Phase transitions in dimer systems

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The solution of the dimer problem on a plane lattice can be reduced to finding the fermion spectrum on the same lattice in a special magnetic field. We find the possible critical indices for phase transitions in dimer systems. We note that in a second-order phase transition point the system in general does not have conformal symmetry.

1. INTRODUCTION

An object occupying two neighboring lattice sites is called a dimer. A dimer covering is a distribution of dimers such that one and only one dimer starts from each lattice site. If we assign to each link in the lattice a well defined statistical weight the weight of a dimer covering is the product of the weights of all links occupied by dimers. The statistical sum of the dimer problem is thus the sum of the weights of all dimer coverings. In the case when the weights of all links are the same the problem is reduced to evaluating the number of possible dimer coverings of a given lattice.

In 1961 Kasteleyn[1] and Temperley and Fisher[2] found independently an exact solution of the dimer problem on a plane lattice by reducing the calculation of the statistical sum to the evaluation of the Pfaffian[1,2] of an antisymmetric matrix. In various dimer systems two phase transitions have been observed: an Ising type transition and a Kasteleyn type transition (see Ref. 3 and the references given there). In the next section we propose, on the basis of Ref. 4, what in our opinion is the most intuitive way for finding an exact solution of the dimer problem and which is ultimately in fact equivalent to the Pfaffian method (see, e.g., Ref. 5). In the later sections we discuss possible phase transitions in dimer systems.

2. DIMERS AND FERMIONS

We consider the dimer problem on a square lattice where z_v is the weight of the vertical links and z_h is the weight of the horizontal links. It is obvious how to change all discussions which follow in this section to other cases. The statistical sum has the form

$$Z=\sum_{P} z_v^{n_v} z_h^{n_h}. \tag{1}$$

The summation here is over all dimer coverings, P, and n_v (n_h) is the number of vertical (horizontal) dimers in the given covering P. It is clear that we have

$$n_v+n_h=N/2, \tag{2}$$

where N is the total number of lattice sites (which must be even). Squaring (1) we have

$$Z^2=\sum_{(P,P')} z_v^{n_v+n_v'} z_h^{n_h+n_h'}. \tag{3}$$

The summation in (3) is over ordered pairs of dimer coverings (P,P'), while n_v, n_h and n_v', n_h' are the numbers of vertical and horizontal dimers in the coverings P and P', respectively. We now consider a pair of typical dimer coverings (see Figs. 1a,b). We superpose the dimer coverings of Figs. 1a,b on one another (see Fig. 1c), obtaining a dimer configuration and closed contours. The "ordered pair of dimer coverings—dimer and closed contour configurations" correspondence is not one-to-one. To make it a one-to-one correspondence we assign a direction to a contour. This can, for instance, be done as follows. We number all lattice sites in an arbitrary way and then assign to each contour the direction in which the dimer from the site with the lowest number on the contour starts in the first dimer covering. For instance, in the case of the numbering corresponding to Fig. 1d, the direction of the contour is counter-clockwise. The summation in (3) can thus be taken over all configurations of closed directed contours. The weight of a configuration is then the product of the weights of all links occurring in the contours. A dimer is considered to be a directed contour going one way and back along the same link. Its contribution to the configuration weight is correspondingly the square of the weight of the link. We thus have

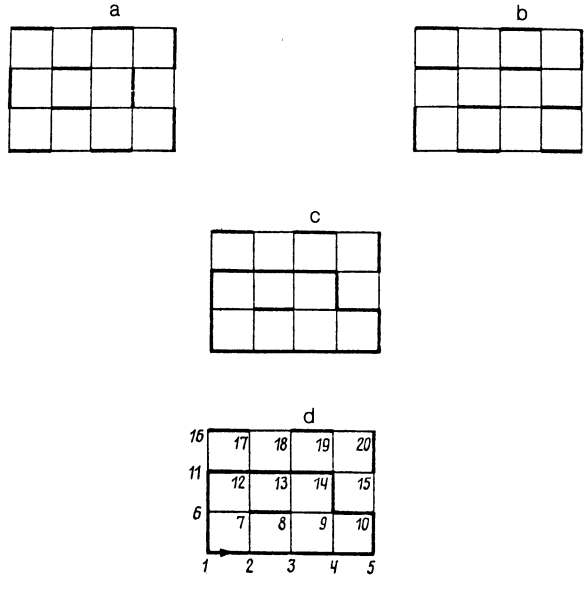

FIG. 1. a: Example of dimer covering with $n_v=4$, $n_h=6$; b: Example of dimer covering with $n_v=2$, $n_h=8$; c: Dimer and closed contour configurations obtained by superimposing the two dimer coverings of Figs. 1a and 1b; d: The closed contour has the direction corresponding to the direction of the dimer in Fig. 1a starting from the smallest number on the contour.

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$$Z=\sum_{C}\prod_{l\in C} z_l. \tag{4}$$

The summation here is over all configurations C of directed contours and the product is over the links l of the lattice which belong to a given contour configuration C; z_l is the statistical weight of link l in the lattice. In the case of the square lattice z_l is equal to z_v (z_h) if the link l is a vertical (horizontal) one. We shall now prove that the directed contours in (4) can be considered as closed world lines of fermions (one of the directions in the lattice plays the role of time, the other that of space). To do that we consider the following integral over the Grassmann variables ψ_i, $\bar{\psi}_i$ (i refers to the lattice sites)

$$Z_F=\int\prod_{i=1}^{N} d\psi_i\, d\bar{\psi}_i \exp\sum_{i,j=1}^{N}\bar{\psi}_i A_{ij}\psi_j, \tag{5}$$

where A_{ij} is an antisymmetric matrix

$$A_{ij}=-A_{ji}. \tag{6}$$

We expand the exponent in (5) in a series and use the rules for integration over Grassmann variables:

$$Z_F=\int\prod d\psi_i\, d\bar{\psi}_i \frac{1}{N!}\Big(\sum_{i,j=1}^{N}\bar{\psi}_i A_{ij}\psi_j\Big)^N, \tag{7}$$

or

$$Z_F=\int\prod d\psi_i\, d\bar{\psi}_i \frac{1}{N!} \times\sum_{i_k=1}^{N}[\bar{\psi}_{i_1}A_{i_1i_2}\psi_{i_2}\bar{\psi}_{i_3}A_{i_3i_4}\psi_{i_4}\dots\bar{\psi}_{i_{2N-1}}A_{i_{2N-1}i_{2N}}\psi_{i_{2N}}]. \tag{8}$$

In the sum in (8) $i_1,i_2,\dots,i_N$ run through all lattice sites independently. Changing the order in the product in (8) and using the integration rules we get from (8)

$$Z_F=\int\prod d\psi_i\, d\bar{\psi}_i \times\sum_{C}\prod_{\substack{c\in C\\ c=[i_1\dots i_n]}}[\bar{\psi}_{i_1}A_{i_1i_2}\psi_{i_2}\bar{\psi}_{i_2}A_{i_2i_3}\psi_{i_3}\dots\bar{\psi}_{i_n}A_{i_ni_1}\psi_{i_1}]. \tag{9}$$

The summation is over the configurations C of directed closed contours and the product over the contours $c=\{i_1,\dots,i_n\}$ from each configuration C. Integrating (9) we have

$$Z_F=\sum_{C}\prod_{\substack{c\in C\\ c=[i_1,\dots i_n]}}[(-1)^{n+1}A_{i_1i_2}A_{i_2i_3}\dots A_{i_ni_1}]. \tag{10}$$

We rewrite (10) in the form

$$Z_F=(-1)^{N/2} \times\sum_{C}\prod_{\substack{c\in C\\ c=[i_1,\dots i_n]}}[\exp[-i(\pi+\pi n/2)]A_{i_1i_2}A_{i_2i_3}\dots A_{i_ni_1}]. \tag{11}$$

Comparing (11) and (4) we see that if the absolute magnitudes of the A_{ij} are equal to the weights of the ij links in the lattice and the phases of the A_{ij} are chosen such that

$$\operatorname{Arg}[A_{i_1i_2}A_{i_2i_3}\dots A_{i_ni_1}]=\pi+\pi n/2 \tag{12}$$

for all closed contours on the lattice which may arise in the sum in (11) we have

$$Z^2=(-1)^{N/2}Z_F. \tag{13}$$

We show that one can make the choice (12) for the phases in the case of plane lattices and that one can choose the A_{ij} to be Hermitean. We choose all A_{ij} to be purely imaginary (satisfying the Hermiticity condition and (6) simultaneously) and in such a way that (12) is satisfied for each elementary plaquette in the lattice. This can be done because the number of variables (" + " and " − " signs on all links in the lattice) is equal to the number of bonds (number of elementary plaquettes in the lattice). We show that it follows that (12) is satisfied for all permissible contours if (12) is satisfied for all elementary plaquettes. We consider an arbitrary closed contour $\{i_1,\dots,i_n\}$ of length n and multiply the phases of all A_{ij} over all counterclockwise plaquettes inside it (see Fig. 2). The phases of the A_{ij} for all interior links cancel because of the Hermiticity of A_{ij} and we obtain

$$\exp[i\operatorname{Arg}[A_{i_1i_2}A_{i_2i_3}\dots A_{i_ni_1}]].$$

On the other hand, the same phase is equal to

$$\exp(i\pi K+i\pi(2L-n)/2),$$

where K is the number of plaquettes and L the total number of links on and inside the contour. From Euler's theorem for the diagram in Fig. 2, according to which $K-L+(V+n)=1$ (V is the number of interior vertices of the diagram), and the fact that V is even for diagrams of a dimer origin, Eq. (12) holds for any allowable contours from (11). The validity of (13) is thus proven and evaluating Z_F we obtain

$$Z^2=(-1)^{N/2}\det A, \tag{14}$$

$$|A_{ij}|=\begin{cases}\text{the weight of the link } ij \text{ if } i,j\\ \quad\text{are nearest neighbors,}\\ 0 \quad\text{otherwise,}\end{cases}$$

$$A_{ij}=A_{ji}^{*},$$

$$A_{i_1i_2}A_{i_2i_3}\dots A_{i_ni_1}=\exp i(\pi+\pi n/2)|A_{i_1i_2}A_{i_2i_3}\dots A_{i_ni_1}|, \tag{15}$$

$[i_1i_2\dots i_n]$ is any arbitrary plaquette in the lattice.

Note that the antisymmetry of A_{ij} is no longer required in (15), as it follows from the gauge invariance of (4) that (14) is completely determined by the conditions (15). The dimer problem is thus reduced to the problem of fermions on a

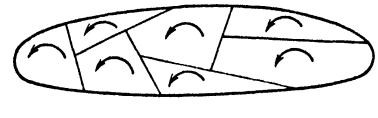

FIG. 2. Diagram formed by a closed contour and the plaquettes inside it. The arrows show the directions in which one must multiply the A_{ij} matrix elements.

lattice with a "magnetic field" with a flux $\Phi = \pi + \pi n/2$ (flux quantum equal to 2π) through an n-gonal plaquette. In concluding this section we note that our calculations are in fact parallel to those in Ref. 1. There is also a construction with a superposition of two dimer coverings upon one another in Ref. 6, albeit in another context.

3. FERMION SPECTRUM AND PHASE TRANSITIONS

We find from (14) the free energy

$$F=-T\ln Z$$

of the dimer system [the temperature T occurs in Z only through the weight of the dimers, for instance, $z_1=\exp(-\varepsilon_l/T)$]:

$$F=-T\,\mathrm{Tr}\ln A \tag{16}$$

apart from an unimportant constant. Having found the fermion spectrum (we assume the distribution of the weights to be periodic and the lattice to be infinite) we have

$$F=-\frac{T}{2}\sum_{\alpha}\iint_{B}\frac{d^2\mathbf{k}}{(2\pi)^2}\ln|E_\alpha(\mathbf{k})|. \tag{17}$$

The summation here is over all branches of the spectrum and the integration is over a Brillouin zone, and $\varepsilon(\mathbf{k})$ is the fermion spectrum. It is more convenient to rewrite (17) in the form

$$F=-\frac{T}{2}\iint_{B}\frac{d^2\mathbf{k}}{(2\pi)^2}\ln|\det A(\mathbf{k})|, \tag{18}$$

where $A(\mathbf{k})$ is the Fourier transform of the matrix A_{ij}. For instance, for the square lattice considered earlier we have

$$A(\mathbf{k})=\begin{pmatrix} -iz_h[\exp ik_1-\exp(-ik_1)] & z_v[\exp ik_2+\exp(-ik_2)] \\ z_v[\exp ik_2+\exp(-ik_2)] & iz_h[\exp ik_1-\exp(-ik_1)] \end{pmatrix} \tag{19}$$

in the gauge corresponding to Fig. 3 and we then find for the free energy

$$F=-\frac{T}{2}\iint_{B}\frac{d^2\mathbf{k}}{(2\pi)^2}\ln(4z_h^2\sin^2k_1+4z_v^2\sin^2k_2), \tag{20}$$

$$B:\ -\pi<k_1<\pi,\ -\pi/2<k_2<\pi/2.$$

We note that the free energy (17) depends in a complicated manner on the temperature because the temperature enters into the shape of the spectrum through the statistical weight of a link. In particular, when we change the temperature one of the branches of the spectrum in (17) may touch the zero of the energy. The temperature dependence of F then becomes singular and a phase transition occurs. We expand $\varepsilon(\mathbf{k})$ near the contact point in a series in the momentum and in the difference between the temperature and the critical temperature, and for the singular part of the free energy we write

$$F_{sing}\sim\iint\frac{d^2\mathbf{k}}{(2\pi)^2}\ln(\tau^2+k_1^2+k_2^2). \tag{21}$$

Here we have chosen the simplest form of tangency, and set $\tau=(T-T_c)/T_c$. Expression (21) formally diverges but we know that the singular part of F is determined by the $\mathbf{k}\approx 0$ region and we have thus

$$F_{sing}\sim\tau^2\ln\tau, \tag{22}$$

and for $C_{\text{sing}}\sim\partial^2F/\partial\tau^2$ we have

$$C_{sing}\sim\ln\tau. \tag{23}$$

We see that (23) is the same as the singular part of the specific heat for the Ising model. We have $\alpha=0$ for the critical index. This is not by accident. We show in the next section that for dimer systems which are equivalent to the Ising model[7] the singular part of the free energy is determined by an expression of the form (21). We consider a somewhat more general form for the singular part of F:

$$F_{sing}\sim\iint\frac{d^2\mathbf{k}}{(2\pi)^2}\ln(\tau^2+k_1^2+k_2^{2n}),\quad n=1,2,\ldots \tag{24}$$

Changing the variables,

$$k_1\to k_1\tau,\quad k_2\to k_2\tau^{1/n}, \tag{25}$$

we have

$$C_{sing}\sim\frac{1}{\tau^{1-1/n}}, \tag{26}$$

and the critical index α for the specific heat is

$$\alpha=1-\frac{1}{n},\quad n=1,2,\ldots \tag{27}$$

For $n=1$ we find, of course, $\alpha=0$ as for the Ising model (to obtain the logarithm in (24) we must proceed more rigorously). For $n=2$ we have $\alpha=\frac{1}{2}$. This index corresponds to the "$\frac{3}{2}$ order" transition or the Kasteleyn transition[5] (see next section). It is clear from (25) that for $n\neq 1$ the wavevectors k_1 and k_2 transform according to different scaling laws. In fact, k_1 has the same dimensionality as τ and k_2 has the dimensionality of $\tau^{1/n}$. This means that at second-order transition points, corresponding to (27) with $n\neq 1$, there is no conformal symmetry which usually is present.[8] This is connected with the pronounced anisotropy of the corresponding dimer model and thus with the different scaling laws along different directions in the lattice. One can, of course, consider also a more general form of (24). For instance, for

$$F_{sing}\sim\iint\frac{d^2\mathbf{k}}{(2\pi)^2}\ln(\tau^{2s}+\tau^{2p}k_1^{2m}+\tau^{2q}k_2^{2n}) \tag{28}$$

we have

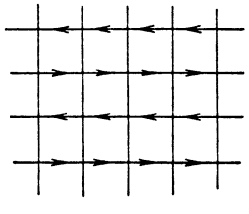

FIG. 3. The A_{ij} corresponding to $\langle ij\rangle$ links without arrows are purely real. If the direction of an $\langle ij\rangle$ link is the same as the arrow, we have $A_{ij}=i|A_{ij}|$, and if it is in the opposite direction, we have $A_{ij}=-i|A_{ij}|$.

$$C_{sing}\sim\tau^{-[2-(s-p)/m-(s-q)/n]}, \tag{29}$$

$$\alpha=2-\frac{s-p}{m}-\frac{s-q}{n}, \quad s,p,q,m,n=1,2,\ldots, s\geqslant p,\ s\geqslant q. \tag{30}$$

In this case only the terms which decrease most slowly as $\tau\to 0$ remain in (28) in the expansion in $\mathbf{k}$. Not all values of s, p, q, m, and n corrrespond to different indices. For instance, the cases

$$\varepsilon(\mathbf{k})\approx\tau^2+k_1^2+k_2^4,\ \varepsilon(\mathbf{k})\approx\tau^2+k_1^2+\tau k_2^2$$

give the same value of α since k_2 has the same scaling dimensionality in both cases. In general, (28) is not the most general formula for the singular part of F. Each term of the expansion in (28) is positive. However, the positive definiteness of the spectrum can be attained also by other means. For instance, a spectrum of the form

$$E(\mathbf{k})=(\tau-k_1^2+k_2^2)^2+k_2^2 \tag{31}$$

is positive definite although there are also negative terms in its expansion. Moreover, the spectrum (31) touches the $\varepsilon=0$ level for $\tau>0$ while for $\tau=0$ only a jump from the zero level can occur. An analysis of the scaling dimensionalities in (31) shows that the transition index is $\alpha=\frac{1}{2}$. It is just this realization of a transition with $\alpha=\frac{1}{2}$ which we deal with in a Kasteleyn transition (see next section).

4. ISING AND KASTELEYN TRANSITIONS

It was shown in Ref. 7 that the model of dimers on a lattice shown in Fig. 4 is equivalent to the Ising model with

$$z_1=\mathrm{th}(J_1/T),\ z_2=\mathrm{th}(J_2/T).$$

Calculations give for $\det A(\mathbf{k})$

$$\det A(\mathbf{k})=(1+z_1^2)(1+z_2^2)+2z_1(z_2^2-1)\times\cos k_1+2z_2(z_1^2-1)\cos k_2. \tag{32}$$

Expression (32) attains its minimum value for $\mathbf{k}=0$ since $z_1<1$, $z_2<1$ hold, and we have

$$\det A(\mathbf{k})\approx\frac{[(1-z_1^2)(1-z_2^2)-4z_1z_2]^2}{(1+z_1^2)(1+z_2^2)+2z_1(1-z_2^2)+2z_2(1-z_1^2)+z_1(1-z_2^2)k_1^2+z_2(1-z_1^2)k_2^2}. \tag{33}$$

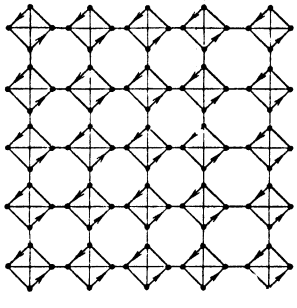

FIG. 4. A lattice the dimer problem of which is equivalent to the Ising model on a square lattice. The phases is the same as for Fig. 3. The magnitude of A_{ij} is equal to 1, if $\langle ij\rangle$ is the side of a square or its diagonal, and it is z_1 (z_2), if $\langle ij\rangle$ is not part of the square and it is horizontal (vertical).

It is clear from (33) that the transition point is determined by the equation

$$\tau\equiv(1-z_1^2)(1-z_2^2)-4z_1z_2=0.$$

One checks easily that this condition is equivalent to the well known condition

$$\mathrm{sh}(2J_1/T)\,\mathrm{sh}(2J_2/T)=1.$$

We thus have near the transition point

$$\det A(\mathbf{k})\approx a_0\tau^2+a_1k_1^2+a_2k_2^2,$$

where a_0, a_1, and a_2 are constants. We have thus verified that (21), indeed, corresponds to the Ising transition.

We now consider the lattice shown in Fig. 5. A phase transition with $\alpha=\frac{1}{2}$ was observed on this lattice in Ref. 7. Indeed, evaluating $\det A(\mathbf{k})$ we have

$$\det A(\mathbf{k})=z^2+4\cos^2 k_1+4z\cos k_1\cos k_2. \tag{34}$$

For $z>2$ (34) attains its minimum value for $(k_1,k_2)=(0,\pi),(\pi,0)$. Expanding near the minimum we have

$$\det A(\mathbf{k})\approx(z-2)^2+2(z-2)k_1^2+\frac{8-z}{6}k_1^4+2zk_2^2. \tag{35}$$

or, writing $\tau\equiv z-2$ and retaining the main terms in τ

$$\det A(\mathbf{k})\approx(\tau+k_1^2)^2+4k_2^2. \tag{36}$$

A scale analysis of (36) gives $\alpha=\frac{1}{2}$ but this conclusion turns out to be invalid because of the "enhanced symmetry" of (36). Substitution of (36) into the expression for the free energy shows no singularity as $\tau\to+0$. As $\tau\to-0$ the scale analysis gives the correct result. In fact, for $\tau<0$ the spectrum touches zero and the contact point moves to the points $(0,\pi),(\pi,0)$ as τ tends to zero, while for $\tau=0$ there is a jump in the spectrum from the zero energy value. One should note that the "enhanced symmetry" of (36) is caused by the additional constant of motion which occurs in the dimer problem on the lattice of Fig. 5 (see Ref. 3). Unfortunately at the present time there are no examples known of dimer systems with a critical behavior differing from Ising and Kasteleyn type transitions. The absence of such examples in the literature indicates either the complexity of such systems or the existence of a rule, unknown to me, forbidding fermions on a lattice with the magnetic field of (15) to have a spectrum which touches the $\varepsilon=0$ level in an arbitrary way.

5. CONCLUSION

In the present paper we have given a formulation of the dimer problem on an arbitrary plane lattice in the language of fermions on the same lattice with a special magnetic field.

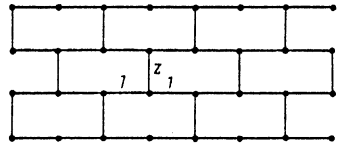

FIG. 5. All A_{ij} are real. We have $|A_{ij}|=z$ if $\langle ij\rangle$ is vertical and $|A_{ij}|=1$ if $\langle ij\rangle$ is horizontal.

Finding the free energy thus reduces to finding the fermion spectrum. We have found the possible values of the critical index α for phase transitions in dimer systems. We have shown that for second-order phase transitions in such systems there may not be conformal symmetry at the transition point. It would be interesting to find examples of critical behavior in dimer systems with α different from $\alpha = 0$ or $\alpha = \frac{1}{2}$. It would also be interesting to find other critical indices which are possible in dimer systems.

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[1]P. W. Kasteleyn, Physica **27**, 1209 (1961).
[2]H. N. V. Temperley and M. E. Fisher, Phil. Mag. **6**, 1061 (1961).
[3]J. F. Nagle and C. S. O. Yokoi, Physica **A163**, 1401 (1990).
[4]S. Samuel, J. Math. Phys. **21**, 2806 (1980).
[5]P. W. Kasteleyn, J. Math. Phys. **4**, 287 (1963).
[6]M. Kohmoto and Y. Shapir, Phys. Rev. **B37**, 9439 (1987).
[7]P. W. Kasteleyn, in *Graph Theory and Theoretical Physics* (ed. F. Harary), Academic, New York.
[8]A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **12**, 538 (1970) [JETP Lett. **12**, 381 (1970)].

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