Distribution density for the energy dissipation rate in turbulent flows

P. L. Van'yan

V. I. Vernadskiĭ Institute for Geochemistry and Analytical Chemistry, Russian Academy of Sciences
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We study a semiempirical model of the evolution equation for the probability distribution density of the energy dissipation rate of flows in an incompressible liquid. We use the invariant simulation method which is a generalization of the "$k-\varepsilon$" model to obtain a universal solution of the equation. We compare it with experiments. The unrefined theory of locally isotropic turbulence is confirmed when the Reynolds number Re increases without bounds. The lognormal distribution for the dissipation turns out to be nonuniform with respect to the parameter of the asymptote. The intermittence index μ is not a universal constant, but depends asymptotically on Re: $\mu \sim \ln(\ln(Re))/\ln(Re)$. Consequences of the proposed model are the transfer equations for the average energy of the pulsations and of the average energy dissipation rate; they contain extra terms as compared to the analogous equations in the traditional semiempirical turbulence theory.

1. INTRODUCTION

The probability distribution density of the dissipation rate of the energy ε is one of the most important characteristics of turbulence in the refined Kolmogorov–Obukhov theory.[1,2] Landau[3] had noted that the statistical properties of the random field $\varepsilon(x,t)$ must determine the probability distributions for the small-scale components of the turbulence. In the original variant of the theory of locally isotropic turbulence[4,5] the average energy dissipation rate was chosen as the single parameter determining the structure of the pulsations in the inertial range of the spectrum.

In the refined theory a turbulent energy dissipation rate ε_r was considered which was averaged over a region of space with a characteristic dimension r referring to the inertial range. On the basis of qualitative ideas about the cascade process of the breaking up of the eddies in turbulent flows, expressed first of all by Richardson,[6] Kolmogorov,[1] and Obukhov,[2] a hypothesis about the logarithmic normality (lognormality) of ε_r (the so-called "third Kolmogorov hypothesis") was proposed. The hypothesis is based on the central limit theorem of probability theory and was first applied by Kolmogorov[7] to the multiplicative break-up process of particles. Novikov and Stewart[8] proposed for the study of the cascade process a model in which a discrete set of embedded spatial regions are considered. Yaglom[9] and Gurvich and Yaglom[10] have shown that such a model leads, under some rather general assumptions, to the logarithmic normality of the ε_r distribution. Novikov, in Refs. 11 and 12, considered a more general method, which does not require discretization of the breaking up, for studying the statistical characteristics of the energy dissipation rate field in the scaling range.

In a number of papers doubts were expressed about the hypothesis of the lognormality of the turbulent energy dissipation rate distribution. Orszag[13] showed that the set of all the moments, which were the same as the corresponding moments of the lognormal distribution, does not determine the probability distribution uniquely and, in the case of dynamical equations the initial values of all the moments, does not enable one to determine their evolution, even if the dynamical problem as a whole has a unique solution. Novikov[11,12] obtained from the scaling hypothesis for the moments of the turbulent energy dissipation rate distribution an inequality which is violated by the leading moments of the lognormal distribution. Kraichnan[14] suggested that one of those possibilities is the cascade process, satisfying scaling, of the breaking up of the turbulent eddies allows different possibilities for the dissipation probability distribution and the lognormal dissipation rate, averaged over small spatial regions. A number of papers (Refs. 8 and 15–18) studied a model (the so-called β-model) a characteristic feature of which is the assumption that there is a finite probability that the turbulent energy dissipation rate tends to zero. Turygin and Chechetkin[19] proposed a restricted lognormal probability distribution for ε_r. In the same paper the β-model and its modifications were subjected to a detailed analysis. It was shown that the experimental results obtained by Meneveau and Sreenivasan[20,21] (see also the summarizing Ref. 22) disagree with the conclusions of the β-model. Schertzer and Lovejoy,[23] using a model of the break up proposed by Mandelbrot,[15] studied the "hyperbolic" or α-model. In this model the turbulent eddies break up into smaller eddies which obtain a strictly defined part of the energy flux. A particular case of this model is the "binomial" or p-model[21] in which the turbulent energy dissipation rate is considered to be a two-scale Cantor set. In Yamazaki's B-model[24] a beta-distribution was proposed for the break-up coefficient. According to the model proposed by Andrews *et al.*[25] ε_r has a gamma-distribution and the distribution density of the unaveraged energy dissipation rate can be expressed in terms of a modified Bessel function ($K-1$ distribution). In Hosokawa's model[26] the square root energy dissipation rate is distributed exponentially.

The models listed here are either approximations of the experimental results or are based upon rather arbitrary assumptions of a statistical nature which are not directly connected with the Navier-Stokes equations. An exception is Kraichnan's paper[27] in which, by analogy with the "turbulence" of the Burgers equation, an equation was proposed for the velocity gradient distribution density (She[28] and She and Orszag[29] have developed the ideas of Ref. 27) and also a paper by Kuznetsov and Sabel'nikov[30] (see also Ref. 31) in which the equation for the ε_r distribution is obtained from the hydro-dynamical equations. The closure of the dynamical equations in Refs. 27–29 is based upon physically plausi-

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ble but in general arbitrary assumptions. The intermittence model considered in Refs. 30 and 31 is close to the β-model and can apparently not be made to agree with the experimental data of Refs. 20–22 when one studies the dissipation distribution moments of "negative" orders. We also refer to the model by Yakhot *et al.*[32] in which refinements of the Kolmogorov-Obukhov theory are obtained by applying the renormalization group method to the Navier-Stokes equations. In Refs. 33–39 and in a number of other papers the characteristics of turbulence in the inertial range of the spectrum are obtained by means of a numerical integration of the hydrodynamical equations or of their simplified analogs. A more detailed analysis of the existing models for the turbulent energy dissipation rate distribution is given in Refs. 19 and 22.

In the present paper we study the possibility of an invariant simulation using the "$k-\varepsilon$" theory for the transfer equation for the energy dissipation rate probability distribution in developed flows of an incompressible liquid. The proposed model is directly based upon ideas of the Kolmogorov and Obukhov theory. We assume that the small-scale structure of the pulsations in any developed turbulent flow is "adjusted" in a universal manner to the evolution of the large-scale (energy-containing) components of the turbulence spectrum. We choose for the parameters of the universal equilibrium distribution the average energy and the average dissipation rate of the turbulent pulsations and the molecular viscosity coefficient.

In statistical hydrodynamics the number of unknowns is always larger than the number of exact equations which can be obtained from the Navier-Stokes equations (see Ref. 40 for the "closure" problem in the theory of turbulence). The unknown functions appearing in the equation for the dissipation distribution density are in the proposed theory not given *ab initio* but are looked for in a rather general form. The actual form of the closed relations is determined by using the invariant simulation method. The essence of this method consists of the following: if some quantity (in this case the dissipation distribution density) depends in a universal manner on well defined parameters this dependence can be determined from a consistent consideration of the appropriate evolution equation in various kinds of flow. As a rule one chooses as a test flows which (due to symmetry properties) are very simple.

The invariant simulation method was first used for the closure of the equations of turbulent motion by Davydov.[41] An attempt at an invariant simulation of the probability distribution of impurity concentrations in developed turbulent flows was made in Ref. 42.

2. EQUATION FOR THE DISSIPATION DISTRIBUTION DENSITY

We can obtain a transport equation for the energy dissipation rate ε from the Navier–Stokes equations

$$\partial_t u_k + u_j \partial_j u_k = -\frac{1}{\rho}\partial_k p + \nu \Delta u_k$$

[where the u_j ($j=1,2,3$) are the components of the velocity along the appropriate axis of the Cartesian coordinate system x_j, p is the pressure, ρ is the density, ν is the kinematic viscosity coefficient, the symbols ∂_t and ∂_j denote, respectively, differentiation with respect to the time t and the x_j coordinate, Δ is the Laplace operator, and summation from 1 to 3 is understood over repeated indices] and the incompressibility condition

$$\partial_j u_j = 0$$

In the case of an incompressible liquid one can use the Stokes formula

$$\varepsilon = \frac{\nu}{2}\sum_{k,j=1}^{3}(\partial_k u_j + \partial_j u_k)^2,$$

to express the ε energy dissipation rate in terms of the velocity gradient and the transport equation for ε has the form

$$\begin{aligned}\partial_t \varepsilon + u_j \partial_j \varepsilon = &-\frac{2}{\rho}\nu \partial_{kj}^2 p (\partial_k u_j + \partial_j u_k)\\ &+\nu^2 (\partial_k u_j + \partial_j u_k)\Delta(\partial_k u_j + \partial_j u_k)\\ &-\nu(\partial_k u_j + \partial_j u_k)(\partial_j u_m \partial_m u_k + \partial_k u_m \partial_m u_j). \qquad (1)\end{aligned}$$

Monin,[43,44] Lundgren,[45] Novikov,[46] Kuznetsov,[47] and a number of other authors have proposed a method for finding the exact evolution equation for the distribution density of a random quantity (velocity, velocity eddy, impurity density) from the transport equation for the same quantity. One can similarly obtain from Eq. (1) an equation for the probability distribution density $P(\varepsilon; x_j, t)$ of the energy dissipation rate ε:

$$\begin{aligned}\partial_t P + \bar{u}_j \partial_j P + \partial_j(v_j P) = -\partial_\varepsilon \Big[P \Big\langle &-\frac{2}{\rho}\nu \partial_{kj}^2 p (\partial_k u_j + \partial_j u_k)\\ &+\nu^2(\partial_k u_j + \partial_j u_k)\Delta(\partial_k u_j + \partial_j u_k)\\ &-\nu(\partial_k u_j + \partial_j u_k)(\partial_j u_m \partial_m u_k + \partial_k u_m \partial_m u_j) \,|\, \varepsilon \Big\rangle \Big]. \qquad (2)\end{aligned}$$

The bar indicates here a probability average and the conditional average for a fixed value of the energy dissipation rate ε is represented as an operator $\langle \ldots | \varepsilon \rangle$,

$$v_j(\varepsilon; x_k, t) = \langle u_j' | \varepsilon \rangle,$$

and the prime indicates the pulsation of the corresponding quantity.

It is immediately clear from Eq. (2) that the evolution of the distribution density $P(\varepsilon; x_j, t)$ is due to the following factors.

1. Transfer of the average velocity (second term on the left-hand side of the equation).

2. Transfer of the pulsational velocity (third term on the left-hand side of the equation).

3. Interaction between the velocity gradients and the pressure (first term on the right-hand side of the equation).

4. Peaking of the velocity gradients due to the action of the molecular viscosity (second term on the right-hand side of the equation).

5. Accentuation of the velocity gradients due to the stretching of the vortex tubes (third term on the right-hand side of the equation).

The physical meaning of all terms in Eq. (2) is completely lucid and their value can, in principle, be determined experimentally. However, the use of this equation for an immediate theoretical analysis offers apparently few perspectives. This is connected with the fact there occur quantities in Eq. (2) which are averaged for a fixed value of ε and which cannot be expressed rigorously in terms of the required function P. The closing of Eq. (2) may turn out to be a more time-consuming problem than determining the energy dissipation rate probability distribution density itself. We note that a separate simulation of the generation and dissipation terms is also difficult. One can show that for large values of the Reynolds number the sum of these terms is finite and comparable to the other terms in the equation. However, separately the generation and dissipation terms do not have an order of magnitude which is comparable with that of the other terms of the equation.

3. INVARIANT SIMULATION OF THE EQUATION FOR THE DISSIPATION DISTRIBUTION DENSITY

The classical Kolmogorov–Obukhov theory is based on a representation of the balanced and localized character of the statistical characteristics of small-scale turbulence. One assumes that these characteristics (amongst which there is also the energy dissipation rate) are in quasiequilibrium with the large-scale structure of the flow and depend implicitly on the latter through a small number of well defined parameters.

Following the hypothesis proposed in Kolmogorov and Obukhov's papers and qualitatively confirmed both by direct and by indirect experiments (see, e.g., Ref. 40) it is natural to assume that the energy dissipation rate probability distribution density in developed turbulent flows (for large values of the Reynolds number) relaxes to a stable universal state which depends solely on the local characteristics of the flow turbulence and on the molecular properties of the liquid.

The study of the relaxation process of an arbitrary initial dissipation distribution to an equilibrium state as well as the analysis of the stability of the limiting distribution is a self-contained problem. In the present paper we study only the characteristics of the equilibrium distribution. The physical basis for the assumption about the quasi-stationarity of the dissipation distribution with respect to the macro-characteristics of the turbulent flow is an analysis of the characteristics for the dissipative frequency range. The characteristic time for changes in the dissipation field is determined by the Kolmogorov microscale time,

$$\tau=(\nu/\bar{\varepsilon})^{1/2}.$$

Since

$$\tau\sim Re^{-1/2},$$

we may assume that the energy dissipation rate distribution density in developed turbulent flows "manages to adjust itself" to the smoother changes of the large-scale quantities. We clearly do not consider here flows formed immediately behind a grid in a tube and similar ones. Such flows must be studied in the framework of a more general approach connected with an analysis of a nonequilibrium dissipation distribution.

In agreement with what has been said above we assume that the equilibrium energy dissipation rate probability distribution density $P(\varepsilon;x_j,t)$, apart from on the probability argument ε, depends only on the average energy dissipation rate per unit mass $\bar{\varepsilon}(x_j,t)$, the average kinetic energy of the velocity pulsations per unit mass $k(x_j,t)$:

$$k={}^1/_2\langle u_j'u_j'\rangle,$$

and on the molecular viscosity coefficient ν. We shall assume that the dissipation distribution density $P(\varepsilon)$ is a universal function of these parameters in any kind of turbulent flows with an arbitrary geometry. The spatial and temporal dependence, on the other hand, we assume to be an implicit one (through $\bar{\varepsilon}$ and k).

The standard dimensional-analysis procedure gives

$$P(\varepsilon;x_j,t)=\frac{1}{\langle\varepsilon\rangle}F(\zeta,Re),\tag{3}$$

where

$$\zeta=\varepsilon/\langle\varepsilon\rangle,\quad Re=k^2/\langle\varepsilon\rangle\nu.$$

Equation (3) significantly restricts the leeway in the choice of the type of probability distribution for the turbulent energy dissipation rate. To find the actual form of the function $F(\zeta,Re)$ one has to adopt some hypotheses which make it possible to model the right-hand side of Eq. (2), i.e., to "close" this equation.

We write the expression within the pointed brackets on the right-hand side of Eq. (2) in the form of a sum

$$\sum_{k=1}^{n}A_kf_k,$$

where the dimensional factors $A_k(x_j,t)$ depend on various averaged characteristics of the flow while the dimensionless functions $f_k(\zeta,Re)$ depend only on the dimensionless arguments of the function F. In that case Eq. (2) becomes

$$\partial_tP+\bar{u}_j\,\partial_jP+\partial_j(v_jP)=-\partial_\varepsilon\Big[P\Big(\sum_{k=1}^{n}A_kf_k\Big)\Big].\tag{4}$$

We assume that each term in the sum on the right-hand side of the semiempirical Eq. (4) reflects some well defined physical process in the transfer of the distribution density $P(\varepsilon)$ in probability space. Each factor A_k, apart from on the average energy dissipation rate, the average kinetic energy of the velocity pulsations, and the viscosity, depends then on the characteristics of the corresponding process.

Let the first term in the sum be determined by the isotropic evolution of the distribution density $P(\varepsilon)$, i.e., the evolution which is not connected with the fact that the flow may be inhomogeneous or anisotropic. We assume that the factor A_1 depends only on $\bar{\varepsilon}$, k, and ν. We assume for A_1 that

$$A_1=\bar{\varepsilon}^2/k,\tag{5}$$

since the dependence of the first term in the sum on the turbulent Reynolds number Re can, without loss of generality, be included in the function f_1.

We connect the second term of the sum with the anisotropy of the pulsational motion and as the characteristic of this we choose the anisotropy tensor a_{ij}:

$$a_{ij}=\langle u_i'u_j'\rangle/k-\delta_{ij}/3.$$

For the factor A_2 we obtain

$$A_2=B_2(a_{ij},Re)\bar{\varepsilon}^2/k,\tag{6}$$

and we require that the function $B_2(a_{ij},Re)$ vanish identically when the anisotropy tensor is equal to zero. Since the energy dissipation rate distribution density is a scalar invariant, the function B_2 depends clearly on the invariants of the anisotropy tensor.

We specify that the action of the shear of the average velocity on the dissipation distribution density describes the third term in the sum. We shall look for the factor A_3 in the form

$$A_3 = B_3\bar{\varepsilon}^2/k, \tag{7}$$

where we require of the dimensionless factor B_3 only that it vanish when there is no average velocity shear, i.e., when we have

$$\partial_i \bar{u}_j = 0.$$

We connect the fourth and fifth terms in the sum of the products $A_k f_k$ with the inhomogeneities, respectively, of the average energy dissipation rate and of the average pulsation energy. We shall assume that the coefficient A_4 vanishes when there is no gradient of $\bar{\varepsilon}$ and that the coefficient A_5 vanishes when there is no gradient of k.

We need in what follows a transport equation for the average kinetic energy of the turbulent pulsations and a transport equation for the average energy dissipation rate. In the case of developed turbulence (neglecting molecular diffusion of averaged quantities) the first of these equations has the form[40]

$$\partial_t k + \bar{u}_j \partial_j k + \partial_j I_j + \partial_j J_j = -\langle u_i' u_j' \rangle \partial_i \bar{u}_j - \bar{\varepsilon}, \tag{8}$$

where we have introduced the notation

$$I_j = \frac{1}{2}\langle u_j' u_m' u_m' \rangle, \qquad J_j = \left\langle u_j' \frac{p'}{\rho} \right\rangle.$$

We assume that the vector J_j is linearly connected with the vectors I_j and Q_j, where

$$Q_j = \langle u_j' \varepsilon' \rangle,$$

and we shall assume the proportionality coefficients to depend on k, $\bar{\varepsilon}$, and ν. From dimensional considerations we have then

$$J_j = \beta_1 I_j + \beta_2 Q_j k/\bar{\varepsilon}, \quad \beta_1 = \beta_1(Re), \quad \beta_2 = \beta_2(Re),$$

and the transfer equation (8) for the turbulence energy becomes

$$\partial_t k + \bar{u}_j \partial_j k + \partial_j [(1+\beta_1) I_j + \beta_2 Q_j k/\bar{\varepsilon}] = \langle u_m' u_j' \rangle \partial_m \bar{u}_j - \bar{\varepsilon}. \tag{9}$$

A model equation for the average energy dissipation rate can be obtained from Eq. (4). Multiplying the latter by ε and integrating over ε we find

$$\partial_t \bar{\varepsilon} + \bar{u}_j \partial_j \bar{\varepsilon} + \partial_j Q_j = \sum_{k=1}^{n} A_k \alpha_k, \tag{10}$$

where

$$\alpha_k(Re) = \int_0^\infty F(\zeta, Re) f_k(\zeta, Re)\, d\zeta.$$

Since the proposed hypotheses must be satisfied in different (arbitrary) flows, i.e., must be invariant with respect to the type of the turbulent flow, we consider consecutively several very simple cases.

First of all we turn to the case of homogeneous and isotropic turbulence. Equations (4), (8), and (10) will have the form

$$\partial_t P = -\partial_\varepsilon [P A_1 f_1], \tag{11}$$

$$\partial_t k = -\bar{\varepsilon}, \tag{12}$$

$$\partial_t \bar{\varepsilon} = \alpha_1 \bar{\varepsilon}^2/k. \tag{13}$$

Substituting the function P in the form (3) into Eq. (11) and using Eqs. (12) and (13) we obtain

$$\alpha_1 \partial_\zeta (\zeta F) + (2+\alpha_1)\partial_\xi F = \partial_\zeta (F f_1), \tag{14}$$

where we have introduced the notation $\xi = \ln(Re)$.

Similar to the preceding case we find for a turbulence which is homogeneous, but not isotropic

$$\alpha_2 \partial_\zeta (\zeta F) + \alpha_2 \partial_\xi F = \partial_\zeta (F f_2). \tag{15}$$

We now apply the invariant simulation method to a stationary turbulent flow with a constant shear. The characteristics of the turbulent quantities are constant along the flow in such a flow: generation is compensated by dissipation. Equations (4) and (10) then become, respectively,

$$\partial_\varepsilon [P(A_1 f_1 + A_2 f_2 + A_3 f_3)] = 0. \tag{16}$$

$$\alpha_1 + \alpha_2 B_2 + \alpha_3 B_3 = 0. \tag{17}$$

Using Eqs. (14) and (15) we find

$$\alpha_3 \partial_\zeta (\zeta F) + \beta \partial_\xi F = \partial_\zeta (F f_3), \tag{18}$$

where

$$\beta(\xi) = \alpha_3 (2 + \alpha_1 + \alpha_2 B_2^*)/(\alpha_1 + \alpha_2 B_2^*).$$

Here B_2^* is the value of the coefficient B_2 in a flow with a constant shear.

To close in Eq. (4) the term describing the turbulent diffusion of the energy dissipation rate probability density we put

$$v_j(\varepsilon; x_i, t) = \varphi_1(\zeta, \xi) Q_j/\bar{\varepsilon} + \varphi_2(\zeta, \xi) I_j/k. \tag{19}$$

One can show that Eq. (19) is a generalization of the gradient hypothesis which connects turbulent flows with the gradients of averaged quantities to the energy dissipation rate distribution density. Substituting Eqs. (14), (15), (18), and (19) into Eq. (4) we obtain

$$\begin{aligned}
&-\partial_\zeta (\zeta F)[\partial_t \bar{\varepsilon} + \bar{u}_j \partial_j \bar{\varepsilon} - (\alpha_1 + \alpha_2 B_2 + \alpha_3 B_3)\bar{\varepsilon}^2/k] \\
&\quad + \partial_\xi F[(\partial_t Re + \bar{u}_j \partial_j Re)\bar{\varepsilon}/Re \\
&\quad + (2 + \alpha_1 + \alpha_2 B_2 + \beta B_3)\bar{\varepsilon}^2/k] + \varphi_1 F \partial_j Q_j \\
&\quad - [\varphi_1 F + \partial_\zeta (\zeta \varphi_1 F)](Q_j \partial_j \bar{\varepsilon})/\bar{\varepsilon} \\
&\quad + \partial_\xi (\varphi_1 F)(Q_j \partial_j Re)/Re + \varphi_2 F \bar{\varepsilon} \partial_j (I_j/k) - \partial_\zeta (\zeta \varphi_2 F)(I_j \partial_j \bar{\varepsilon})/k \\
&\quad + \partial_\xi (\varphi_2 F)(I_j \partial_j Re)\bar{\varepsilon}/(kRe) = -\partial_\zeta \left[F \left(\sum_{k=4}^{n} A_k f_k \right) \right].
\end{aligned} \tag{20}$$

We apply a Mellin transform[48] with respect to the ζ variable to Eq. (20). Using Eqs. (9) and (10) we get

$$\begin{aligned}
&q\Phi \left(\partial_j Q_j - \sum_{k=4}^{n} A_k \alpha_k \right) + d_\xi \Phi \left(E - \sum_{k=4}^{n} A_k \alpha_k \right) \\
&\quad + \Psi_1 \partial_j Q_j + (q-1)\Psi_1 (Q_j \partial_j \bar{\varepsilon})/\bar{\varepsilon} \\
&\quad + d_\xi \Psi_1 (Q_j \partial_j Re)/Re + \Psi_2 \bar{\varepsilon} \partial_j (I_j/k) + q\Psi_2 (I_j \partial_j \bar{\varepsilon})/k \\
&\qquad + d_\xi \Psi_2 (I_j \partial_j Re)\bar{\varepsilon}/(kRe) = -\sum_{k=4}^{n} A_k \chi_k,
\end{aligned} \tag{21}$$

where

$$\Phi(q,\xi)=\int_0^\infty \zeta^q F(\zeta,\xi)\,d\zeta, \qquad \Psi_j(q,\xi)=\int_0^\infty \zeta^q F(\zeta,\xi)\varphi_j(\zeta,\xi)\,d\zeta,$$

$$\chi_j(q,\xi)=\int_0^\infty \zeta^q \partial_\zeta[F(\zeta,\xi)f_j(\zeta,\xi)]\,d\zeta,$$

$$E=(\beta-\alpha_3)A_3-2\langle u_i'u_j'\rangle\partial_i\bar{u}_j\bar{\varepsilon}/k$$
$$-2\partial_j[(1+\beta_1)I_j+\beta_2 Q_j k/\bar{\varepsilon}]\varepsilon/k+\partial_j Q_j.$$

Let

$$\theta_k(q,\xi)=\chi_k-\alpha_k[d_\xi\Phi-q\Phi].$$

Using the definition of the quantity E and expressing the turbulent Reynolds number Re in terms of the average energy and its average dissipation rate we transform Eq. (21):

$$d_\xi\Phi[(\beta-\alpha_3)A_3-2\langle u_i'u_j'\rangle\partial_i\bar{u}_j\bar{\varepsilon}/k]$$
$$+[-q\Phi+(1-2\beta_2)d_\xi\Phi+\Psi_1]\partial_j Q_j$$
$$+[-2(1+\beta_1)d_\xi\Phi+\Psi_2](\bar{\varepsilon}\partial_j I_j)/k+[2(\beta_2+d_\xi\beta_2)d_\xi\Phi-d_\xi\Psi_1$$
$$+(q-1)\Psi_1](Q_j\partial_j\bar{\varepsilon})/\bar{\varepsilon}+[-4d_\xi\beta_1 d_\xi\Phi-\Psi_2+2d_\xi\Psi_2](\bar{\varepsilon}I_j\partial_j k)/k^2$$
$$+[-2(\beta_2+2d_\xi\beta_2)d_\xi\Phi+2d_\xi\Psi_1](Q_j\partial_j k)/k+[2d_\xi\beta_1 d_\xi\Phi+q\Psi_2$$
$$-d_\xi\Psi_2](I_j\partial_j\bar{\varepsilon})/k=-\sum_{k=4}^{n}A_k\theta_k. \tag{22}$$

In the general case the shear of the average velocity, the gradient of the average energy dissipation rate, and the gradient of the average turbulence energy are linearly independent. Therefore, and also by virtue of the assumption that the coefficients A_4 and A_5 vanish when the gradients of the dissipation and of the energy are respectively equal to zero, Eq. (22) splits into a set of equations

$$(\beta-\alpha_3)A_3-2\langle u_i'u_j'\rangle\partial_i\bar{u}_j\bar{\varepsilon}/k=0, \tag{23}$$
$$-q\Phi+(1-2\beta_2)d_\xi\Phi+\Psi_1=0, \tag{24}$$
$$-2(1+\beta_1)d_\xi\Phi+\Psi_2=0, \tag{25}$$
$$2(\beta_2+d_\xi\beta_2)d_\xi\Phi-d_\xi\Psi_1+(q-1)\Psi_1=-\theta_4, \tag{26}$$
$$-4d_\xi\beta_1 d_\xi\Phi-\Psi_2+2d_\xi\Psi_2=-\theta_5, \tag{27}$$
$$-2(\beta_2+2d_\xi\beta_2)d_\xi\Phi+2d_\xi\Psi_1+\gamma[2d_\xi\beta_1 d_\xi\Phi+q\Psi_2-d_\xi\Psi_2]$$
$$=-\gamma_1\theta_4-\gamma\gamma_2\theta_5, \tag{28}$$
$$\theta_k=0, \quad k>5, \tag{29}$$

and

$$A_4=(Q_j\partial_j\bar{\varepsilon})/\bar{\varepsilon}+\gamma_1(Q_j\partial_j k)/k, \tag{30}$$
$$A_5=(\bar{\varepsilon}I_j\partial_j k)/k^2+\gamma_2(I_j\partial_j\bar{\varepsilon})/k, \tag{31}$$
$$I_j\partial_j\bar{\varepsilon}=\gamma Q_j\partial_j k, \tag{32}$$

where

$$\gamma=\gamma(\xi), \quad \gamma_1=\gamma_1(\xi), \quad \gamma_2=\gamma_2(\xi).$$

We have thus shown that if the proposed hypotheses about the form of the energy dissipation rate probability distribution density in developed turbulent flows are valid the sum on the right-hand side of Eq. (10) for the average energy dissipation rate must contain at least five terms. In that case Eq. (10) will have the following form:

$$\partial_t\bar{\varepsilon}+\bar{u}_j\partial_j\bar{\varepsilon}+\partial_j Q_j=[\alpha_1+\alpha_2 B_2(a_{ij})$$
$$+2\alpha_3\langle u_i'u_j'\rangle\partial_i\bar{u}_j/[(\beta-\alpha_3)\bar{\varepsilon}]]\bar{\varepsilon}^2/k$$
$$+\alpha_4[(Q_j\partial_j\bar{\varepsilon})/\bar{\varepsilon}+\gamma_1(Q_j\partial_j k)/k]+\alpha_5[(\bar{\varepsilon}I_j\partial_j k)/k^2+\gamma_2(I_j\partial_j\bar{\varepsilon})/k]. \tag{33}$$

Equation (33) can in principle contain additional terms which on the right-hand side of Eq. (21) correspond to functions χ_k of the form

$$\chi_k=\alpha_k[d_\xi\Phi-q\Phi],$$

similar to the function χ_2 which is connected with the anisotropy of the velocity pulsations.

Starting with a paper by Davydov[41] (see Refs. 49, 50, and others) the action of the mean velocity shear has been described in semiempirical transfer equations for the average energy dissipation rate by an expression similar to (23). However, Lumley[51] suggested the need to supplement this expression with additional terms depending on the average velocity shear. If the hypotheses proposed above are valid there is no need to generalize Eq. (23).

There occur in Eq. (33) new terms, as compared with similar equations in semiempirical theories, which vanish only when the coefficients α_4 and α_5 are equal to zero. In contrast to those theories Eq. (33) is not postulated a priori but is obtained as a consequence of a model based upon the theory of locally isotropic turbulence. This equation is the result of a more general application of the widely used "$k-\varepsilon$" method for closing the set of equations describing turbulent flows.

Using (24)–(27) to eliminate the functions θ_4, θ_5, Ψ_1, and Ψ_2 in Eq. (28) we get for the functions Φ of the normalized dissipation moments the equation

$$d_{\xi\xi}^2\Phi+2(\lambda_1+\lambda_2 q)d_\xi\Phi+\lambda_3 q(q-1)\Phi=0, \tag{34}$$

where

$$2\lambda_1=[2\beta_2+\gamma_1-2\gamma\gamma_2(1+\beta_1)]/[2(1-2\beta_2)+2\gamma(1+\beta_1)$$
$$+\gamma_1(1-2\beta_2)+4\gamma\gamma_2(1+\beta_1)],$$
$$\lambda_2=[-1-\gamma_1(1-\beta_2)-\gamma(1+\beta_1)]/[2(1-2\beta_2)$$
$$+2\gamma(1+\beta_1)+\gamma_1(1-2\beta_2)+4\gamma\gamma_2(1+\beta_1)],$$
$$\lambda_3=\gamma_1/[2(1-2\beta_2)+2\gamma(1+\beta_1)+\gamma_1(1-2\beta_2)+4\gamma\gamma_2(1+\beta_2)].$$

The coefficients λ_j depend in the general case on ξ, i.e., on the turbulent Reynolds number Re.

4. SOLUTION OF THE EQUATION

We shall show in Sec. 6 that the dependence of the coefficients λ_j on the Reynolds number is important, albeit weak. For a comparison with experiments we therefore assume that the functions $\lambda_1(\xi)$, $\lambda_2(\xi)$, and $\lambda_3(\xi)$ have a finite asymptotic behavior for large ξ (as $Re\to\infty$). The general solution of Eq. (34) will in that case be of the form

$$\Phi(\xi,q)=C_1(q)\exp\{[-(\lambda_1+\lambda_2 q)+((\lambda_1+\lambda_2 q)^2$$
$$-\lambda_3 q(q-1))^{1/2}]\xi\}$$
$$+C_2(q)\exp\{[-(\lambda_1+\lambda_2 q)-((\lambda_1+\lambda_2 q)^2-\lambda_3 q(q-1))^{1/2}]\xi\}. \tag{35}$$

Since the coefficient of ξ in the index of the exponent of the function Φ is positive for values of the parameter q which are large in absolute magnitude, the coefficient C_1 does not vanish identically. We can then for large ξ, which are the only ones to be considered, neglect the second term in (35) as compared to the first one. Applying an inverse Mellin transformation[48] to Eq. (35) we find a formal expression for the function F:

$$F(\zeta, Re) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} C_1(q)\, Re^{\{-(\lambda_1+\lambda_2 q)+[(\lambda_1+\lambda_2 q)^2-\lambda_3 q(q-1)]^{1/2}\}} \zeta^{-q-1} dq .$$

5. COMPARISON WITH EXPERIMENTS

The statistical characteristics of the energy dissipation rate in the inertial range of the spectrum have been studied experimentally in Refs. 20–22. Let us see how the conclusions reached in the present paper are related to the results of those experiments.

If we use the scaling hypothesis of Refs. 11 and 12, we can state that the normalized moment $E_q(r)$ of order q of the energy dissipation rate, averaged over a region of dimension r,

$$E_q(r) = \langle \varepsilon_r^q \rangle / \langle \varepsilon \rangle^q,$$

behaves as follows:

$$E_q(r) \sim (r/L)^{-\mu_q} \text{ when } \eta \ll r \ll L.$$

$$E_q(r) = \Phi(q, \xi) \text{ when } r \ll \eta.$$

Here L is the external turbulence scale:

$$L \sim k^{3/2}/\varepsilon,$$

η is the Kolmogorov microscale:

$$\eta = \langle \varepsilon \rangle^{-1/4} \nu^{3/4},$$

and the coefficient μ_q depends on the order of the corresponding moment. We introduce a scale η_q:

$$(\eta_q/L)^{-\mu_q} = \Phi(q, \xi). \tag{36}$$

From scaling considerations it follows that

$$\eta_q = \eta l(q).$$

Since $\eta/L \sim Re^{-3/4}$ we find, by comparing Eqs. (35) and (36), that

$$\mu_q = {}^4/_3\{-(\lambda_1+\lambda_2 q) + [(\lambda_1+\lambda_2 q)^2 - \lambda_3 q(q-1)]^{1/2}\}. \tag{37}$$

For values of the parameter q which have a large absolute magnitude the increase in the exponents μ_q is linear, which agrees with Novikov's analysis[12] of the way the dissipation moments depend on their order.

The main attention in Refs. 20–22 was paid to the multifractal structure of the energy dissipation rate field. The generalized dimensionalities D_q are defined by the equation

$$D_q = D + \frac{\mu_q}{1-q},$$

where D is the effective dimensionality of the space in the experiment. Since in Refs. 20–22 a one-dimensional (temporal) series of data was studied, we have $D = 1$. We then get from Eq. (37)

$$D_q = 1 + \frac{4}{3(1-q)}\{-(\lambda_1+\lambda_2 q) + [(\lambda_1+\lambda_2 q)^2 - \lambda_3 q(q-1)]^{1/2}\}. \tag{38}$$

We show in Fig. 1 the experimental values[20–22] of the dimensionality D_q and the function (38) approximating them. The values of the parameters of the distribution are $\lambda_1 \approx 2.37$, $\lambda_2 \approx 0.0$, $\lambda_3 \approx -0.43$.

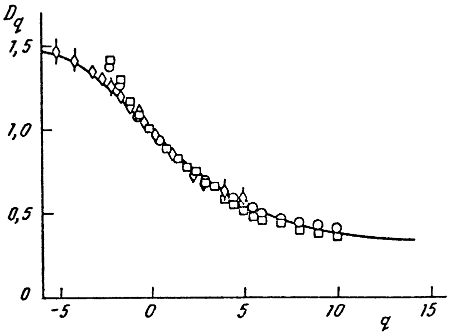

FIG. 1. The generalized dimensionality D_q as function of q. The experimental points are from Ref. 22 and the solid curve is according to Eq. (38).

In the study of the multifractal properties of developed turbulence there has recently been widespread use of an analysis of "$f-\alpha$" curves, or of the so-called multifractal spectrum[22,52] (see also the references in those papers) which is defined by the relations

$$f(\alpha(q)) = q\alpha(q) - (q-1)D_q, \tag{39}$$

$$\alpha(q) = \frac{d}{dq}[(q-1)D_q]. \tag{40}$$

Substituting expression (38) for D_q in the defining functions (39) and (40) we find for $f(\alpha(q))$ and $\alpha(q)$

$$f(\alpha(q)) = 1 + q(\alpha-1) + {}^4/_3\{-(\lambda_1+\lambda_2 q) + [(\lambda_1+\lambda_2 q)^2 - \lambda_3 q(q-1)]^{1/2}\}, \tag{41}$$

$$\alpha(q) = 1 + \frac{2}{3}\left[2\lambda_2 - \frac{2\lambda_2(\lambda_1+\lambda_2 q) - \lambda_3(2q-1)}{[(\lambda_1+\lambda_2 q)^2 - \lambda_3 q(q-1)]^{1/2}}\right]. \tag{42}$$

From Eq. (42) we can find the function $q(\alpha)$:

$$q(\alpha) = \frac{1}{2} - \frac{3}{2}(\alpha-1)\left[\frac{4\lambda_1^2+\lambda_3}{16\lambda_3^2+9\lambda_3(\alpha-1)^2}\right]^{1/2}. \tag{43}$$

To simplify the formulae we take the parameter λ_2 to be equal to zero. Using Eq. (43) to eliminate the parameter q in Eq. (41), we get an explicit expression for the multifractal spectrum:

$$f(\alpha) = 1 - \frac{4}{3}\lambda_1 + \frac{\alpha-1}{2} - \frac{1}{6\lambda_3}[(9\lambda_3(\alpha-1)^2 + 16\lambda_3^2][4\lambda_1^2+\lambda_3]]^{1/2}. \tag{44}$$

Substituting $q = \pm\infty$ in Eq. (42), we find the range within which α varies:

$$\alpha_{min} = 1 - {}^4/_3(-\lambda_3)^{1/2}, \quad \alpha_{max} = 1 + {}^4/_3(-\lambda_3)^{1/2}.$$

We show in Fig. 2 the multifractal spectrum according to the data of Refs. 20–22 and the model expression for it given by Eq. (44).

Equation (37) makes it possible to determine the power index ξ_n of the velocity structure function in the inertial range. According to Ref. 22 we have

$$\langle (\delta u_r)^n \rangle \sim [\bar{\varepsilon} L]^{n/3} (r/L)^{\xi_n},$$

where

$$\xi_n = {}^1/_3 n + ({}^1/_3 n - 1)(D_{n/3} - D),$$

and δu_r is the value of the velocity difference in points at a

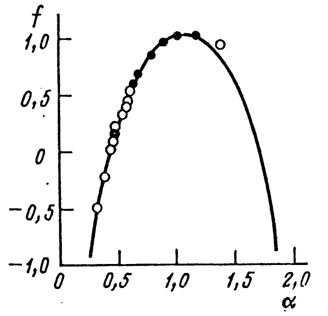

FIG. 2. Multifractal spectrum. The experimental points are from Ref. 22 and the solid curve is according to Eq. (44).

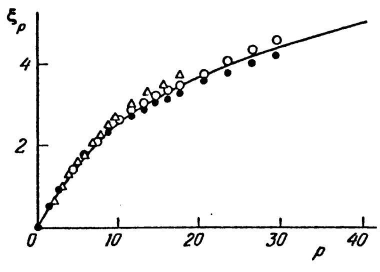

FIG. 3. The power index ξ_p of the velocity structure function as function of the order p. The experimental points are from Ref. 22 (data from Refs. 22 and 54) and the solid curve is according to Eq. (45).

distance r from one another. In accordance with Eq. (38) we have

$$\xi_n=\frac{1}{3}n-\frac{4}{3}\left\{-\left(\lambda_1+\lambda_2\frac{n}{3}\right)+\left[\left(\lambda_1+\lambda_2\frac{n}{3}\right)^2-\lambda_3\frac{n}{3}\left(\frac{n}{3}-1\right)\right]^{1/2}\right\}. \quad (45)$$

We show in Fig. 3 the results of the experimental determination[22,53] of the index ξ_n and also the theoretical expression given by Eq. (45).

6. ANALYSIS OF THE RESULTS

Notwithstanding the good agreement between the experimental data and the model functions we must note that the solution (35), obtained under the assumption that the coefficients λ_j are constant, cannot be deemed to be exact, but must be considered to be an approximation. This is connected with the fact that, as we shall show in what follows, Eq. (37) for μ_q is in contradiction to the scaling hypothesis of the field ε_r.

We consider the probability distribution density $p(M,b)$ of the break-up coefficient $M(M=\varepsilon_r/\varepsilon_{r/b})$ when we decrease the scale by a factor b, where the scales r and r/b lie within the scaling range. We then have (see Refs. 12, 22, and 54)

$$\int_0^b p(M,b)M^q\,dM=b^{\mu_q}. \quad (46)$$

Through the substitutions

$$y=-\ln(M/b),$$
$$s=q-{}^1/_2,$$
$$p(M,b)=g(y,b)\exp\{3y/2-({}^3/_2+4\lambda_1/3)\ln b\},$$

Eq. (46) is transformed into

$$\int_0^\infty g(y,b)e^{-sy}\,dy=\exp\left\{\ln(b)\left[-s+\frac{4}{3}(-\lambda_3)^{1/2}(s^2-\lambda_1^2/\lambda_3-{}^1/_4)^{1/2}\right]\right\}. \quad (47)$$

Applying an inverse Laplace transform[48] to Eq. (47) we find the function $g(y,b)$. Changing from the variable y to M we obtain

$$p(M,b)=\begin{cases} M^{-3/2}b^{-4\lambda_1/3}\left\{\delta(M-M_1)-\frac{4}{3}\left(\frac{\lambda_1^2}{\lambda_3}+\frac{1}{4}\right)\ln(b)J_1(z^{1/2})/z^{1/2}\right\}, & M\leqslant M_1\\ 0, & M>M_1; \end{cases} \quad (48)$$

where

$$z=-\left(\frac{\lambda_1^2}{\lambda_3}+\frac{1}{4}\right)\ln(b)\left[\ln^2(M)+\frac{16}{9}\lambda_3\ln^2(b)\right],$$
$$M_1=b^{(4/3)(-\lambda_3)}$$

J_1 is a first-order Bessel function and δ is a Dirac delta function.

We show in Fig. 4 the function $p(M,2)$. When $M<M_0\,(M_0\approx0.3)$ this function is alternating[55] which is apparently a consequence of the assumption that the λ_j coefficients in Eq. (34) are constant. One can show easily that this conclusion remains valid when $\lambda_2\neq0$.

We note that when the hypotheses proposed in the present paper are valid the function μ_q cannot be independent of the turbulent Reynolds number Re. However, Eq. (37) is not only not exact, but it also cannot be the main term of an expansion of the function μ_q which is uniform in the parameter q as $Re\to\infty$.

The main term $\tilde{\Phi}(\xi,q)$ of the asymptotic expression for $\xi\gg1$ of the fundamental solution of Eq. (34) in which we are interested has the form[56]

$$\tilde{\Phi}(\xi,q)=S^{-1/4}\exp\{\textstyle\int[-(\lambda_1+\lambda_2q)+S^{1/2}]\,d\xi\}, \quad (49)$$

where

$$S=(\lambda_1+\lambda_2q)^2-\lambda_3q(q-1)+d_\xi(\lambda_1+\lambda_2q).$$

The analysis given in this paper shows that as $Re\to\infty$ the index $\mu_q\to0$. We then find from Eq. (49) that for large Re the parameter λ_1 can no longer be a constant and that the parameters λ_2 and λ_3 tend to zero. In order that the dispersion of the logarithm of the dissipation increases when the Reynolds number increases it is then necessary that the coefficient λ_3 (for a power law asymptotic behavior of ξ) de-

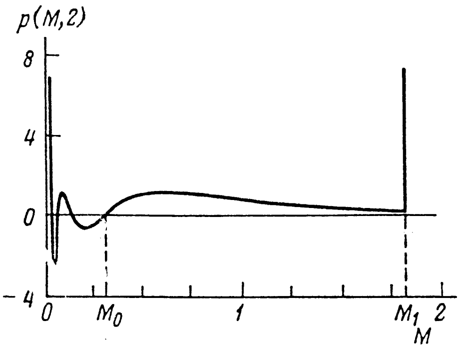

FIG. 4. The probability distribution density of the break-up coefficient when the scale is reduced by a factor 2, according to Eq. (48).

crease not faster than ξ^{-1}. We assume that the functions $\lambda_j(\xi)$ can for $\xi \gg 1$ be expanded in negative powers of ξ. As we are only interested in the main terms of the expansion we shall assume that

$$\lambda_1=\text{const}, \quad \lambda_2=0, \quad \lambda_3=\lambda/\xi, \quad \lambda=\text{const}. \tag{50}$$

In that case the main term of the asymptotic expansion of the function Φ which is uniform in q has the following form:

$$\Phi(\xi,q)=C(q)\exp\{-\lambda q(q-1)/[\lambda_1+(\lambda_1^2-\lambda q(q-1)/\xi)^{1/2}] -(\lambda/\lambda_1)q(q-1)\ln[\xi^{1/2}(\lambda_1+(\lambda_1^2-\lambda q(q-1)/\xi)^{1/2})]\}. \tag{51}$$

From the normalization conditions we have $C(0)=C(1)=1$. In agreement with Eq. (36) we get

$$\mu_q=\frac{1}{3\xi/4-\ln(l(q))}\{-\lambda q(q-1)/[\lambda_1+(\lambda_1^2-\lambda q(q-1)/\xi)^{1/2}] -(\lambda/\lambda_1)q(q-1)\ln[\xi^{1/2}(\lambda_1+(\lambda_1^2-\lambda q(q-1)/\xi)^{1/2})]+\ln[C(q)]\}. \tag{52}$$

The parameter l_q can in the general case depend on the order q of the corresponding moment. However, since the minimum scale of the eddies which can exist in the turbulent flow is of order η, we have $l_q \sim 1$. For the further asymptotic analysis it is sufficient that the logarithm of l_q be bounded.

In order that the function (52) for q with a large modulus satisfy the above mentioned Novikov inequality[12] and agree with experiment it is necessary that

$$\ln[C(q)]-(\lambda/\lambda_1)q^2\ln[(-\lambda)^{1/2}|q|]\leqslant\text{const}|q|.$$

Expression (52) shows that for fixed q and as $\xi \to \infty$ the moments of the turbulent energy dissipation rate probability distribution are described with arbitrary accuracy by the corresponding functions of the lognormal distribution. The intermittency parameter $\mu(\mu=[\partial^2_{qq}\mu_q]|_{q=0})$ has the asymptotic form

$$\mu\sim\ln(\xi)/\xi \quad \text{for} \quad \ln(\xi)\gg 1. \tag{53}$$

In actual experiments the logarithm of the logarithm of the Reynolds number reaches a maximum value of $\approx$3–4. For such values of ξ we find in accordance with Eq. (52)

$$\mu\sim[\ln(\xi)+\text{const}]/\xi \quad \text{for} \quad \xi\gg 1. \tag{54}$$

It is clear from Eq. (49) that the solution (37) obtained by assuming that the exponents μ_q are independent of the Reynolds number is no more than a convenient approximation of the experimental data. We note that the numerical values of the parameters used in the comparison with the results of measurements are not estimates of the coefficients in Eq. (34). The solution (37) satisfies the normalization conditions

$$\mu_0=\mu_1=0,$$

and for values of the parameter q with a large modulus there is a linear growth. The solution depends weakly (logarithmically) on the Reynolds number and it is therefore not surprising that good agreement with experiment is reached using constant fitting coefficients.

The experimental values of the intermittency index μ obtained by a number of authors (see, e.g., Ref. 31) vary considerably: $\mu \approx 0.2$–0.5. In the paper by Kuznetsov *et al.*[57] the smallest known value of the parameter, $\mu=0.15$, was found. The spread of the experimental data exceeds, apparently, the accuracy of the measurements. The nonuniversality of μ is determined in Ref. 57 by the variation of the external intermittence coefficient. The presently available experimental data do not allow us to accept or reject the hypothesis that μ is universal. We note that the range of variation of μ in the experiments is in qualitative agreement with Eq. (54): when the turbulent Reynolds number changes by several orders μ changes by a factor of two to three.

A model was proposed in Ref. 58 in which the parameters of the lognormal dissipation distribution are not universal constants but depend on the logarithm of the Reynolds number. One shows easily that the hypothesis of a power law dependence of the dispersion of the dissipation logarithm on the ratio r/L, assumed in the present paper, contradicts the scaling hypothesis of Ref. 12. In the proposed model, the scaling dependence of the dissipation moments, which follows from the scaling for $\eta \ll r \ll L$, is preserved. The dissipation distribution turns out to depend not only on the external scale L, a dependence which is the basis of the refinement of the theory of locally isotropic turbulence, but also on the internal (Kolmogorov) scale, i.e., on the Reynolds number.

7. CONCLUSION

We have shown the possibility of a semiempirically closing of the transfer equation for the energy dissipation rate probability distribution density. Using the invariant simulation method (a modification of the "$k-\varepsilon$" model) we found that when the turbulent Reynolds number increases without bounds the small-scale turbulence structure is described by the non-corrected 1941 theory of Kolmogorov and Obukhov.

The lognormal distribution for the dissipation is uniform for the leading moments of the asymptotic expression: one can indicate for the moment of any order a Reynolds number for which the moment is the same with prescribed accuracy as the asymptotic one. However, for any fixed Re there is a moment which differs arbitrarily strongly from the corresponding moment of the lognormal distribution. It is clear from Eq. (52) that the applicability region of this asymptotic behavior increases as the root of the logarithm of the Reynolds number. The intermittency parameter μ is not a universal constant,

$$\mu \sim [\ln(\ln(Re))+\text{const}]/\ln(Re) \quad \text{for} \quad \ln(Re)\gg 1.$$

The excess coefficient K of the velocity gradient has the following asymptotic form:

$$K \sim \ln(Re)+\text{const} \quad \text{for} \quad Re\gg 1.$$

We found the asymptotic expression for the moments of the distribution which satisfy the Novikov inequality and which agree well with experiment.

The proposed method enables us to model the ("$k-\varepsilon$" type) set of equations for turbulent motion starting from the theory of locally isotropic turbulence. The term describing the generation and dissipation in the semiempirical transport equation for the average energy dissipation rate must contain quantities connected with the inhomogeneity of the average energy and the average dissipation in order that the proposed model be valid.

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[1] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
[2] A. M. Obukhov, J. Fluid Mech. **13**, 77 (1962).
[3] L. D. Landau and E. M. Lifshitz, *Mechanics of Continuous Media*, Gostekhizdat, Moscow (1953) [English translation published by Pergamon, Oxford].
[4] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941).
[5] A. M. Obukhov, Dokl. Akad. Nauk SSSR **32**, 22 (1941).
[6] L. F. Richardson, Proc. R. Soc. London **A756**, 709 (1926).
[7] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **31**, 99 (1941).
[8] E. A. Novikov and R. W. Stewart, Izv. Akad. Nauk SSSR Ser. Geofiz. **3**, 408 (1964).
[9] A. M. Yaglom, Dokl. Akad. Nauk SSSR **166**, 49 (1966) [Sov. Phys. Dokl. **11**, 26 (1966)].
[10] A. S. Gurvich and A. M. Yaglom, Phys. Fluids, Suppl., p. 59 (1967).
[11] E. A. Novikov, Dokl. Akad. Nauk SSSR **168**, 1279 (1966) [Sov. Phys. Dokl. **11**, 497 (1966)].
[12] E. A. Novikov, Prikl. Mat. Mekh. **35**, 266 (1971).
[13] S. A. Orszag, Phys. Fluids **13**, 2211 (1970).
[14] R. H. Kraichnan, J. Fluid Mech. **62**, 305 (1974).
[15] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[16] U. Frisch, P.-L. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978).
[17] U. Frisch and G. Parisi, *Turbulence and Predictability in Geophysical and Climate Dynamics*, North-Holland, Amsterdam (1985) p. 84.
[18] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. **A17**, 3521 (1984).
[19] A. Yu. Turygin and V. P. Chechetkin, Zh. Eksp. Teor. Fiz. **98**, 146 (1990) [Sov. Phys. JETP **71**, 80 (1990)].
[20] C. Meneveau and K. R. Sreenivasan, Nucl. Phys., Suppl. **B2**, 49 (1987).
[21] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).
[22] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).
[23] D. Schertzer and S. Lovejoy, *Turbulent Shear Flows*, Springer, Berlin (1985) Vol. 4, p. 7.
[24] H. Yamazaki, J. Fluid Mech. **219**, 181 (1990).
[25] L. C. Andrews, R. L. Phillips, B. K. Shivamoggi *et al.*, Phys. Fluids **A1**, 999 (1989).
[26] I. Hosokawa, Phys. Fluids **A1**, 186 (1989).
[27] R. H. Kraichnan, Phys. Rev. Lett. **65**, 575 (1990).
[28] Z.-S. She, Phys. Rev. Lett. **66**, 600 (1991).
[29] Z.-S. She and S. A. Orszag, Phys. Rev. Lett. **66**, 1701 (1991).
[30] V. P. Kuznetsov and V. A. Sabel'nikov, Usp. Mekh. **4**, Nr 3, 81 (1981).
[31] V. P. Kuznetsov and V. A. Sabel'nikov, *Turbulence and Combustion* [in Russian], Nauka, Moscow (1986).
[32] V. Yakhot, Z.-S. She, and S. A. Orszag, Phys. Fluids **A1**, 289 (1989).
[33] E. D. Siggia, Phys. Rev. **A15**, 1730 (1977).
[34] E. D. Siggia, Phys. Rev. **A17**, 1166 (1978).
[35] K. Ohkitany and M. Yamada, Progr. Theor. Phys. **81**, 329 (1989).
[36] I. Hosokawa and K. Yamamoto, Phys. Fluids **A2**, 889 (1990).
[37] T. Higuchi, Physica **D46**, 254 (1990).
[38] M. H. Jensen, G. Paladin, and A. Vulpiani, Phys. Rev. **A43**, 798 (1991).
[39] J. Eggers and S. Grossmann, Phys. Lett. **A156**, 156 (1991).
[40] A. S. Monin and A. M. Yaglom, *Statistical Hydrodynamics*, Nauka, Moscow, Vol. 1 (1965); Vol. 2 (1967) [English translation published by MIT Press].
[41] B. I. Davydov, Dokl. Akad. Nauk SSSR **136**, 47 (1961) [Sov. Phys. Dokl. **6**, 10 (1961)].
[42] P. L. Van'yan, *Contemporary Problems in the Mechanics of Continuous Media and Geocosmic Studies*, MFTI, Moscow (1987) p. 13.
[43] A. S. Monin, Dokl. Akad. Nauk SSSR **177**, 1036 (1967) [Sov. Phys. Dokl. **12**, 1113 (1968)].
[44] A. S. Monin, Prikl. Mat. Mekh. **31**, 1057 (1967).
[45] T. S. Lundgren, Phys. Fluids **10**, 968 (1967).
[46] E. A. Novikov, Dokl. Akad. Nauk SSSR **177**, 299 (1967) [Sov. Phys. Dokl. **12**, 1006 (1968)].
[47] V. P. Kuznetsov, Prikl. Mat. Mekh. **31**, 1069 (1967).
[48] A. Erdélyi (ed.), *Tables of Integral Transforms* (California Institute of Technology H. Bateman MS Project), Vol. 1, McGraw–Hill, New York (1954).
[49] K. Hanjalic and B. E. Launder, J. Fluid Mech. **52**, 609 (1972).
[50] B. E. Launder and D. B. Spalding, Comput. Meth. Appl. Mech. Eng. **3**, 269 (1974).
[51] D. L. Lumley, *Calculation Methods for Turbulent Flows* (Russian translation), Mir, Moscow (1984) p. 7.
[52] G. Paladin and A. Vulpiani, Phys. Rept. **156**, 147 (1987).
[53] F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984).
[54] E. A. Novikov, Phys. Fluids **A2**, 814 (1990).
[55] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1964).
[56] M. V. Fedoryuk, *Asymptotic Methods for Linear Ordinary Differential Equations*, Nauka, Moscow (1983).
[57] V. P. Kuznetsov, A. A. Praskovskiĭ, and V. A. Sabel'nikov, Izv. Akad. Nauk, Mekh. Zhidk. Gaza **6**, 51 (1988).
[58] B. Castaing, Y. Gagne, and E. J. Hopfinger, Physica **D46**, 177 (1990).

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