Dynamics of nonuniform excitations in one-dimensional spin systems

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We obtain exact results about the dynamics of spatially nonuniform states in a one-dimensional X–Y model of $S = 1/2$ spin. We show that in the $t \to \infty$ limit the system changes, as a rule, to a spatially uniform state due to spin-spin interactions. An exception is the behavior of excitations with a $Q = \pi$ wave vector in the isotropic model when the spatial nonuniformity is retained for long times. We establish that the damping of nonuniform excitations takes place as $t \to \infty$ according to a $(t/\tau)^{-\nu}$ power law where the exponent ν depends on the form of the initial state, the parameters of the models, and the wave vector Q. When we vary Q, the exponent ν changes discontinuously at a certain value $Q = Q_c$, where Q_c is determined solely by the parameters of the basic Hamiltonian.

1. INTRODUCTION

In theoretical studies of the static and dynamic properties of one-dimensional spin systems a great deal of attention has been paid to exactly soluble models.[1–10] This is connected with the great theoretical value of exact solutions in general and also with the fact that traditional methods of studying many-body systems with strong spin-spin interactions have turned out to be badly applicable to the one-dimensional case. The most appropriate amongst the exactly soluble models for the study of the dynamics of one-dimensional systems has apparently been the linear X–Y model of $S = 1/2$ spin with Hamiltonian

$$\mathcal{H} = -\hbar\omega_0 \sum_{j=1}^{N} S_j^z - \hbar\omega_e \sum_{j=1}^{N} \{(1+\gamma) S_j^x S_{j+1}^x + (1-\gamma) S_j^y S_{j+1}^y\}, \quad (S_{N+1} \equiv S_1), \tag{1}$$

where ω_0 and ω_e are the Zeeman and the exchange frequencies, γ is the anisotropy parameter ($0 \leqslant \gamma \leqslant 1$), and N is the number of spins in the chain.

First of all, this model allows an exact quantum statistical description of both the static and of a number of dynamic properties.[2–8] Secondly, the elementary excitations in the X–Y model are of the nature of spin waves which connects it with the Heisenberg model, which is more widely used in the theory of exchange coupled spin systems, and enables us to hope that the results obtained for the X–Y model are of a more general nature. Thirdly, it is well known[11–13] that the X–Y model is suitable for describing the low-temperature properties of a number of quasi-one-dimensional compounds such as $PrCl_3$, PrES, and CsH_2PO_4 which in principle makes it possible experimentally to check the theoretical conclusions reached.

Up to recently the main interest in studying the dynamics of the X–Y model was focused on finding the time-dependent spin correlation functions of the form $\langle A(t) B \rangle$, where A and B are spin operators,

$$A(t) = \exp(i\mathcal{H}t/\hbar) A \exp(-i\mathcal{H}t/\hbar),$$

and $\langle \ldots \rangle$ indicates an equilibrium average with the Hamiltonian (1), in terms of which one can in the linear response approximation express many observable quantities (susceptibility, scattering cross-sections, and so on).[4,9] Moreover, several exact results were obtained for a number of time-dependent spin averages of the form

$$\langle A \rangle_t = \mathrm{Sp}[\rho(t) A],$$

[$\rho(t)$ is the density matrix] which are needed for a description of processes which start far from equilibrium.[2,5]

A characteristic feature of those latter results is that they were found under the assumption that the initial state (the initial excitation) of the system was spatially uniform. This specific fact was decisive for the fact that applying the methods used in Refs. 2 and 5 one could obtain exact expressions for the required quantities. On the other hand, the problem of the dynamics of spatially nonuniform excitations has practically not been considered. However, its study is of interest both for the theory of nonequilibrium phenomena in spin systems and for experiments since such excitations may be produced in a real spin system by external, for instance, acoustic, actions.

We consider in the present paper the dynamics of a one-dimensional, thermally isolated spin system, which is described by the X–Y model of (1), when the initial excitation is spatially nonuniform. We show that it is in principle possible to solve exactly the problem stated here and we study in detail the features of the temporal evolution of a number of spin averages in various situations.

2. STATEMENT OF THE PROBLEM AND BASIC RELATIONS

Let the system considered initially, at time $t = 0$, be in a state described by the density matrix $\rho(0)$ and let its evolution for $t > 0$ be determined by the Hamiltonian (1). The averages $\langle A \rangle_t$ can then be written in the form

$$\langle A \rangle_t = \mathrm{Sp}[\rho(0) A(t)] \equiv \langle A(t) \rangle_0.$$

Below we consider the dynamics of the spatial Fourier components of the magnetization along the Z-axis, $\langle S_Q^Z \rangle_t$, and of the correlation functions $\langle \mathscr{E}_Q^{\alpha\beta} \rangle_t$ ($\alpha, \beta = X, Y$) of the X- and the Y-components of nearest neighbor spins which in the $Q = 0$, $\alpha = \beta$ case describe the exchange interaction of the system. Here we have

$$S_Q^z \equiv \sum_{j=1}^{N} S_j^z \exp(iQj), \qquad \mathscr{E}_Q^{\alpha\beta} \equiv \sum_{j=1}^{N} S_j^\alpha S_{j+1}^\beta \exp(iQj),$$

$Q = 2\pi n/N$, $n = -\frac{1}{2}N + 1, -\frac{1}{2}N + 2, \ldots, \frac{1}{2}N$, and N is an even number. To find the time-dependence of the spin opera-

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tors we need to diagonalize the Hamiltonian (1). The diagonalization procedure has been well described in Ref. 2. It essentially consists in changing from the spin operators to Fermi operators C_j^+, C_j using the Jordan–Wigner formulae

$$S_j^z = C_j^+ C_j - 1/2, \quad S_j^y = (C_j^+ - C_j) L_j / 2i, \quad S_j^x = (C_j^+ + C_j) L_j / 2, \quad (2)$$

where

$$L_j = \prod_{l=1}^{j-1} (2C_l^+ C_l - 1), \qquad L_1 \equiv 1,$$

after which the Hamiltonian (1) takes the form

$$\mathcal{H} = \mathcal{H}^+ P^+ + \mathcal{H}^- P^-,$$

where

$$\mathcal{H}^\pm = -\hbar\omega_0 \sum_{j=1}^{N} (C_j^+ C_j - 1/2)$$
$$-(\hbar\omega_e/2) \sum_{j=1}^{N-1} [C_j^+ C_{j+1} + C_{j+1}^+ C_j + \gamma (C_j^+ C_{j+1}^+ - C_j C_{j+1})]$$
$$\pm (\hbar\omega_e/2) [C_N^+ C_1 + C_1^+ C_N + \gamma (C_N^+ C_1^+ - C_N C_1)],$$
$$P^\pm = (1 \pm L_{N+1})/2, \quad (P^\pm)^2 = P^\pm, \quad P^+ P^- = 0.$$

The Hamiltonian $\mathcal{H}^-$ is brought to diagonal form by changing to a Fourier representation using the formulae

$$C_j^\pm = (1/N)^{1/2} \sum_p C_p^\pm \exp(\pm ipj), \qquad C_j^- \equiv C_j \qquad (3)$$

and a canonical transformation to the Fermi operators η_p^+, η_p:

$$C_p^+ = U_p \eta_p^+ - V_p \eta_{-p}, \quad C_p^- = U_p \eta_p + V_p \eta_{-p}^+,$$

where

$$U_p^2 = (E_p + \varepsilon_p)/2E_p, \quad V_p = i\xi_p U_p / (E_p + \varepsilon_p),$$
$$\varepsilon_p = -\hbar(\omega_e \cos p + \omega_0), \quad \xi_p = -\gamma\hbar\omega_e \sin p, \qquad (4)$$
$$E_p \equiv \hbar\omega_p = \{\varepsilon_p^2 + \xi_p^2\}^{1/2}.$$

As a result $\mathcal{H}^-$ takes the form of the Hamiltonian of a system of noninteracting fermions

$$\mathcal{H}^- = \sum_p \hbar\omega_p \eta_p^+ \eta_p. \qquad (5)$$

[The ground state energy was put equal to zero in Eq. (5).]

The Hamiltonian $\mathcal{H}^+$ is diagonalized using the same Eqs. (3)–(5). The only difference is that we must put the wave vector equal to $p = (2n+1)\pi/N$.

Using the properties of the $P^\pm$ operators the expression for $\langle A \rangle_t$ takes the form[6,7]

$$\langle A \rangle_t = \langle A(t_-) P^- \rangle_0 + \langle A(t_+) P^+ \rangle_0, \qquad (6)$$

where

$$A(t_\pm) = \exp(i\mathcal{H}^\pm t/\hbar) A \exp(-i\mathcal{H}^\pm t/\hbar).$$

One easily finds the time-dependence of the A operator by changing to the $\eta_p^\pm$ operators ($\eta_p^- = \eta_p$) and using the formula

$$\eta_p^\pm(t) = \eta_p^\pm \exp(\pm i\omega_p t).$$

In the last stage of finding $\langle A \rangle_t$ we average the operators $A(t_\pm) P^\pm$ over the initial state which in this part of the paper is assumed to be arbitrary and which will be specified only in the concluding stage of the calculations.

To avoid unjustifiedly complicating the calculations we shall concentrate our effort in what follows on two limiting cases: the isotropic X–Y model ($\gamma = 0$) and its extreme anisotropic variant ($\gamma = 1$) corresponding to the Ising model in a transverse field, and restricting then the range of values of the vector Q to $0 < Q \leqslant \pi$.

3. ISOTROPIC X–Y MODEL

In the case of the isotropic X–Y model ($\gamma = 0$) the diagonalization of the Hamiltonians $\mathcal{H}^\pm$ simplifies considerably. It is completed by the transformation (3) after which the Hamiltonians $\mathcal{H}^\pm$ take the form

$$\mathcal{H}^\pm = \sum_p \hbar\omega_p C_p^+ C_p,$$

where $\omega_p = -\omega_0 - \omega_e \cos p$ while we have, respectively, $p = 2\pi n/N$ and $p = (2n+1)\pi/N$ for $\mathcal{H}^-$ and $\mathcal{H}^+$.

We illustrate the main points of the further calculations by the example of finding the average $\langle S_Q^z(t) P^- \rangle$. Using Eqs. (2) and carrying out the transformation (3) we obtain

$$\langle S_Q^z(t_-) P^- \rangle_0 = -(N^{1/2}/2)\delta_{Q,0} \langle P^- \rangle_0 + N^{-1/2}$$
$$\times \sum_p \langle C_{p+Q/2}^+ C_{p-Q/2} P^- \rangle_0 \exp(i\Omega_Q t \sin p),$$

where $\Omega_Q = 2\omega_e \sin(Q/2)$. To find the final result we must solve the problem of taking into account the initial states. Following Ref. 7 we consider three kinds of initial states defined, respectively, by the density matrices

$$\rho^\mu(0) = F[S^\mu], \quad \mu = X, Y, Z,$$

each of which is a functional of only one component of the spin operators while otherwise being arbitrary. Under those assumptions the required quantities can be evaluated exactly using a method which we consider in detail for the $\rho(0) = \rho^z(0)$ case.

Inverting Eqs. (3) and (2) and bearing in mind that $L_j^2 = 1$ we express the $C_p^\pm$ operators in terms of the spin operators of the model:

$$C_p^\pm = \sum_{j=1}^{N} \prod_{l=1}^{j-1} (2S_l^z) S_j^\pm \exp(\mp ipj), \qquad S_j^\pm = S_j^x \pm i S_j^y.$$

Using this we find

$$\langle C_{p+Q/2}^+ C_{p-Q/2} P^- \rangle_0 = 1/2 \delta_{Q,0} \langle P^- \rangle_0 + N^{-1/2} \langle S_Q^z P^- \rangle_0 -$$
$$- N^{-1} \sum_{m=1}^{N-1} \sum_{j=1}^{N-m} \exp(-iQj)$$
$$\times \left\{ \exp[-i(p+Q/2)m] \left\langle S_j^- \prod_{l=1}^{m-1} (2S_{j+l}^z) S_{j+m}^+ P^- \right\rangle_0 \right.$$
$$\left. + \exp[i(p-Q/2)m] \left\langle S_j^+ \prod_{l=1}^{m-1} (2S_{j+l}^z) S_{j+m}^- P^- \right\rangle_0 \right\}. \qquad (7)$$

In the sums on the right-hand side of Eq. (7) only those terms remain in which the averaged expression does not con-

tain X- or Y-components of the spin operators, because here $\langle \ldots \rangle_0 \equiv \mathrm{Tr}[\rho^z(0)\ldots]$. Using this we obtain

$$\langle C^{+}_{p+Q/2} C_{p-Q/2} P^- \rangle_0 = {}^1/_2 \delta_{Q,0} \langle P^- \rangle_0 + N^{-1/2} \langle S_Q^z P^- \rangle_0 .$$

We must thus also calculate the averages $\langle C^{\pm}_{p+Q/2} C^{\pm}_{Q/2-p} P^- \rangle_0$ which are needed for finding the required averaged of the $\mathscr{E}^{\alpha\beta}_Q(t)$.

Substituting the expressions found into Eq. (7) and changing in the limit as $N \to \infty$ from summation over p to integration we find

$$\langle S_Q^z(t_-) P^- \rangle = \langle S_Q^z P^- \rangle_0 J_0(\Omega_Q t),$$

where $J_m(x)$ is a Bessel function of the first kind.

Using the same method to calculate averages of the form $\langle A(t_+) P^+ \rangle_0$ and using Eq. (6) we get finally

$$\langle S_Q^z \rangle_t = \langle S_Q^z \rangle_0 J_0(\Omega_Q t). \tag{8}$$

Similarly we can also evaluate the averages $\langle \mathscr{E}^{\alpha\beta}_Q \rangle_t$ which in the $\rho(0) = \rho^z(0)$ case have the form

$$\langle \mathscr{E}_Q^{xx} \rangle_t = \langle \mathscr{E}_Q^{yy} \rangle_t = 0,$$
$$\langle \mathscr{E}_Q^{xy} \rangle_t = {}^1/_2 i \exp(iQ/2) \langle S_Q^z \rangle_0 J_1(\Omega_Q t).$$

In the $\rho(0) = \rho^\mu(0)$, $\mu = X, Y$ cases the required averages can be written as follows:

$$\langle S_Q^z \rangle_t = 0, \quad \langle \mathscr{E}_Q^{xx} \rangle_t = \langle \mathscr{E}_Q^{\mu\mu} \rangle_0 f^{\pm}(t), \quad \langle \mathscr{E}_Q^{yy} \rangle_t = \langle \mathscr{E}_Q^{\mu\mu} \rangle_0 f^{\mp}(t), \tag{9}$$
$$\langle \mathscr{E}_Q^{xy} \rangle_t = -\langle \mathscr{E}_Q^{xy} \rangle_0 \frac{J_1(v_Q t)}{v_Q t} \sin(2\omega_0 t),$$

where

$$f^{\pm}(t) = \frac{J_1(\Omega_Q t)}{\Omega_Q t} \pm \frac{J_1(v_Q t)}{v_Q t} \cos(2\omega_0 t), \qquad v_Q = 2\omega_e \cos\left(\frac{Q}{2}\right), \tag{10}$$

while in the formulae for $\langle \mathscr{E}^{xx}_Q \rangle_t$ and $\langle \mathscr{E}^{yy}_Q \rangle_t$ the upper sign in $f(t)$ is taken for $\mu = X$ and the lower sign for $\mu = Y$. Knowing the averages $\langle \mathscr{E}^{xx}_Q \rangle_t$ and $\langle \mathscr{E}^{yy}_Q \rangle_t$ we can determine also the dynamics of the Fourier transform of the exchange energy $\langle \mathscr{E}_Q \rangle_t$ which is proportional to the sum of these quantities,

$$\langle \mathscr{E}_Q \rangle_t = -\hbar\omega_e (\langle \mathscr{E}_Q^{xx} \rangle_t + \langle \mathscr{E}_Q^{yy} \rangle_t).$$

Using (9) and (10) we obtain

$$\langle \mathscr{E}_Q \rangle_t = -2\hbar\omega_e \langle \mathscr{E}_Q^{\mu\mu} \rangle_0 J_1(\Omega_Q t)/\Omega_Q t.$$

For the further discussion of the results we write down the asymptotic expressions for the averages found that are nonvanishing as $t \to \infty$. For $\mu = Z$ we have

$$\langle S_Q^z \rangle_t \sim \langle S_Q^z \rangle_0 (2/\pi)^{1/2} (t/\tau_z)^{-1/2} \cos(t\Omega_Q - \pi/4) \quad (0 < Q \leqslant \pi),$$
$$\langle \mathscr{E}_Q^{xy} \rangle_t \sim (-i/2) \langle S_Q^z \rangle_0 (2/\pi)^{1/2} e^{iQ/2} (t/\tau_z)^{-1/2} \sin(t\Omega_Q - \pi/4). \tag{11}$$

For $\mu = X, Y$ we have

$$\langle \mathscr{E}_Q^{xx} \rangle_t \sim -(2/\pi)^{1/2} \langle \mathscr{E}_Q^{xx} \rangle_0 ((t/\tau_z)^{-1/2} \sin(t\Omega_Q - \pi/4)$$
$$\pm {}^1/_2 (t/\tau_x)^{-1/2} \{\sin[(v_Q + 2\omega_0)t - \pi/4]$$

$$\langle \mathscr{E}_Q^{xy} \rangle_t \sim (2/\pi)^{1/2} \langle \mathscr{E}_Q^{xy} \rangle_0 (t/\tau_x)^{-1/2} (\cos[(v_Q - 2\omega_0)t - \pi/4] \tag{12}$$
$$-\cos[(v_Q + 2\omega_0)t - \pi/4]) \quad (0 < Q < \pi),$$

where $\tau_z = |\Omega_Q|^{-1}$ and $\tau_x = |v_Q|^{-1}$.

It is clear from Eqs. (11) and (12) that the evolution of the required averages at long times is in the general case described by a sum of terms, oscillating at different frequencies with amplitudes which decrease according to a power law of the form

$$\langle \ldots \rangle_t \sim (t/\tau)^{-\nu}, \tag{13}$$

where the index ν equals $\frac{1}{2}$ for $\mu = Z$ and $\nu = \frac{3}{2}$ for $\mu = X, Y$.

4. ANISOTROPIC X–Y MODEL

In the $\gamma = 1$ case the procedure for evaluating the required quantities remains in principle the same as for $\gamma = 0$, but it becomes more cumbersome. Omitting inessential mathematical details we give in a unified form the final results for the averages $\langle S^z_Q \rangle_t, \langle \mathscr{E}^{\alpha\alpha}_Q \rangle_t (\alpha = X, Y)$:

$$\langle A_r \rangle_t = (1/4\pi)\Gamma^r(Q) \left[\int_0^\pi dp \varphi_r^+(p, Q) \cos[t\Omega^+(p, Q)] \right.$$
$$\left. + \int_0^\pi dp \varphi_r^-(p, Q) \cos[t\Omega^-(p, Q)] \right], \tag{14}$$

where the index r takes the values X, Y, Z, and where $A_z \equiv S^z_Q$, $\Gamma^z(Q) = \langle S^z_Q \rangle_0$, $A_\eta \equiv \mathscr{E}^{\eta\eta}_Q$, $\Gamma_\eta(Q) = \langle \mathscr{E}^{\eta\eta}_Q \rangle_0$, $(\eta = X, Y)$, $\Omega^{\pm}(p, Q) = \omega_{p+Q/2} \pm \omega_{p-Q/2}$.

The form of the functions $\varphi_r^{\pm}(p, Q)$ depends both on the value of the index r and also on the choice of the initial density matrix $\rho^\mu(0)$ (the value of the index μ). For instance, for $r = Z$ we have

$$\varphi_z^{\pm}(p, Q) = \begin{cases} -4e^{-iQ/2}\{[1 \mp \Phi_0(p, Q)]\cos p \mp \Phi_1(p, Q)\sin p\}, & \mu = X, \\ -4e^{-iQ/2}\{[1 \mp \Phi_0(p, Q)]\cos p \pm \Phi_1(p, Q)\sin p\}, & \mu = Y, \\ 2[1 \mp \Phi_0(p, Q)], & \mu = Z. \end{cases}$$

For the $r = Y$ case the corresponding formulae have the form

$$\varphi_y^{\pm}(p, Q) = \begin{cases} 2[\cos 2p \mp \Phi_0(p, Q)], & \mu = X, \\ 2[1 \mp \Phi_0(p, Q)\cos 2p \pm \Phi_1(p, Q)\sin 2p], & \mu = Y, \\ -\{[1 \mp \Phi_2(p, Q)]\cos p \pm \Phi_1(p, Q)\sin p\} e^{iQ/2}, & \mu = Z. \end{cases}$$

And, finally, for $r = X$ we obtain

$$\varphi_x^{\pm}(p, Q) = \begin{cases} 2[1 \mp \Phi_0(p, Q)\cos 2p \pm \Phi_1(p, Q)\sin 2p], & \mu = X, \\ 2[\cos 2p \mp \Phi_0(p, Q)], & \mu = Y, \\ -e^{iQ/2}\{\cos p \mp \Phi_0(p, Q)\cos p \mp \Phi_1(p, Q)\sin p\}, & \mu = Z. \end{cases}$$

Here

$$\Phi_0(p, Q) = [\omega_0^2 - 2\omega_0\omega_e \cos p \cos(Q/2) + \omega_e^2 \cos 2p]/\omega_{Q/2-p}\omega_{Q/2+p},$$
$$\Phi_1(p, Q) = [\omega_e^2 \sin 2p - 2\omega_0\omega_e \cos(Q/2)\sin p]/\omega_{Q/2-p}\omega_{Q/2+p}.$$

Because of the complexity of the integrands $\varphi_r^{\pm}(p, Q)$ and $\Omega^{\pm}(p, Q)$ it is not possible to obtain expressions for the averages (14) in explicit form. We therefore restrict our discussion to their asymptotic behavior at long times, using for this the stationary phase method.[14]

It is well known[14] that the asymptotic form of integrals

containing fast oscillating factors is determined by the stationary points of the phase functions [$\Omega^{\pm}(p,Q)$ in our case]. One can show that the function $\Omega^{+}(p,Q)$ for any values of the parameters ω_0 and ω_e and for $Q>Q_c$, where

$$Q_c=2\arccos\delta \quad (\delta=\min\{\lambda, 1/\lambda\},\ \lambda=\omega_0/\omega_e). \tag{15}$$

has three nondegenerate stationary points:

$$p_1=0,\ p_2=\pi,\ p_3=\arccos(\delta^{-1}\cos(Q/2)).$$

In the $Q=Q_c$ case the stationary points p_1 and p_3 merge into a single degenerate point ($p_1=p_3=0$):

$$\frac{d^n}{dp^n}\Omega^{+}(p,Q_c)\,|_{p=Q_c}=0 \qquad (n=1,2,3),$$

$$\frac{d^4}{dp^4}\Omega^{+}(p,Q_c)\,|_{p=Q_c}\neq 0,$$

while for $Q<Q_c$ there is no p_3 point.

The phase function $\Omega^{-}(p,Q)$ has a single nondegenerate stationary point $p_0=\arccos(\delta\cos(Q/2))$ for any values of ω_0, ω_e, and Q.

We illustrate the features of the asymptotic behavior (as $t\to\infty$) of the averages (14) by the most characteristic example of the evolution of the Fourier component $\langle S^z_Q\rangle_t$ for $\rho(0)=\rho^y(0)$. The time-dependence of this average has a different form for different values of the parameters λ and Q. For instance, for $\lambda<1$, $0<Q<Q_c$ we have

$$\begin{aligned}\langle S_Q{}^z\rangle_t\sim(8\pi)^{-1/2}\Gamma^y(Q)\{&G(0,Q)(t/\tau_1)^{-3/2}\cos[t\Omega^{+}(0,Q)+3\pi/4]\\&+G(\pi,Q)(t/\tau_2)^{-3/2}\cos[t\Omega^{+}(\pi,Q)-3\pi/4]+\\&+G_1(p_0,Q)(t/T)^{-3/2}\cos[t\Omega^{-}(p_0,Q)+3\pi/4]\}.\end{aligned} \tag{16}$$

In the $\lambda<1$, $Q=Q_c$ case we have

$$\begin{aligned}\langle S_Q{}^z\rangle_t\sim&-4\pi^{-1}(3/2)^{3/4}\Gamma(3/4)\Gamma^y(Q_c)(t/\tau_c)^{-3/4}\\&\times\cos[t\Omega^{+}(0,Q_c)+3\pi/8],\end{aligned} \tag{17}$$

where $\Gamma(x)$ is the gamma function.

For $\lambda<1$, $Q_c<Q\leqslant\pi$ we have

$$\begin{aligned}\langle S_Q{}^z\rangle_t\sim&-8/(2\pi)^{1/2}\Gamma^y(Q)e^{-iQ/2}\sin^2 p_3\cos p_3(t/\tau_3)^{-1/2}\\&\times\cos[t\Omega^{+}(p_3,Q)+\pi/4],\end{aligned} \tag{18}$$

For $\lambda=1$ we have for $0<Q\leqslant\pi$

$$\begin{aligned}\langle S_Q{}^z\rangle_t\sim&3/(8\pi)^{1/2}\Gamma^y(Q)\left\{\frac{\cos(4\omega_0 t\sin Q/4+\pi/4)}{(\omega_0 t\sin Q/4)^{3/2}}\right.\\&\left.-\frac{\cos(4\omega_0 t\cos Q/4+\pi/4)}{(\omega_0 t\cos Q/4)^{3/2}}\right\}.\end{aligned} \tag{19}$$

And, finally, for $\lambda>1$, $0<Q\leqslant\pi$ we have

$$\begin{aligned}\langle S_Q{}^z\rangle_t\sim&-8/(2\pi)^{-1/2}\Gamma^y(Q)\\&\times e^{-iQ/2}\sin^2 p_0\cos p_0(t/T)^{-1/2}\cos[t\Omega^{-}(p_0,Q)+\pi/4].\end{aligned} \tag{20}$$

In Eqs. (16)–(20) we have

$$T^{-1}=\left|\left[\frac{d^2}{dp^2}\Omega^{-}(p,Q)\right]_{p=p_0}\right|,$$

$$\tau_n^{-1}=\left|\left[\frac{d^2}{dp^2}\Omega^{+}(p,Q)\right]_{p=p_n}\right|, \quad n=1,2,3,$$

$$\tau_c^{-1}=\left|\left[\frac{d^4}{dp^4}\Omega^{+}(p,Q_c)\right]_{p=p_3}\right|,$$

$$G(p,Q)=\frac{d^2}{dp^2}[\cos p-\Phi_0(p,Q)\cos p+\Phi_1(p,Q)\sin p],$$

$$G(p,Q)=\frac{d^2}{dp^2}f(p,Q)-T\frac{d}{dp}f(p,Q)\frac{d^3}{dp^3}\Omega\ (p,Q),$$

where

$$f(p,Q)=\cos p+\Phi_0(p,Q)\cos p-\Phi_1(p,Q)\sin p.$$

It is clear from Eqs. (16)–(20) that the evolution as $t\to\infty$ of the nonuniform Z-component of the magnetization of the system considered proceeds similarly to the evolution of the averages in the isotropic model. However, the index ν which according to (13) characterizes the damping rate of the averages depends in this case on the values of the parameters λ and Q. One can show that the time-dependence of the average $\langle S^z_Q\rangle_t$ and also of the averages $\langle\mathscr{E}^{xx}_Q\rangle_t$ and $\langle\mathscr{E}^{yy}_Q\rangle_t$ have the same character also for other initial conditions. To illustrate this we show schematically in Fig. 1 for different initial conditions the ranges of the parameters λ and Q in which the index ν has different values.

5. DISCUSSION OF THE RESULTS

Turning to a discussion of the exact results obtained we note first of all that in the one-dimensional X–Y model independently of the initial conditions and the degree of anisotropy all spatially nonuniform quantities considered tend to zero as $t\to\infty$ for $0<Q<\pi$ which indicates that the system goes over into a spatially uniform state. (The case $Q=\pi$ is special and is discussed below.) It is clear from Eqs. (12) and (16)–(20) that the damping of the excitations occurs in a complex manner and is non-exponential in character.

Details of the time-dependence of the nonuniform excitations depend both on the initial state of the system under study and on the values of the parameters in the Hamiltonian (1) and the wave vector Q which determines the degree of inhomogeneity of the excitations considered. For instance, the characteristics of the initial state in terms of which the required quantities can be expressed change if one changes the component S^{μ} of the spin operators on which the density matrix of the initial state depends. For instance, for $\rho(0)=\rho^z(0)$ ($\mu=Z$) all averages found are proportional to the initial excitation of the nonuniform Z-component $\langle S^z_Q\rangle_0$ of the magnetization while for $\mu=X,Y$ they are proportional to the initial values of the averages $\langle\mathscr{E}^{\mu\mu}_Q\rangle_0$. Moreover, the character itself of the time-dependence of the averages considered changes, depending on μ. This fact can be traced using the example of the isotropic model by comparing Eqs. (8) and (9). For instance, the Z-component $\langle S^z_Q\rangle_t$ of the magnetization decreases in the $\mu=Z$ case from the value $\langle S^z_Q\rangle_0$ to zero as $t\to\infty$, while for $\mu=X,Y$ it remains always equal to zero, although the operator S^z_Q is not an integral of the motion for $Q\neq 0$. On the contrary, the averages $\langle\mathscr{E}^{xx}_Q\rangle_t$ and $\langle\mathscr{E}^{yy}_Q\rangle_t$ are equal to zero for $\mu=Z$ and nonvanishing for $\mu=X,Y$.

When μ changes, the index ν, which is the main characteristic of the asymptotic behavior of the required quantities as $t\to\infty$, also changes [see Eqs. (11)–(13) and Fig. 1]. For instance, for the average $\langle\mathscr{E}^{xy}_Q\rangle_t$ we have in the isotropic model $\nu=\frac{1}{2}$ for $\mu=Z$ and $\nu=\frac{3}{2}$ for $\mu=X,Y$. In the case of the anisotropic X–Y model ($\gamma=1$) the change in the index ν when the nature of the initial state changes and for constant

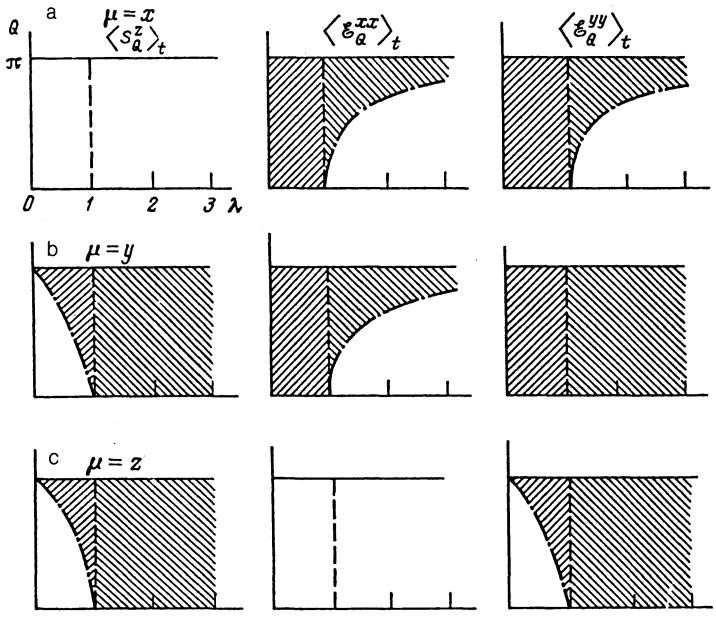

FIG. 1. Values of the index ν under various initial conditions for the averages $\langle S_Q^z \rangle_t$, $\langle \mathscr{E}_Q^{xx} \rangle_t$, and $\langle \mathscr{E}_Q^{yy} \rangle_t$. The shaded regions correspond to $\nu = \frac{1}{2}$, the unshaded ones to $\nu = \frac{3}{2}$, the dash-dot line to $\nu = \frac{3}{4}$, and the dashed line to $\nu = \frac{3}{2}$.

λ and Q is illustrated in Fig. 1. In particular, we have for the average $\langle S_Q^z \rangle_t$, when $\lambda < 1$ and $Q > Q_c$: $\nu = \frac{3}{2}$ for $\mu = X$ and $\nu = \frac{1}{2}$ for $\mu = Y, Z$.

The time-dependence of the spin excitations in the X–Y model turns out to be greatly affected also by the magnitude of the vector Q. This effect is most clearly pronounced in the anisotropic model and manifests itself in the Q dependence of the value of the index ν. For instance, for $\lambda < 1, \mu = Z$ we have for the average $\langle S_Q^z \rangle_t$

$$\nu = \begin{cases} {}^3/_2, & Q < Q_c, \\ {}^3/_4, & Q = Q_c, \\ {}^1/_2, & Q > Q_c. \end{cases} \tag{21}$$

A similar behavior also occurs for other averages (see Fig. 1). It is clear from Eqs. (21) and Fig. 1 that the change in the index ν in those cases when it occurs takes place discontinuously at some critical value Q_c of the wave vector Q. This critical value, given by Eq. (15), is connected neither with the type of the initial state nor with a particular average, and it is determined solely by the parameters of the Hamiltonian (1). We note that in the isotropic case there is no such Q dependence of the index ν.

The effect of the quantity Q on the dynamics of the nonuniform excitations in the X–Y model manifests itself also in the Q dependence of the times τ_i which characterize the time scale of the damping of the excitations. For instance, in the isotropic case the time τ_z increases when Q decreases and for small Q we have $\tau_z \sim Q^{-1}$ which indicates the stability of those long-wavelength excitations, the damping time scale of which is equal to τ_z (for instance, $\langle S_Q^z \rangle_t$ for $\mu = Z$). On the other hand, the time τ_x has its minimum value for $Q = 0$ and tends to infinity for $Q = \pi$. This latter fact leads to the circumstance that those excitations with a wave vector $Q = \pi$ the damping time scale of which is determined by the quantity τ_x do not tend to zero as $t \to \infty$. For instance, in the isotropic model for $Q = \pi, \mu = X$ we have

$$\begin{aligned} &\langle \mathscr{E}_\pi^{xx} \rangle_t = -\langle \mathscr{E}_\pi^{yy} \rangle_t = {}^1/_2 \langle \mathscr{E}_\pi^{xx} \rangle_0 \cos(2\omega_0 t) \quad (t \to \infty), \\ &\langle \mathscr{E}_\pi^{xy} \rangle_t = -{}^1/_2 \langle \mathscr{E}_\pi^{xx} \rangle_0 \sin(2\omega_0 t) \quad (t\text{—arbitrary}). \end{aligned} \tag{22}$$

This fact contradicts intuitive ideas about the nature of the evolution of physical quantities in many-body spin systems under the action of spin-spin interactions.

The traditional view of this problem consists in that any physical quantity with an operator which does not commute with the main Hamiltonian must as $t \to \infty$ take on a stationary value which is independent of the time.[15] Then in that sense unusual behavior of the averages in the X–Y model is connected with the peculiarities of its energy spectrum. Indeed, for $Q = \pi$ the sum of the eigenfrequencies

$$\omega_{p+Q/2} + \omega_{p-Q/2} = -[2\omega_0 + \nu_Q \cos(p + Q/2)]$$

no longer depends on Q since $\nu_Q = 0$. In the expressions determining the time-dependence of the averages $\langle \mathscr{E}_Q^{\alpha\beta} \rangle_t$

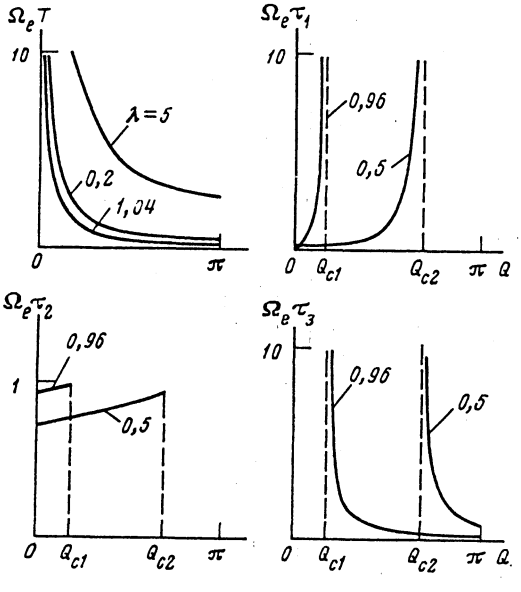

FIG. 2. Characteristic behavior of the times T, τ_1, τ_2, and τ_3 as functions of the degree of spatial inhomogeneity of the initial state for different values of the parameter λ: Q_{c1} and Q_{c2} are the critical values of the wave vector Q corresponding, respectively, to $\lambda = 0.96$ and $\lambda = 0.5$.

there is therefore, instead of a sum of terms oscillating at frequencies which are close to one another, which tends to zero as $t \to \infty$, a sum of terms which oscillate at one and the same frequency $2\omega_0$.

We note also that for $Q = \pi$ in the $\mu = X, Y$ cases the average $\langle \mathscr{E}_Q^{xy} \rangle_t$ no longer depends on the magnitude of the exchange frequency ω_e [see Eq. (22)]. The time-dependence of $\langle \mathscr{E}_Q^{xy} \rangle_t$ then has exactly the same form as in a system of noninteracting spins in a magnetic field parallel to the Z axis. This means from a physical point of view that in the system considered one can "eliminate" the effect of the exchange interaction on the dynamics of several quantities by the choice of the initial conditions.

The way the times τ_i depend on the wave vector Q in the anisotropic X–Y model ($\gamma = 1$) is shown in Fig. 2 for different values of the parameter λ ($\lambda \neq 1$). The most interesting feature of these functions is apparently the fast increase in the times τ_1 and τ_3 when Q approaches the critical value Q_c. This indicates the slowing down of the relaxation process as $Q \to Q_c$ for those physical quantities the damping time scale of which is determined by the times τ_1 and τ_3 (for instance, $\langle S_Q^z \rangle_t$ for $\lambda < 1$).

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