(1+\sigma)^{1/2}. \tag{22}$$

For $k<0$ and $0<t<2T$ the exponent in the exponential function in the integrand of (21) is negative and to evaluate the integral by the theory of residues the integration contour must be closed in the lower half-plane. The only singular point inside the contour, $-i\sqrt{1+\sigma}$, yields the following value for the integral with $k\leqslant-1$ and $0\leqslant t\leqslant2T$:

$$\tilde\Phi_k(t,\sigma)=\frac{\pi}{\rho}\frac{(-\sigma)^{|k|}}{(1+\rho)^{2|k|}}\exp\left[\frac{\rho(t-2|k|T)}{\tau}\right]. \tag{22'}$$

From Eqs. (22) and (22') it follows that at $\sigma=0$ the only function $\tilde\Phi_k$ that is not identically zero is

$$\tilde\Phi_0(t,0)=\pi\exp(-t/\tau). \tag{22''}$$

Substituting (18), (20)–(22'), and (22") into (16), we find the following expression for the correlation function $K(t)$ when $0\leqslant t<2T$:

$$\begin{aligned}K(t)=&\exp\left(\frac{-t}{\tau}\right)-\frac{1}{\rho}\exp\left(-\frac{\rho t}{\tau}\right)\\&+\frac{1}{\rho}\exp\left(-\frac{\rho t}{\tau}\right)\left\{1+\frac{(1+\rho)^2}{\sigma}\right.\\&\left.\times\exp\left[\frac{2\rho}{\sigma}-2iS(E_0)\right]\right\}^{-1}+\frac{1}{\rho}\\&\times\exp\left(\frac{\rho t}{\tau}\right)\left\{1+\frac{(1+\rho)^2}{\sigma}\right.\\&\left.\times\exp\left[\frac{2\rho}{\sigma}+2iS(E_0)\right]\right\}^{-1}.\end{aligned} \tag{23}$$

The probability amplitude $p(t)$ defined in (19) is determined by the only nonzero function $\tilde{\Phi}_0(t,0)$ [Eq. (22″)] and is given by the following formula:

$$p(t)=\exp(-t/\tau), \quad 0<t<2T. \tag{24}$$

At $t=0$ the correlation function $K(t)$ is a real quantity:

$$K(0)=1-\frac{1}{\rho}+\frac{2}{\rho}\operatorname{Re}\left\{1+\frac{(1+\rho)^2}{\sigma}\exp\left[\frac{2\rho}{\sigma}-2iS(E_0)\right]\right\}^{-1}. \tag{23′}$$

In the limit as $\sigma\to\infty$ ($\tau\gg T$),

$$K(0)=1. \tag{23″}$$

The quantity $K(0)$ determines how much time asymptotically the system spends in the initial state corresponding to time $t=0$. The limiting case considered here corresponds to a situation in which the system asymptotically is entirely in the initial state $\psi_0(x)$ with energy E_0. This mode is described by the Fock–Krylov oscillation theorem,[5] where the wave function of the initial state, oscillating in time with a phase factor iE_0t, goes from this state, which is not an eigenstate for the total two-well system, to a coherent eigenstate of the same system whose energy differs from E_0 by an exponentially small quantity of order $\sim\tau^{-1}=\omega_0e^{-2D}/4\pi$. Here there is virtually no dynamics in the system and the metastable state E_0 does not decay. It must also be noted that in the case considered, in addition to the characteristic time τ, there emerges a characteristic correlation-decay time $\tau/\rho=\sqrt{T\tau}$ [see Eq. (23)]. This latter quantity corresponds to the establishment of correlations when coherent band motion forms in the system. It differs markedly from τ in both the exponential function and its coefficient.

In the opposite limiting case $\sigma\to 0(\tau\ll T)$,

$$K(0)=\tau/(2T). \tag{23‴}$$

In this limit the initial metastable state of the system decays via Eq. (24) with a half-decay time τ given by the Gamow law,[4] and the process takes a long time t compared to the half-decay time ($t\sim 2T\gg\tau$). In this situation the level separation in the right well, on the order of $\sim 1/2T$, is much smaller than the width $1/\tau$ of the metastable level in the left well. The value of $K(0)$ is exactly equal to the ratio of these two quantities and shows that in the process of decay the initial state spreads over all the levels in the right well that find themselves within the width $1/\tau$ of the metastable state E_0 in the left well. In the limit $T\to\infty$ decay obeys the exponential Gamow law (24).

4. DIMENSIONAL PHENOMENA IN THE DECAY OF A METASTABLE STATE

Equation (23) determines the behavior of $K(t)$ of (14) at times t satisfying the inequalities $0\leqslant t<2T$. Here we will consider the behavior of $K(t)$ and $p(t)$ of (4) at large times. To this end we introduce the following quantity:

$$K_1(t,\sigma)=\pi^{-1}\sum_{k=-\infty}^{\infty}\Phi_k(t,\sigma)\exp[2ikS(E_0)]. \tag{25}$$

The correlation function $K(t)$ of (16) and the transition-probability amplitude $p(t)$ of (19) are expressed in terms of this quantity in the following manner:

$$K(t)=K_1(t,0)-K_1(t,\sigma)$$
$$p(t)=K_1(t,0). \tag{26}$$

To obtain $K_1(t,0)$ we must calculate $\tilde{\Phi}_k(t,0)$:

$$\Phi_k(t,0)=\int_{-\infty}^{\infty}\frac{(1+ix)^{k-1}}{(1-ix)^{k+1}}\exp(ixt_k)\,dx \tag{27}$$

over time intervals t satisfying the inequalities $2Tm<t<2T(m+1)$, $m=0,1,2,\ldots$. For $k\geqslant 1$ there is only one pole in the integrand, $x=-i$. Since in the argument of the exponential function in the integrand of (27), we have $t_k>0$, the integration contour must be closed in the upper half-plane; consequently, the pole is not inside the contour. Hence, $\tilde{\Phi}_k(t,0)=0$ holds for $k\geqslant 1$. On the other hand, we have $t_k<0$ for $k\leqslant-(m+1)$, which means that in this case the integration contour in (27) must be closed in the lower half-plane; consequently, the only pole of the integrand, $x=i$, does not lie inside the contour. Hence, $\tilde{\Phi}_k(t,0)=0$ holds for $k\leqslant-(m+1)$, too. All this means that $\tilde{\Phi}_k(t,0)$ is nonzero only for k satisfying the inequalities $-m\leqslant k\leqslant 0$ and has the form

$$\Phi_k(t,0)=2\pi i\operatorname{Res}f_k^0(i),$$

where $f_k^0(x)$ is the integrand in (27). Summing 0 to $-m$ from the resulting expression, which is a geometric series, we get

$$K_1(t,0)=2i\operatorname{Res}F^0(i),$$
$$F^0(x)=[1-r^{m+1}](1-r)^{-1}(1+x^2)^{-1}\exp(ixt/\tau),$$
$$r(x)=\exp[-i2Tx/\tau-2iS(E_0)](1-ix)/(1+ix). \tag{28}$$

Note that the unity in the expression in brackets in $F^0(x)$ contributes nothing to the residue in the expression for $K_1(t,0)$ since that term has no singularity at point i.

Reasoning along similar lines, we can calculate the value of $\tilde{\Phi}_k(t,\sigma)$ at $\sigma\neq 1$ in intervals of values of t satisfying the inequalities $2Tm<t<2T(m+1)$, $m=0, 1, 2, \ldots$. In this case, $\tilde{\Phi}_k(t,\sigma)$ for $k\geqslant 0$ is determined by formula (22) and for $k\leqslant-(m+1)$ by (22′). For $-m\leqslant k\leqslant-1$ we have $t_k>0$, and the integration contour in (21) must be closed in the upper half-plane. Since in this case $k<0$ holds, we have

$$\Phi_k(t,\sigma)=\int_{-\infty}^{\infty}\left(\frac{1-ix}{1+ix}\right)^{|k|}\exp(ixt_k)\frac{dx}{1+\sigma+x^2}. \tag{21′}$$

Two poles lie inside the integration contour: a simple pole at point $i(1+\sigma)^{1/2}$ and a pole of order k at the point i. The simple pole contributes $\tilde{\Phi}_k^{(1)}(t,\sigma)$ to $\tilde{\Phi}_k(t,\sigma)$, and this contribution is exactly equal to $\tilde{\Phi}_k(t,\sigma)$ of (22) but for $k<0$. As for the $|k|$th order pole, we write its contribution as

$$\Phi_k^{(2)}(t,\sigma)=2\pi i\operatorname{Res}f_k(i), \quad -m\leqslant k\leqslant-1,$$

where $f_k(x)$ is the integrand in (21′). The contribution to $K_1(t,\sigma)$ from the simple pole at $i(1+\sigma)^{1/2}$ can be represented in the form of an infinite sum of combinations of (22) and (22′), which reduces to a geometric series:

$$K_1^{(1)}(t,\sigma)=\frac{1}{\rho}\exp\left(-\frac{\rho t}{\tau}\right)\frac{q^{2m+1}+q^{-(2m+1)}}{q+1/q}+\frac{1}{\rho}\frac{(-\sigma)^m}{(1+\rho)^{2m}}$$
$$\times\exp\left[-\frac{\rho t}{\tau}-\frac{2m\rho}{\sigma}+2imS(E_0)\right]\left\{1+\frac{(1+\rho)^2}{\sigma}\right.$$

$$\times \exp\left[\frac{2\rho}{\sigma} - 2iS(E_0)\right]\Big\}^{-1}$$

$$+\frac{1}{\rho}\frac{(-\sigma)^m}{(1+\rho)^{2m}}\exp\left[\frac{\rho t}{\tau} - \frac{2m\rho}{\sigma} - 2imS(E_0)\right]$$

$$\times\left\{1 + \frac{(1+\rho)^2}{\sigma}\exp\left[\frac{2\rho}{\sigma} + 2iS(E_0)\right]\right\}^{-1},$$

$$q = \sigma^{1/2}(1+\rho)^{-1}\exp[-\rho/\sigma + iS(E_0)], \tag{29}$$

where $q = \sigma^{1/2}(1+\rho)^{-1}\exp[-\rho/\sigma + iS(E_0)]$.

Substituting (28) and (29) with $m = 0$ into (26), we immediately obtain $K(t)$ for times t satisfying the condition $0 \leqslant t < 2T$, or simply Eq. (23). The contribution to $K_1(t,\sigma)$ from the $|k|$th order pole at the point i is the sum of m terms $\tilde{\Phi}_k^{(2)}(t,\sigma)$ at values of k satisfying the condition $-m \leqslant k \leqslant -1$ and constitutes the sum of a geometric series:

$$K_1^{(2)}(t,\sigma) = 2i \operatorname{Res} F(i),$$

$$F(x) = r(1+\sigma+x^2)^{-1}\exp(ixt/\tau)[1-r^m]/(1-r). \tag{29'}$$

The final expression for $K_1(t,\sigma)$ is the sum of (29) and (29′):

$$K_1(t,\sigma) = K_1^{(1)}(t,\sigma) + K_1^{(2)}(t,\sigma). \tag{30}$$

As in Eq. (28), the unity in the brackets in $F(x)$ contributes nothing to $K_1^{(2)}(t,\sigma)$ since that term has no singularity at point i.

Equations (29), (29′), and (30) for $K_1(t,\sigma)$ and (28) for $K_1(t,\sigma)$ make it possible to write the explicit expressions for the probability amplitude $p(t)$ and the correlation function $K(t)$ defined in (26). For instance, in the time interval $2T < t < 4T (m = 1)$,

$$p(t) = \exp\left(\frac{-t}{\tau}\right) + 2\left(\frac{t-2T}{\tau}\right)\exp\left[-\frac{t-2T}{\tau} - 2iS(E_0)\right]. \tag{31}$$

In addition to the ordinary exponential term (24), this equation contains a term[11] corresponding to the interference of the primary flux in the decay of the state in the well *1* and the waves reflected from the right edge of well *2* when these waves interact near the barrier separating the wells. This term also has an exponential function, but with an exponent containing time measured from when the waves interact. We see that although $p(t)$ is a continuous function, its first time derivative $\dot{p}(t)$ experiences a jump at point $2T$.

Let us examine in greater detail the limiting case $2T/\tau \gg 1$ when $2T < t < 4T$. In these conditions the second term in (31) is much larger than the first, since the first term has time to exponentially decay. At times t satisfying the condition $0 < t < 2T$ the Gamow decay law (24) operates. At $t = 2T$ the decay process terminates and the function $p(t)$ virtually becomes equal to its initial value at $t = 0$ and then falls off exponentially but more slowly than in the $0 < t < 2T$ interval. The ratio

$$\frac{p(t+2T)}{p(t)} \approx \frac{t}{\tau}\exp[-2iS(E_0)] \tag{32}$$

is much greater than unity for all values of t in the interval $(0,2T)$ except in the region $t \lesssim \tau$, since $t \sim T \gg \tau$. Actually, a relation of the form (32) holds not only in the interval $(0,2T)$ but for all positive values of $t > 0$. Thus, starting at $t = 2T$, the quantity $p(t)$ ceases to decrease according to the exponential law (24) and at $t = 2Tm$ $(m = 1,2,...)$ its derivatives experience jumps up to the mth order inclusive. At greater values of t at a distance of the order of $\sim\tau$ from $2Tm$ the quantity increases to its value at ~ 1 and amounts to roughly unity, and then exponentially decreases, but more slowly than in the interval $(0.2T)$, since in the mth interval it contains an algebraic factor $(t - 2Tm)^m$, which results in an additional factor of order $\sim (T/\tau)^m$. This behavior of the probability amplitude $p(t)$ affects the time dependence of $K(t)$, which according to (29) and (30) dramatically changes its value at $t = 2T, 4T, \ldots$. The sections below describe this problem in greater detail.

5. SPECTRAL REPRESENTATION OF THE PROBABILITY AMPLITUDE AND CORRELATION FUNCTION

The explicit expressions (28)–(30) for the probability amplitude $p(t)$ and the correlation function $K(t)$ obtained in Sec. 4 have different analytic representations in different intervals of time t, increase in complexity as t grows and become practically incomprehensible. They have similar structure within different time intervals, which suggests that they become much simpler after a Fourier transformation, and this also makes it possible to study their asymptotic behavior at large times. Technically it is more convenient to employ the Laplace transform, since this is more suitable for studying the analytic properties and establishing the asymptotic behavior. To obtain the transitions of $p(t)$ and $K(t)$ it is expedient to begin with the initial expressions (4) and (14).

The Laplace transform of the probability amplitude $p(t)$ is defined in the following manner:[12]

$$P(s) = \int_0^\infty p(t)e^{-st}dt. \tag{33}$$

Applying (4) directly, we can easily show that $|p(t)| \leqslant 1$, whereby the growth index s_0 of this function is zero. The transform $P(s)$ is analytic in the half-plane $\operatorname{Re} s > s_0 = 0$ and has the form

$$P(s) = i\sum{}' |C_E|^2/(\delta E + is). \tag{34}$$

By employing the Poisson transformation, we can express $P(s)$ as follows:

$$P(s) = \sum_{k=-\infty}^{\infty} H_k, \quad H_k(s) = i\int_0^\infty dn \exp(2\pi ikn)|C_E|^2/(\delta E_n + is). \tag{35}$$

Using Eq. (17), we arrive at an expression for $H_k(s)$ similar to (21):

$$H_k(s) = \frac{i\tau}{\pi}\exp[2ikS(E_0)] \times \int_{-\infty}^{\infty} \frac{(1+ix)^{k-1}}{(1-ix)^{k+1}} \exp\left(\frac{2ikTx}{\tau}\right)\frac{dx}{x+is\tau}. \tag{36}$$

We evaluate (36) for $\operatorname{Re} s > 0$. For $k \geqslant 1$ the integral in (36) must be closed by a contour in the upper half-plane.

Since the poles of the integrand at $x = -i$ and $x = -is\tau$ lie in the lower half-plane, we have $H_k(s) \equiv 0$ in this case ($\mathrm{Re}\, s > 0$ and $k \geqslant 1$).

For $k < 0$ the contour for the integral in (36) must be closed in the lower half-plane, which contains the pole at $x = -is\tau$ of the integrand. The pole provides the following contribution:

$$H_k(s) = 2\tau \exp[-2i|k|S(E_0) - 2|k|Ts](1-\tau s)^{|k|-1}/(1+\tau s)^{|k|+1}.$$

At $k = 0$ we easily find $H_0(s) = \tau/(1 + \tau s)$. Substituting this $P(s)$ into (35), we get a geometric series whose range of convergence is the half-plane $\mathrm{Re}\, s > 0$, where

$$P(s) = \tau\{\exp[2iS(E_0) + 2Ts] + 1\}\{(1+\tau s) \times \exp[2iS(E_0) + 2Ts] - (1-\tau s)\}^{-1}. \qquad (37)$$

This function $P(s)$ is analytic and has no singularities in the right half-plane $\mathrm{Re}\, s > 0$. All poles of this function, as can easily be verified directly, lie on the imaginary axis $\mathrm{Re}\, s = 0$.

For $K(t,\sigma)$ of (25) we introduce the Laplace transform, which is similar to (35) and (36):

$$K_1(s,\sigma) = \sum_{k=-\infty}^{\infty} K_{1k}(s,\sigma), \quad K_{1k}(s,\sigma) = i\frac{\tau}{\pi}\exp[2ikS(E_0)] \cdot \times \int_{-\infty}^{\infty} \exp\left(\frac{2ikTx}{\tau}\right) \frac{dx}{x + is\tau} \frac{(1+ix)^k}{(1-ix)^k} (1 + \sigma + x^2)^{-1}. \qquad (38)$$

As $P(s)$, the function $K_1(s,\sigma)$ is analytic in the half-plane $\mathrm{Re}\, s > s_0 = 0$. For $k \geqslant 0$, the integration contour in (38) must be closed in the upper half-plane, with

$$K_{1k}(s,\sigma) = \frac{\tau}{\rho}(s\tau + \rho)^{-1}\left(\frac{1-\rho}{1+\rho}\right)^k \exp\left[2ikS(E_0) - 2k\rho\frac{T}{\tau}\right]. \qquad (39)$$

For $k < 0$ we must close the contour in the lower half-plane, with

$$K_{1k}(s,\sigma) = \frac{\tau}{\rho}(s\tau - \rho)^{-1}\left(\frac{1-\rho}{1+\rho}\right)^{|k|} \exp\left[-2i|k|S(E_0) - 2|k|\rho\frac{T}{\tau}\right] - \frac{2\tau}{(s\tau)^2 - \rho^2}\left(\frac{1-s\tau}{1+s\tau}\right)^{|k|} \exp[-2i|k|S(E_0) - 2|k|Ts]. \qquad (40)$$

Substituting (39) and (40) into (38) and summing the geometric progression obtained, we arrive at the final result

$$K_1(s,\sigma) = \frac{\tau}{\rho(s\tau+\rho)}\left\{1 - \frac{1-\rho}{1+\rho}\exp\left[2iS(E_0) - 2\rho\frac{T}{\tau}\right]\right\}^{-1} + \frac{\tau}{\rho(s\tau-\rho)}\left\{1 - \frac{1-\rho}{1+\rho}\exp\left[-2iS(E_0) - 2\rho\frac{T}{\tau}\right]\right\}^{-1} - \frac{\tau}{\rho(s\tau+\rho)} - \frac{2\tau}{(s\tau)^2 - \rho^2}\left\{1 - \frac{1-s\tau}{1+s\tau} \times \exp[-2iS(E_0) - 2Ts]\right\}^{-1}. \qquad (41)$$

At $\sigma = 0$ we have $\rho = 1$ and $K_1(s,\sigma)$ of (41) is identically equal to $P(s)$ of (37). According to (26), the image function $K(s)$ of the correlation function (16) is equal to the difference between $K_1(s,0) = P(s)$ and $K_1(s,\sigma)$ of (41).

6. THE ASYMPTOTIC BEHAVIOR OF THE PROBABILITY AMPLITUDE AND CORRELATION FUNCTIONS AND THE POSSIBILITY OF A STOCHASTIC REGIME IN THE DECAY OF A QUASISTATIONARY STATE

Let us study the decay of a quasistationary state in a two-well potential (Fig. 1). Initially the particle is in the left well with an energy E_0. The decay of the quasistationary state is determined by two characteristic times τ and T, with

$$\tau^{-1} = (4\pi e^{2D}/\omega_0)^{-1}$$

the width of level E_0 and T^{-1} the level separation in the right well.

At times t satisfying the condition $0 < t < 2T$ the probability amplitude $p(t)$ for the transition from level E_0 to the states of a two-well potential behaves according to an exponential law in accordance with the Gamow theory:

$$p(t) = \exp(-t/\tau),$$

where τ contains no characteristics of the right well, into which states the quasistationary state decays. Starting at $t = 2T$, the period of motion in the right well, the particle begins to feel the finiteness of the right well, and in the time interval from $2T < t < 4T$ there appears, in addition to the exponential factor corresponding to the Gamow decay of the quasistationary state, a second term also describing the decay of the quasistationary state but with an exponential function measured from the moment $t = 2T$ [Eq. (31)]. This exponential function already contains the characteristics of the right well in the form of the oscillating factor $\exp[2iS(E_0)]$.

The decay phenomenon manifests itself most vividly in the limit $\tau \ll 2T$. In this case the exponential function (24) corresponding to the decay of the state falls off completely and there remains only the second exponential function (31) measured from the time $t = 2T$. At times t close to $2T$ the value of this exponential function is of order unity; then it decays with a time constant τ, just as it did in the interval $0 < t < 2T$, and, hence, at t close to $4T$ constitutes and exponentially small quantity. At $t = 4T$ there emerges another exponential function with an amplitude close to unity, which then also falls off with a time constant τ. In general, at $t = 2kT$ (with k an integer) $p(t)$ is of order unity and then decays with a time constant τ in the time interval $2kT < t < 2(k+1)T$. Thus, $p(t)$ is quasiperiodic with sharp maxima at $t = 2kT$ followed by an exponential decay with a time constant τ. Here the peak amplitude before the multiple harmonics depends on the size of the right well owing to the oscillating factor $\exp\{2ikS(E_0)\}$. For $\tau \gg 2T$, the decay has no time to manifest itself fully at times $t \sim 2T$ and the probability amplitude $p(t)$ has no sharp minima at points $t = 2kT$ and is always of order unity. This mode corresponds to Fock–Krylov oscillations.

The asymptotic behavior of the probability amplitude can be clarified by analyzing the temporal correlation function $K(t)$ of Eq. (14). Within a single period $2T$ of motion in the right well the correlation function is not only characterized by the time constant τ, as in $p(t)$, but contains a second time constant τ/ρ, with $\rho = \sqrt{1+\sigma}$ and $\sigma = \tau/T$ [see Eq.

(23)]. For $\tau \ll T$ these two constants coincide, but for $\tau \gg T$ the value of $\tau/\rho = \sqrt{\tau T}$ differs substantially from τ and gives the time required for coherent motion to develop in the system, and for an infinite number of wells describes the time of formation of band motion. Here $\tau/\rho \ll \tau$, that is, in this limiting case the decay time τ is much longer than the time for coherent motion to set in. While the time motion τ gives the correlation-decay time, τ/ρ characterizes not only the decay but the strengthening of correlations with the passage of time. Hence, the correlation function reaches its minimum in the middle of the period of motion in the right well, at $t = T$. The same situation occurs in the middle of multiples of the period [see Eq. (29)].

An important quantity specifying the behavior of the probability amplitude $p(t)$ at infinity is the value of the correlation function $K(t)$ at $t = 0$. This quantity $K(0)$ determines the fraction of time spent by the system in the asymptotic region in the initial state. Generally, this time experiences rapid oscillations related to the size of the right well [see Eq. (23')]. The most graphic answer is presented in the limiting cases of $\tau \gg T$ and $\tau \ll T$. In the first case $K(0) = 1$, decay is practically absent, and the system is characterized by oscillatory dynamics, in accordance with the Fock–Krylov oscillation theorem.[5] In the second limiting case, $K(0) = \tau/2T$ is the ratio of the width τ^{-1} of the decaying level to the level separation $1/2T$ in the right well. Thus, $K(0)$ is inversely proportional to number of levels lying within the level width τ^{-1} (Fig. 2), and in the asymptotic region it is equally probable for the system to stay on one of these levels. In both limiting cases, $K(0)$ does not experience oscillations as $S(E_0)$ does. Oscillations occur only at $\sigma = \tau/2T \sim 1$, when the period of motion in the right well $2T$, is on the order of the half-decay time τ.

A more subtle characteristic of the asymptotic behavior of the system is the behavior of higher-order correlation functions, such as

$$K^{(2)}(t) = \lim_{t' \to \infty} \frac{1}{t'} \int_0^{t'} K^{(1)}(t'') K^{(1)*}(t''+t)\, dt'', \qquad (42)$$

where $K^{(1)}(t)$ is the first-order correlation function $K(t)$. For the second-order correlation function there exists a formula similar to (14):

$$K^{(2)}(t) = \sum |C_E|^8 \exp(it\delta E), \quad \delta E = E - E_0. \qquad (43)$$

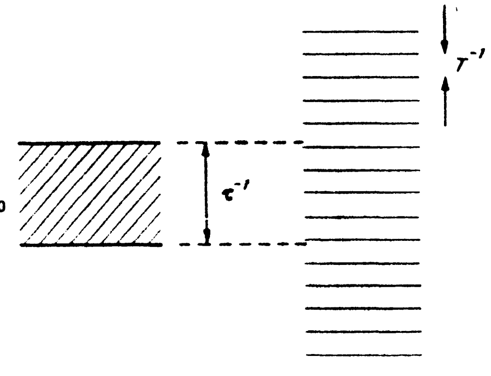

FIG. 2. The metastable state spreads out over the states of well *2* (Fig. 1) that lie in a band τ^{-1} wide.

This correlation function can be expressed in terms of σ-derivatives of $K^{(1)}(t)$ of (23):

$$K^{(2)}(0) = \frac{\sigma^3}{2} \frac{d^2}{d\sigma^2} \Big[1 - \frac{1}{\rho} + \frac{2}{\rho} \times \mathrm{Re} \Big\{ 1 + \frac{(1+\rho)^2}{\sigma} \exp\Big[2\rho \frac{T}{\tau} - 2iS(E_0) \Big] \Big\}^{-1} \Big], \qquad (44)$$

where in finding σ-derivatives one must bear in mind that $\rho = \sqrt{1+\sigma}$. If we let σ go to infinity in (44), we get $K^{(2)}(0) = 1/2$, which refines the limit that formula (23") yields, $K(0) = 1$, and shows that this limiting case corresponds to the Fock–Krylov mode when only one level is located within a band of width τ^{-1}. If we let σ go to zero, we have $K^{(2)}(0) = (\sigma/2)^3$, which is related to the expression $K(0) = \tau/2T$ in this limit in the following manner: according to (42), this is the square of $K(0)$ divided by $2T/\tau$, which is the number of levels lying within a band τ^{-1} wide (Fig. 2), that is, $K^{(2)}(0) = (\tau/T)^3 = K^3(0)$. Other higher-order correlation functions can be analyzed similarly.

The system most suited for studying experimentally the laws established above is a SQUID. In most systems where the decay of a quasistationary state can be observed the tunneling is not one-dimensional (for instance, the decay of a radioactive nucleus takes place in three-dimensional space). In view of this inhomogeneity, the above laws are averaged over the size of the second well because the well has different dimensions along different directions. For example, averaging over the action S of the correlation function $K(t)$ of (23) yields

$$\langle K(t) \rangle_S = \exp(-t/\tau) - \rho^{-1} \exp(-\rho t/\tau), \quad 0 < t < 2T. \qquad (45)$$

Averaging over S means averaging over the size of the right well or over the spread of the external field and leads to exclusion of all dimensional effects in the problem. It is most convenient to find the average value of the probability amplitude for all values of t by employing the transform $P(s)$ of (37):

$$\langle P(s) \rangle_S = (s + 1/\tau)^{-1}.$$

Going over to original functions, we arrive at the following expression for the probability amplitude averaged over S:

$$\langle p(t) \rangle_S = \exp(-t/\tau), \qquad (46)$$

which corresponds to the Gamow decay law but, in contrast to (24), operates not only in the interval $0 < t < 2T$ but for all values of t. Thus, eliminating dimensional effects leads at once to an exponential decay law over the entire time range.

Averaging of $K_1(s,\sigma)$ over S yields

$$\langle K_1(s, \sigma) \rangle = (\tau/\rho)(s\tau + \rho)^{-1}.$$

Using (26), we obtain formula (45) for the averaged correlation function, $\langle K(t) \rangle_s$, valid for all times t and not only for $0 < t < 2T$. Thus, the averaged value of $K(t)$ is nonzero for all values of t. As the size of the right well increases without limit, $\rho = \sqrt{1 + \tau/T}$ tends to unity. In this case the period of motion T becomes infinite and $\langle K(t) \rangle_s$ vanishes, since this limit corresponds to the transition to the continuous spectrum, which means that all correlations cease. In the other limiting case, where the right well shrinks ($\sigma \to \infty$ and

$\rho \to \infty$), we have $\langle K(t) \rangle_s = \langle p(t) \rangle_s$. This occurs in the Fock–Krylov oscillation mode, when there is no real dynamic decay of the initial state. The above reasoning suggests that in three-dimensional space all dependence on S becomes smeared in the lowest-order approximation and the Gamow law (43) is valid for the averaged probability amplitude $\langle p(t) \rangle_s$ at all times t even in a finite system. Here the spatially averaged correlations behave according to the law (45).

Only a one-dimensional system is suitable for observing dimensional effects as functions of S, and a SQUID is just such a system because its parameters that determine the potential field in the decay of a quasistationary state can be easily controlled. This usually presupposes the use of the modulation method.[6]

Let us consider two limiting cases that emerge in the modulation method. In the mode where the modulation of a parameter ΔS is much smaller than unity, the method makes it possible to measure the finer dynamic characteristics of the system, say, the derivatives of $p(t)$ and $K(t)$. The opposite limiting case of $\Delta S \gg 1$ involves measuring averaged characteristics, for instance, (45) and (46), in the same way as during self-averaging in the three-dimensional case. If in this method, for $\Delta S \gg 1$, one measures the S-derivatives of $p(t)$ and $K(t)$, the result is a random quantity. The reason is that for $\Delta S \gg 1$ modulation means averaging not over an integer number of periods of the function $\exp(2iS)$ but over an interval that apart from an integer number of periods contains a noninteger part random in length. Since the average over an integer number of periods is independent of S, its derivative with respect to S vanishes and there remains only the derivative of the random quantity. This may be the reason why for certain values of the parameters of the system randomization is observed in experiments involving SQUIDs.[6]

Another source of randomization may be the internal properties of the system, properties not related to modulation. As is well known,[13] even in classical dynamics there is a dimensionless critical parameter K that determines whether the motion of the system is stable ($K < 1$) or whether randomization develops ($K > 1$). In this case tunneling between the two wells becomes possible, and the effect of the modulating field on the system makes the system quite ripe for the stochastic mode to appear in the classical motion in two wells, the transition between which is a purely quantum nature, that is, quantum tunneling. The resulting situation is similar to the vibrations of a nonlinear pendulum in the neighborhood of the separatrix. Hence, a purely classical type of stochastization is possible in a two-mode system with the modes coupled by quantum transitions.

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