Use of the renormalization-group method to describe intermittency and to derive the corrections to the exponents in Kolmogorov turbulence theory

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The renormalization-group method is used to calculate corrections to the Kolmogorov exponents for intermittency effects. These corrections incorporate the dependence of the spectrum on the external scale of the turbulence. An ir cutoff parameter serves as this external scale. A renormalization-group function is derived in the two-loop approximation. The form of this function implies the existence of a nontrivial fixed point which is stable in the uv region. The corrections to the spectrum are manifested as an anomalous dimension. The anomalous dimensionality of the correlation function of the effective random forces is predominant. The anomalous dimension of the effective viscosity is smaller by a factor of 3. The value $\mu = 0.153$ is found for the exponent which characterizes the dissipation spectrum. This value agrees with experimental data.

1. INTRODUCTION

The basic features of active turbulence are described by Kolmogorov's semiphenomenological theory, which is based on the assumption that the spectral energy flux remains constant in the inertial interval. However, intermittency effects and the associated fluctuations of the spectral flux give rise to small corrections to the Kolmogorov exponents, and they demote the constants characterizing the active turbulence from their universal status. Methods based solely on self-similarity considerations are incapable of deriving these corrections.

To calculate these corrections, it is necessary to describe the turbulence by a hydrodynamic method based on a study of statistical solutions of the Navier-Stokes equations. If we look at a turbulent fluid as a dynamic system, we see that it is characterized by a very large number of excited modes and a strong intermode interaction. The situation is quite reminiscent of that in quantum field theory. In methods of quantum field theory, the Kolmogorov spectrum can be derived under the assumption that the correlation function of the effective random forces, which determines the pumping of energy to a given mode by nonlinear interactions with other modes, is

$$\langle f_i(\mathbf{k},\omega) f_j(\mathbf{k}',\omega')\rangle = D_{ij}(\mathbf{k})(2\pi)^d \delta(\mathbf{k}+\mathbf{k}')(2\pi)\delta(\omega+\omega'), \tag{1.1}$$

where

$$D_{ij}(\mathbf{k}) = \delta_{ij} D_0 k^{-y}$$

and d is the dimensionality of the space.

With $y = d$, the constant D_0 has the dimensionality of an energy dissipation rate. According to Kolmogorov's ideas, this is the only parameter of importance in determining the universal turbulence regime in the inertial interval.[1] In this sense we will regard the value $y = d$ as corresponding to the "real" theory.[2] The renormalization-group method[3,4] can be used to derive an expression corresponding to (1.1) for the turbulent viscosity σ which determines the decay rate of the response of turbulent fluctuations to an instantaneous perturbation. According to Refs. 5–7 we have

$$\sigma(k) = A_0 D_0^{1/3} k^{-2/3}, \tag{1.2}$$

where A_0 is a constant related to the Kolmogorov constant.[6,8] Actually, the results in (1.1) and (1.2) are a consequence of scale invariance, i.e., of the absence of a characteristic parameter with the dimensionality of a length in the inertial interval. It was shown in Refs. 9 and 10 that scale-invariant solutions with Kolmogorov exponents would be compatible with the Dyson equations if divergences did not appear on the diagrams corresponding to Kolmogorov solutions at either small wave numbers (an ir divergence) or large wave numbers (a uv divergence). Since the ir divergence turns out to be stronger than logarithmic, a divergence of this type cannot be removed by a renormalization procedure. The presence of an ir divergence means that there is an additional important parameter with the dimensionality of a length, L (the external scale of the turbulence), in the inertial interval. The dependence on this parameter as $L \to \infty$ is nonanalytic. In this case we are dealing with an incomplete self-similarity (a self-similarity of the second kind in the terminology of Ref. 11).

The renormalization-group method can be used to calculate the exponents of the asymptotic behavior in a situation with an incomplete self-similarity. In particular, Goldenfeld *et al.*[12] have shown how this method can be used to find the exponent of incomplete self-similarity in a nonlinear-diffusion problem. In the case of turbulence, however, the problem is more complex, since the divergences which arise in the use of a perturbation theory are power-law rather than logarithmic. In a case of this sort, it is customary to use an ε expansion. In that method, the calculations are carried out in a space with a dimensionality which differs from that of real physical space. In a space of this dimensionality, the uv divergences are logarithmic, and the theory is renormalizable.

The presence of a logarithmic singularity leads to the appearance of pole singularities along ε in the coefficients of the perturbation-theory series. The transition to real space is made by analytically continuing this pole along ε to the point corresponding to the dimensionality of the space. A similar procedure is used in the theory of critical phenomena to find

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critical exponents and in turbulence theory to determine the exponents of the power-law behavior of the spectrum in the ir region. It is asserted in Ref. 13 that the use of a continuation along ε in the renormalization-group method corresponds to incorporating spatially localized wave numbers of intermode interactions which are responsible for cascade interaction-transport processes. It also filters out nonlocal wave numbers.

The use of the renormalization-group method to describe the behavior of a turbulent fluid in the ir limit actually reduces to calculating the scale dimensionalities of physical quantities.[2] This simply duplicates the results of the Kolmogorov theory, although in several cases the renormalization-group method also leads to several nontrivial results. For example, it shows how a solution goes into the self-similar regime[7] and reveals numerical amplitide factors.[5,6,8]

The use of the renormalization-group method in the uv limit, in which the average effect of the large-scale modes on the behavior of the small-scale modes is examined in the inertial interval $L \gg k^{-1}$, requires a preliminary solution of the problem of eliminating the ir divergences. According to Kolmogorov's ideas,[14] the ir divergences are associated with an incorrect treatment of the interaction with large-scale modes (soft quanta) in the use of low-order perturbation theories. The situation is reminiscent of that in QED.[15] In electrodynamics, the problem is solved by restructuring the perturbation-theory series in such a way that the interaction with soft quanta is taken into account exactly. For example, one could use the Bloch-Nordsieck model here. In hydrodynamics, the summed interaction with soft quanta reduces to a transport and a Doppler frequency shift. These effects do not alter the characteristics of the spectrum in the inertial interval.

We see thus that when the renormalization-group method is used in the uv limit there is the problem of first distinguishing the weak dynamic interactions which shape the spectrum and which occur against the background of the strong kinematic transport effects that are eliminated upon the transformation to a moving coordinate system. In the few studies which have used the renormalization-group method in the uv limit,[16,17] essentially the only result has been to show that the strongest ir divergences reduce to a transport. After the transport is eliminated, the external scale of the turbulence drops out of the picture, and scale-invariance arguments make it possible to find the exponents which characterize the spectrum. A shortcoming of this analysis is that after the transport effects have been eliminated there is still the weak logarithmic dependence on the external scale (a logarithmic ir divergence); this dependence has not been studied. According to Wilson,[18] however, the presence of a logarithmic divergence is evidence that the intermode interactions are localized in the space of scales (or wave numbers), and it is evidence of a cascade mechanism for the interactions between modes with very different scales. The dependence on the external scale L "penetrates" via the cascade mechanism into the uv part of the spectrum, corresponding to the inertial interval. This dependence can be determined by summing an infinite subsequence of the perturbation-theory series; this is done by the renormalization-group method. In turbulence theory, the renormalization-group method is thus a satisfactory method of describing this problem.

2. STATEMENT OF THE PROBLEM

We consider the model of a viscous, incompressible fluid described by the Navier-Stokes equations in an external random force which is a Gaussian white noise. We treat the characteristics of the hydrodynamic field—the pressure p and the velocity components v_i at the point $1 = \{\mathbf{r}_1, t_1\}$—as components of a $(d+1)$-dimensional vector in a space of d dimensions in accordance with the definition

$$\psi_\alpha(1)=\{\psi_0(1),\ \psi_i(1)\}=\{p(\mathbf{r}_1, t_1),\ v_i(\mathbf{r}_1, t_1)\},$$
$$\alpha=0,1,\ldots,d,\quad i=1,\ldots,d.$$

In the "field-doubling" formalism,[19] this system is specified by the action[8]

$$S[\psi,\hat\psi]=-\hat\psi_\alpha(1)L_{\alpha\beta}(12)\psi_\beta(2)+{}^1/_2 i\hat\psi_\alpha(1)D_{\alpha\beta}(12)\hat\psi_\beta(2)-\tfrac{1}{2}\lambda_0\hat\psi_\alpha(1)V_{\alpha\beta\gamma}(123)\psi_\beta(2)\psi_\gamma(3),\qquad(2.1)$$

where

$$L_{\alpha\beta}(12)=\begin{bmatrix}0 & \partial_j^{(1)}\\ \partial_i^{(1)} & (\partial_t^{(1)}-\nu_0\Delta)\delta_{ij}\end{bmatrix}\delta(1-2),$$
$$D_{ij}(12)=\delta_{ij}D(\mathbf{r}_1-\mathbf{r}_2)\delta(t_1-t_2),\qquad(2.2)$$
$$V_{ijk}(123)=-[\delta_{ij}\partial_k^{(2)}+\delta_{ik}\partial_j^{(3)}]\delta(1-2)\delta(1-3),$$

ν_0 is the molecular viscosity, and λ_0 is a formal expansion parameter, which should be set equal to one in the final result. It follows from the Dyson equations that the effect of the nonlinear intermode interactions reduces to the replacement of the correlation function of the random external forces, localized in the interval containing the energy, by the correlation function of effective random forces (1.1) and the replacement of the molecular viscosity $\sigma = \nu_0 k^2$ by the turbulent viscosity (1.2). These functions will be used as a zeroth approximation in deriving the corrections to the Kolmogorov theory for the effect of the external scale of the turbulence on the characteristics of the turbulent field in the inertial interval. In a sense, the Kolmogorov theory should be thought of as an analog of Landau's mean field theory in the physics of critical phenomena, while the corrections to the mean field theory incorporate the fluctuations in the governing parameters such as A_0 and D_0. Incorporating these fluctuations reduces to replacing the parameters A_0 and D_0 by certain functions of the wave number k and of the external scale of the turbulence, L. The deviation from analyticity with respect to L, which stems from the ir divergences, is confined to these functions.

As zeroth approximations of the renormalized perturbation theory for the correlation function of the effective random forces and the turbulent viscosity we adopt the expressions

$$D(k)=Dk^{-d},\quad \sigma(k)=Ak^{2/3},\qquad(2.3)$$

where the renormalized values of the parameters, D and A, are related by $D = Z_1 D_0$ and $A = Z_2 A_0 D_0^{1/3}$ to the original values D_0 and A_0. The renormalization constants Z_1 and Z_2 are found from the requirement that the corrections to the renormalized parameter values, which arise in various orders of perturbation theory, cancel out with the contributions of the counterterms introduced in the renormalization procedures at the normalization point $k = \eta, \omega = 0$. In addition to the renormalization of these parameters, one should also carry out a frequency renormalization[8] corresponding to the multiplicative renormalization of the amplitudes of the Green's functions, vertices, and correlation functions of random forces which is allowed by the system of Dyson equations (it follows from the Galilean invariance of the hydrodynamic system that the multiplicative-transformation group of the Dyson equations is a one-parameter group[20]).

After transport effects are eliminated, the ir divergences which arise in the perturbation theory are logarithmic and can be removed by a renormalization. Consequently, the singular dependence on the external parameter L is confined to the renormalization constants. In the limit $kL\to\infty$ (which corresponds to the inertial interval) the physical quantities are independent of L. In the renormalized perturbation theory, however, the dependence on the external scale is replaced by a dependence on the normalization momentum η; this dependence is determined by the renormalization-group method.

In Fourier space, the renormalized action is specified by Fourier-transformed relation (2.1), which contains $\sigma(k)$ in place of $\nu_0 k^2$, which has $D(k)$ as in (2.3), and which has $\lambda = Z_3^{-1}\lambda_0$, where Z_3 is the renormalization constant of the field amplitude $\hat{\psi}$. The counterterm which cancels the renormalization effect enters the perturbation in the form

$$\delta S[\psi,\hat{\psi}]=-\int\frac{d\mathbf{k}}{(2\pi)^d}\frac{d\omega}{(2\pi)}\,\delta_{ij}\hat{\psi}_i(-\mathbf{k},-\omega)$$
$$\times[-i\omega(Z_3^{-1}-1)\psi_j(\mathbf{k},\omega)$$
$$+(Z_2^{-1}-1)Z_3^{-1}\sigma(\mathbf{k})\psi_j(\mathbf{k},\omega)+{}^1/_2 iD(\mathbf{k},\omega)$$
$$\times(Z_1^{-1}-1)Z_3^{-2}\hat{\psi}_j(\mathbf{k},\omega)]. \qquad (2.4)$$

3. CONDITIONS FOR RENORMALIZATION INVARIANCE

To calculate the corrections to the Kolmogorov theory we seek a solution in the form corresponding to the replacement of the renormalized parameters A and D by some functions which depend on the wave number k; on the renormalized parameter values A, D, and λ; and on the normalization momentum η. Taking dimensionality considerations into account, along with the results of an analysis of the low orders of perturbation theory, we can write these functions in the form

$$D(k,A,D,\lambda;\eta)=Df_1(k/\eta,h),$$
$$A(k,A,D,\lambda;\eta)=Af_2(k/\eta,h), \qquad (3.1)$$
$$h=\lambda^2D/A^3,\quad f_{1,2}(1,h)=1.$$

The condition for renormalization invariance is that the result of the calculation of the Green's function and of the binary correlation function of the velocity be independent of the choice of the normalization point η for the values of the renormalized parameters corresponding to this point. These conditions can be written in the form[8]

$$Df_1(k/\eta,h)=D_1f_3^2(\eta_1/\eta,h)f_1(k/\eta_1,h_1),$$
$$Af_2(k/\eta,h)=f_3(\eta_1/\eta,h)A_1f_2(k/\eta_1,h_1). \qquad (3.2)$$

In (3.2) we have introduced the new function $f_3(\eta_1/\eta,h)$, which determines the change in the renormalization constant of the field amplitude $\hat{\psi}$ upon a change in the normalization point:

$$f_3(\eta_1/\eta,h)=Z_3(A_1,D_1,\lambda_1;\eta_1,L)/Z_3(A,D,\lambda;\eta,L). \qquad (3.3)$$

It follows from definition (3.3) that the function f_3 satisfies the group composition law

$$f_3(\eta_2/\eta,h)=f_3(\eta_2/\eta_1,h_1)f_3(\eta_1/\eta,h). \qquad (3.4)$$

Using the relation $\lambda^2=f_3^2(\eta_1/\eta,h)\lambda_1^2$, we find from (3.2) and (3.4)

$$\frac{hf_1(k/\eta,h)}{f_2^3(k/\eta,h)f_3(k/\eta,h)}=\frac{h_1f_1(k/\eta_1,h_1)}{f_2^3(k/\eta_1,h_1)f_3(k/\eta_1,h_1)}. \qquad (3.5)$$

The function

$$\bar{h}(x,h)=hf_1(x,h)/f_2^3(x,h)f_3(x,h) \qquad (3.6)$$

is thus an invariant of the renormalization-group transformation. It is the actual parameter of the series expansion of the renormalized perturbation theory. It is analogous to the invariant charge in quantum field theory.[3,4] Using the condition $\bar{h}(1,h)=h$, we see that the invariant charge satisfies the renormalization-group functional equation

$$\bar{h}(x,h)=\bar{h}(x/t,\bar{h}(t,h)). \qquad (3.7)$$

Working from (3.2) and (3.4), and eliminating the parameters A and D, we find functional equations for the functions f_i:

$$f_i(x,h)=f_i(t,h)f_i(x/t,\bar{h}(t,h)). \qquad (3.8)$$

4. THE RENORMALIZATION-GROUP METHOD

In accordance with the renormalization-group method,[3] we switch from functional equations (3.7) and (3.8) to the differential equations

$$\left[-x\frac{\partial}{\partial x}+\beta(h)\frac{\partial}{\partial h}\right]\bar{h}(x,h)=0,$$
$$\left[-x\frac{\partial}{\partial x}+\beta(h)\frac{\partial}{\partial h}\right]f_i(x,h)=-\gamma_i(h)f_i(x,h), \qquad (4.1)$$
$$\beta(h)=\left.\frac{\partial\bar{h}(x,h)}{\partial x}\right|_{x=1},\qquad \gamma_i(h)=\left.\frac{\partial f_i(x,h)}{\partial x}\right|_{x=1}.$$

We see from (4.1) that in order to find the functions $f_i(x,h)$ we need to first specify the so-called renormalization-group functions $\gamma_i(h)$ and $\beta(h)=h(\gamma_1-3\gamma_2-\gamma_3)$, which are determined by the behavior of the functions $f_i(x,h)$ near the normalization point $x=1$. The solution of the equations then makes it possible to find these functions over their entire range, in the way that finite-transformation operators are determined in the theory of continuous Lie groups on the basis of the known infinitesimal-transformation operators. In the renormalization-group method, the renormalization-group functions are calculated by a renormalized perturbation theory. A hydrodynamic perturbation theory for the correlation functions and Green's functions is customarily constructed through a multiple iteration of the system of Navier-Stokes equations, followed by a multiplication of the resulting series and by a term-by-term averaging of these

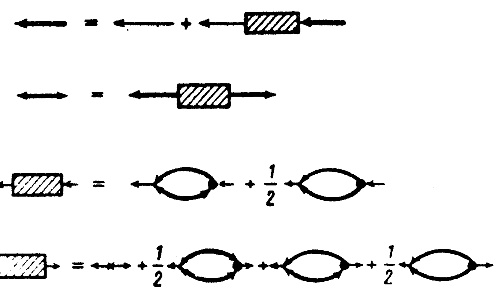

FIG. 1.

series in a random external force.[21] Such a procedure is exceedingly complicated and laborious. A more efficient method is to use Dyson's equations, which can be constructed outside a perturbation theory on the basis of the formalism of a characteristic (generating) functional.[19,22] These equations are shown in diagram form in Fig. 1 (Ref. 22). Here the lines with a single arrowhead correspond to the Green's function of the unperturbed system,

$$G_{ij}^{(0)}(\mathbf{k},\omega)=P_{ij}(\mathbf{k})\left[-i\omega+\sigma(k)\right]^{-1}$$

(a light line) or the complete Green's function

$$G_{ij}(\mathbf{k},\omega)=P_{ij}(\mathbf{k})\left[-i\omega+\sigma(k)-\Sigma(\mathbf{k},\omega)\right]^{-1}$$

(a heavy line). Here $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ is the transverse-projection operator. A heavy line with two arrowheads corresponds to the binary correlation function of the velocities, $B_{ij}(\mathbf{k},\omega)$, while a light line corresponds to the unperturbed correlation function

$$B_{ij}^{(0)}=P_{ij}(\mathbf{k})D(k)\left[\omega^2+\sigma^2(k)\right]^{-1}.$$

A point with two incoming lines and one outgoing line corresponds to the simple vertex $V_{ijk}(\mathbf{k})$. The cross with two outgoing lines corresponds $D(k)$, the functions $\sigma(k)$ and $D(k)$ are defined by Eqs. (2.3). A rectangle with one incoming and one outgoing line corresponds to the self-energy operator $\Sigma(\mathbf{k},\omega)$. A rectangle with two outgoing lines corresponds to the correlation function of the effective random forces, $\tilde{D}(\mathbf{k},\omega)$. The exact Dyson diagram equations contain complete vertices of three types, corresponding to the creation of one, two, and three quanta. Martin *et al.*[19] pointed out the need to consider the vertices of all three types. The vertices of the second and third types arise only in higher orders of the perturbation theory.

The effective turbulent viscosity $\tilde{\sigma}(k) = \sigma(k) \times f_2(k/\eta, h)$ determines the position of the pole of the Green's function in the complex frequency plane. The residue at this pole determines the renormalization coefficient for the field amplitude $\hat{\psi}$, in such a way that we have

$$f_3(k/\eta,\ h)=1+(\partial\Sigma(k,\ \omega)/i\partial\omega)_{\omega=-i\tilde{\sigma}(k)}. \qquad (4.2)$$

As was shown in Refs. 16 and 17, in low orders of perturbation theory the result of a calculation of each diagram is written as the sum of powers of the large parameter $(kL)^{2/3}$ and of logarithms of this parameter. The terms containing positive powers of kL, however, can be interpreted as representing the sequential absorption of quanta with zero frequencies and wave numbers (null quanta). The summation of an infinite subsequence of the perturbation-theory series with null quanta reduces to a transport.[16,17,23,24] This transport can be eliminated by transforming to the comoving coordinate system. Terms containing negative powers of this parameter vanish in the uv limit $(kL\to\infty)$. We are left with only the powers of $\ln(kL)$; they determine the local intermode interactions. After the renormalization procedure is carried out, $\ln(kL)$ is replaced by $\ln(k/\eta)$. Only the first powers of the logarithms contribute the renormalization-group function.

According to the Ward identity which follows from the requirement of Galilean invariance of the hydrodynamic system, the correction $(\delta\Gamma)$ to the vertex corresponding to the absorption of a null quantum is given by the exact expression[20]

$$\delta\Gamma_{ijk}(\mathbf{k},\omega\,|\,\mathbf{k},\omega:0,0)=\lambda k_k\frac{\partial\Sigma_{ij}(\mathbf{k},\omega)}{\partial\omega}. \qquad (4.3)$$

Since transport effects associated with the absorption of null quanta are eliminated by a transformation to the comoving coordinate system, the corrections to vertex (4.3) in this coordinate system vanish. It is then legitimate to ignore the frequency dependence of the eigenenergy operator. This procedure is equivalent to the hypothesis of a frozen turbulence, according to which the frequency dependence of the statistical moments of the turbulent fluctuations of the velocity stems from exclusively the transport of the spatially nonuniform velocity field.[1] It follows from (4.2) and (4.3) that effects of a renormalization of the field amplitude $\hat{\psi}$ and of the frequency can be ignored; i.e., we can set $f_3 \equiv 1$. In addition, instead of using the normalization conditions on the dispersion curve, $\omega = -i\tilde{\sigma}(k)$, which contain singularities, we can carry out a renormalization at $\omega = 0$. This justifies Kraichnan's determination of the turbulent viscosity in terms of the eigenenergy operator at a zero frequency[25] (see Ref. 26 for a discussion of these questions).

5. SOLUTION OF THE RENORMALIZATION-GROUP EQUATIONS

In contrast to the use of the renormalization-group method to describe turbulence in the ir limit, in which case the expansion of the renormalization-group function begins with the term proportional to εh (Refs. 2, 6–8, 24, 27, and 28), in the uv limit the expansion begins with the term proportional to h^2. In low-order perturbation theory there exists only a trivial fixed point for the renormalization-group function. The invariant charge disappears slowly in the limit (uv asymptotic freedom), and the correction to the Kolmogorov spectrum is logarithmic.[17] (In this case we are talking about the uv asymptotic freedom with respect to perturbations of the Kolmogorov regime by fluctuations of the spectral flux.) Flux fluctuations prevail over the entire inertial interval. They are seen as corrections to the Kolmogorov exponents for the spectrum. These corrections stem from the existence of a nontrivial fixed point. The dependence on the external scale L should be seen as an anomalous dimension[3,4] or a self-similarity dimension of the second kind.[11] The existence of a nontrivial fixed point $h^* > 0$ requires that in the representation

$$\beta(h)=ah^2+bh^3 \qquad (5.1)$$

for the renormalization-group function the parameters a and b differ in sign. The requirement of uv stability of the

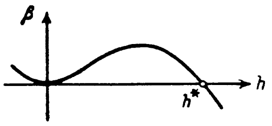

FIG. 2.

fixed point leads to the condition $a>0$ (Fig. 2). A calculation by perturbation theory shows that these conditions are satisfied.

The fixed point of the renormalization-group transformation, h^*, is found from the condition $\beta(h^*)=0$. According to (5.1), we then find $h^*=-a/b$. Near the fixed point we have

$$\beta(h)=bh^{*2}(h-h^*). \tag{5.2}$$

The large-x asymptotic solution of the renormalization-group differential equation for the invariant charge $\bar{h}(x,h)$ can be found by the standard methods.[3,4] This solution is given implicitly by

$$\frac{\bar{h}(x,h)-h^*}{h-h^*}=x^{bh^{*2}}. \tag{5.3}$$

According to Ref. 3, the solution of the renormalization-group differential equation (5.1) for $f_i(x,h)$ is given by a formula of the type

$$f_i(x,h)=\exp\left\{\int_h^{\bar{h}(x,h)} dh'\,\frac{\gamma_i(h')}{\beta(h')}\right\}$$

$$\approx\exp\left\{\frac{\gamma_i(h^*)}{bh^{*2}}\int_h^{\bar{h}(x,h)}\frac{dh'}{h'-h^*}\right\}. \tag{5.4}$$

Carrying out the integration in (5.4), and using (5.3), we find the following result for the asymptotic behavior at large x:

$$f_i(x,h)=x^{\delta_i},\quad \delta_i=(a_i+b_ih^*)h^*. \tag{5.5}$$

6. CALCULATION OF THE PARAMETERS OF THE RENORMALIZATION-GROUP FUNCTION

The functions f_1 and f_2 are given by

$$f_1(x,h)=\tilde{D}^{(R)}(k,0)/D(k),\quad f_2(x,h)=1-\Sigma^{(R)}(k,0)/\sigma(k), \tag{6.1}$$

where the renormalized self-energy operator $\Sigma^{(R)}$ corresponds to the rectangle with incoming and outgoing arrows in Fig. 1, while the correlation function of the effective random forces, $\tilde{D}^{(R)}$, corresponds to the rectangle with two outgoing arrows. We should also add to these diagrams the contribution of the counterterms which satisfy the normalization conditions.

To illustrate the method for calculating the parameters a_i and b_i, we use the example of the diagrams of second-order perturbation theory. According to Fig. 1, the self-energy operator in this approximation is given by

$$\Sigma_{ij}^{(2)}(\mathbf{k},\omega)=\lambda^2V_{inm}(\mathbf{k})$$

$$\times\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{d\Omega}{2\pi}G_{nn'}(\mathbf{k}-\mathbf{q},\omega-\Omega)B_{mm'}(\mathbf{q},\Omega)V_{n'm'j}(\mathbf{k}-\mathbf{q})$$

$$=\lambda^2\int\frac{d\mathbf{q}}{(2\pi)^d}b_{ij}(\mathbf{k},\mathbf{q})D(q)$$

$$\times\int\frac{d\Omega}{2\pi}\frac{1}{-i(\omega-\Omega)+\sigma(\mathbf{k}-\mathbf{q})}\frac{1}{\Omega^2+\sigma^2(\mathbf{q})},$$

$$=\lambda^2\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{b_{ij}(\mathbf{k},\mathbf{q})D(q)}{2\sigma(\mathbf{q})[-i\omega+\sigma(\mathbf{k}-\mathbf{q})+\sigma(\mathbf{q})]}, \tag{6.2}$$

$$b_{ij}(\mathbf{k},\mathbf{q})=V_{inm}(\mathbf{k})P_{nn'}(\mathbf{k}-\mathbf{q})P_{mm'}(\mathbf{q})V_{n'm'j}(\mathbf{k}-\mathbf{q})$$

(the range of the integration over $\mathbf{q}$ is limited by the condition $q\geqslant m=1/L$).

From (6.2) with $D(q)=q^{-d}$ we conclude that the integral is dominated by small values $q\simeq m\ll k$. We can thus set $\mathbf{k}-\mathbf{q}\simeq\mathbf{k}$ and carry out the integration over the directions of the vector $\mathbf{q}$. As a result we find, for $\sigma(k)\gg\sigma(m)$,

$$\Sigma_{ij}^{(2)}(\mathbf{k},0)=-\frac{3}{2}\varkappa\lambda^2Dk^2\int_{\sigma(m)}^{\infty}\frac{d\sigma}{\sigma^2}\frac{1}{\sigma(k)+\sigma}=-\frac{3}{2}\frac{\varkappa\lambda^2Dk^2}{\sigma(k)\sigma(m)}+\varkappa h\sigma(k)\ln\left[\frac{\sigma(k)}{\sigma(m)}\right],$$

$$\sigma=Aq^{2/3},\quad \varkappa=\frac{d-1}{2d}\frac{s_d}{(2\pi)^d},\quad s_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{6.3}$$

The first term in (6.3) contains a power-law singularity as $m\to0$. According to the discussion above, this term is the first term of an infinite subsequence whose sum reduces to a Doppler frequency shift and can therefore be discarded. The second term, after the addition of the counterterm which leads to the satisfaction of the normalization conditions, leads to a term of the type $\varkappa h\sigma(k)\ln(k/\eta)$. Using (6.2), we find $a_2=-\varkappa$. Examining the second-order diagram in a corresponding way for the correction to the correlation function of the effective random forces, we find $a_1=-\varkappa$.

The diagrams of fourth-order perturbation theory for the corrections to the self-energy operator and the correlation function of the effective random forces are shown in Figs. 3 and 4, respectively. Calculating the two-loop diagrams is a complicated problem, but we are interested in the part proportional to the first power of $\ln(k/m)$, since it alone contributes to the renormalization-group function. Our analysis of the diagrams for $\Sigma^{(4)}$ shows that logarithmic singularities arise because of interior lines of the velocity correlation function which carry a small momentum (or wave number). These singularities are suppressed if the small momentum corresponds to a propagator line going

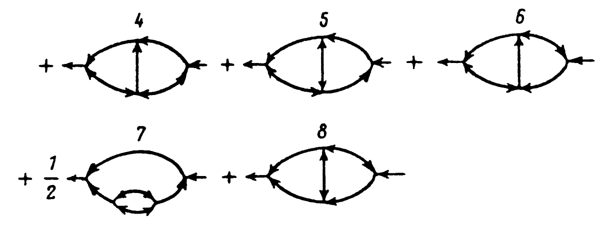

FIG. 3.

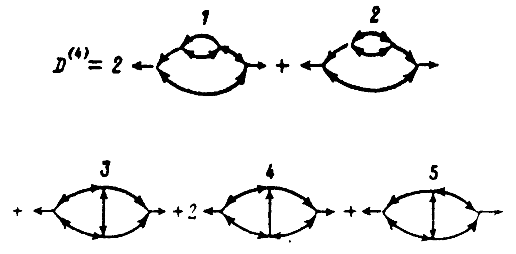

FIG. 4.

away from a vertex. In calculating the parameter b_2 it is thus sufficient to consider only the contributions from diagram *1* in Fig. 3 (found by inserting the self-energy part in the propagator line in Fig. 1) and from diagrams *4–6*, which incorporate the third-approximation corrections to the vertex. In calculating the corrections to the vertex we use Ward's hydrodynamic identity[20] written in the form

$$\Gamma^{(3)}_{ijh}(\mathbf{k}, \omega+\Omega\,|\,\mathbf{k}, \omega; 0, \Omega) = \lambda k_h \frac{\Sigma_{ij}(\mathbf{k}, \omega+\Omega) - \Sigma_{ij}(\mathbf{k}, \omega)}{\Omega}. \tag{6.4}$$

A calculation yields

$$b_2 = 8\varkappa^2 + 6\varkappa^2 = 14\varkappa^2. \tag{6.5}$$

The first term in this expression is determined by diagram *1* in Fig. 3, while the second is determined by diagrams *4–6*.

Analysis of the fourth-order diagrams in Fig. 4 for the correlation function of the effective random forces reveals that diagrams *2* and *5* contain second powers of $\ln(k/m)$. They can accordingly be ignored in the determination of the renormalization-group function. Diagrams *1* are found by inserting the self-energy part in one of the propagator lines in Fig. 1. There are four such diagrams, but they are weighted by a factor of 1/2. Diagrams *3–5* are found by substituting the vertices of all three types in third-order perturbation theory into the diagrams equations in Fig. 1. This process reduces to a doubling of the contribution of the vertex of the first type. Ward's identity (6.4) can be used again in calculating this contribution. The corresponding calculation yields

$$b_1 = {}^{47}/_4\varkappa^2 - {}^{9}/_4\varkappa^2 = {}^{19}/_2\varkappa^2, \tag{6.6}$$

where the first term is the contribution from diagram *1*, and the second term is the contribution from diagrams *3* and *4*.

Using the parameter values found here, we find, according to (5.5),

$$\delta_1 = -2{,}56\cdot 10^{-2}, \quad \delta_2 = -0{,}85\cdot 10^{-2}. \tag{6.7}$$

The spectral energy density of the turbulent fluctuations of the velocity can be found in the usual way:

$$E(k) \sim k^{d-1}\tilde{D}(k)/\tilde{\sigma}(k) \sim k^{-5/3} f_1(k/\eta)/f_2(k/\eta) = k^{-5/3}(k/\eta)^{\delta}.$$

Using (6.7), we find

$$\delta = \delta_1 - \delta_2 = -1{,}71\cdot 10^{-2}.$$

In discussions of corrections to the exponents, the quantity δ is usually written in the form $\delta = -\mu/9$. The quantity μ determines the exponent of the dependence of the correlation function of the fluctuations of the dissipation field.[1] It can be determined experimentally. While the value $\mu \approx 0.4$ has been used previously, a more recent refined analysis of the experimental data has yielded the estimate[29] $0.15 \leqslant \mu \leqslant 0.25$. From (6.7) we find $\mu = 0.153$.

7. CONCLUSION

In our analysis, the dependence on the external scale L is manifested as an anomalous dimension. In this regard, our study differs from most previous studies, in which the renormalization-group method has actually been used to justify the Kolmogorov hypotheses which make it possible to find the exponents of the asymptotic behavior on the basis of dimensionality considerations. In cases in which the scale dimension of the single dimensional parameter which exists in the inertial interval—the energy dissipation rate—is fixed in some way or other, the renormalization-group equation tells us nothing about the spectral exponents beyond what we can learn from simply dimensionality considerations (see, for example, the review by Kraichnan[30]).

In particular, in a study of the properties of a turbulent fluid in the ir limit by the renormalization-group method, the dimensionality of the parameter D is fixed by the choice of the "physical" value of ε in the procedure of the continuation along ε. The use of the renormalization-group method in the uv limit[16] also presupposes that the parameter D is unchanged by renormalization-group transformations. For this reason, effects of fluctuations in the energy dissipation are actually eliminated from consideration, as are the related intermittency effects, which are responsible for the appearance of corrections to the Kolmogorov exponents for the spectrum. These effects are described by higher-order perturbation theories, beginning with the fourth. Although the effect of the high-order approximations was studied in Refs. 16, 17, and 23, the analysis was aimed at proving that the sum of these approximations reduces to a transport. The logarithmic corrections, which describe cascade processes from our point of view, were not included in that study. As a result, the attempts to find the corrections to the Kolmogorov exponents by the renormalization-group method failed. In Ref. 17, for example, the corrections turned out to be simply logarithmic, while in Refs. 16 and 31 the value $\mu \approx 1.2$ was found. That value differs from the experimental data by an order of magnitude. Our calculations show that the correction to the exponent of the spectrum of turbulent-fluctuation energy is governed primarily by the anomalous dimension of the correlation function of the random forces (the dimension of this quantity was assumed to be canonical in the previous studies). Our calculations also show that the contribution of the anomalous dimension of the effective viscosity to the correction for the spectrum is much smaller while it has previously been regarded as predominant.

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