Nonlinear equations for Green's functions and calculation of turbulent diffusion coefficients

N. A. Silant'ev

Main Astronomical Observatory, Russian Academy of Sciences

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A hierarchy of nonlinear equations is constructed for average Green's functions describing the transport of passive fields (a particle-concentration field, a magnetic field, or a temperature field) in a given turbulent medium. It is not assumed that the ensemble of turbulence velocities is Gaussian. In the case of an incompressible medium, these equations can be solved effectively by a numerical method analogous to the method of successive iterations in the calculation of continued fractions. Solutions are derived for the simplest nonlinear equation, with a quadratic nonlinearity. These solutions are then used to calculate the turbulent diffusion coefficients for a Kolmogorov turbulence spectrum and for a δ-function spectrum. This method can be used to calculate the diffusion coefficient for spectra with arbitrary values of the parameter $\xi = u_0\tau_0/R_0$, where u_0, τ_0, and R_0 are typical values of the velocity, the lifetime, and the size of the turbulent fluctuations.

1. INTRODUCTION

The diffusion coefficients for (passive) impurity fields (a particle concentration field, a magnetic field, or a temperature field) must be calculated for various turbulent media in order to describe phenomena which occur in astrophysics, meteorology, geophysics, hydrodynamics, and other settings. An expression for the turbulent diffusion coefficient D_T was derived in the Lagrangian representation in the well-known paper by Taylor.[1] In most cases, however, the turbulent velocity field $\mathbf{u}(\mathbf{r},t)$ of the host liquid or gas is known in the Eulerian representation, and it becomes necessary to calculate D_T in this representation specifically. Going over from the Eulerian representation to the Lagrangian representation is known to be a difficult problem, which has not been solved in its general form. The problem of calculating D_T has been taken up in many papers (e.g., Refs. 2–5). The basic results which have been derived are given along with a comprehensive bibliography in some monographs.[6–8]

The turbulent diffusion coefficient D_T depends strongly on the parameter $\xi = u_0\tau_0/R_0 \equiv \tau_0/t_0$, where u_0, τ_0, and R_0 are typical values of the velocity, the lifetime, and the size of the turbulence fluctuations, with $t_0 = R_0/u_0$. The diffusive mixing length L is known to be related to D_T and to the time t by $L^2 \approx D_T t$. On the other hand, turbulent diffusion stems from the transport of a passive field by a fluid element, and the quantity L must be proportional to the characteristic velocity u_0 of the convective motion, if we take the time t to be the shorter of the time scales τ_0, t_0. In this case, L becomes the mean free path of a liquid particle. In the limit $\tau_0 \ll t_0 (\xi \ll 1)$ we have $L \approx u_0\tau_0$ and $t \simeq \tau_0$, and we find $D_T \approx u_0^2\tau_0$. In the opposite case, $\tau_0 \gg t_0 (\xi \gg 1)$, we have $L \approx u_0 t_0 \approx R_0$, and $t \simeq t_0$, and we find $D_T \simeq u_0 R_0$. In other words, the turbulent diffusion coefficient is independent of τ_0. An additional possible combination of the parameters R_0^2/τ_0 which has the dimensions of a diffusion coefficient is not physical, since it does not include the scale velocity u_0, which is the only dynamic parameter of the turbulent motion which determines the spatial transport of the impurity field. The ratio of the limiting values of the diffusion coefficients, $u_0^2\tau_0/u_0 R \approx \xi$, can be extremely large, so the problem of deriving a theory which predicts the correct values of D_T in both limiting cases would appear to be a rather difficult one.

In most of the studies which have been carried out, the numerical and analytic methods have been valid for calculation D_T only in the case $\xi \ll 1$. In Ref. 5, as well as Refs. 9 and 10, a procedure for renormalizing the equation for the fluctuational part of the impurity field was proposed. That renormalization makes it possible to calculate D_T up to $\xi \approx 1$. In this method, the new equation for the Green's function has as a free term not the molecular Green's function

$$G_m(R,\tau) = \theta(\tau)(4\pi D_m\tau)^{-3/2}\exp(-R^2/4D_m\tau)$$

but the function $M(R,\tau)$, which gives a more direct description of the convective nature of the impurity transport. In particular, for large values of R, this function is the same as

$$G_T(R,\tau) = \theta(\tau)(4\pi D_T\tau)^{-3/2}\exp(-R^2/4D_T\tau)$$

with the diffusion coefficient

$$D_T = u_0{}^2\tau_0/3$$

(we are assuming the case $D_m \ll D_T$).

In the present paper we generalize Ref. 5. Instead of the function $M(R,\tau)$ we use a solution of the nonlinear equation for the average Green's function $\langle G(R,\tau)\rangle$. It thus becomes possible to find D_T for turbulence spectra with an arbitrary value of the parameter $\xi (0 \ll \xi < \infty)$. For the case of an incompressible medium, finding a solution of the nonlinear equation reduces to a rapidly converging procedure similar to the method of successive iterations in the treatment of continued fractions. The calculation becomes elementary in the computational sense. In addition, the numerical solutions found here as well as the analytic asymptotic form of the nonlinear equations may prove useful for describing the

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time evolution of the fluctuations in impurity fields in a nondiffusive approximation.

2. FORMULATION OF THE PROBLEM

For impurity fields of all types, the evolution of the impurity is described by the equation

$$\left(\frac{\partial}{\partial t}-D_m\nabla^2\right)f(\mathbf{r},t)\equiv\mathscr{L}_0 f=\mathscr{L}(\mathbf{r},t)f(\mathbf{r},t), \qquad (1)$$

where D_m is the molecular diffusion coefficient, $f(\mathbf{r},t)$ is the impurity field, and the operator $\mathscr{L}(\mathbf{r},t)$ describes the effect of the background liquid or gas on the impurity transport process. For the impurity concentration field $n(\mathbf{r},t)$ we have

$$\mathscr{L}n=-\mathrm{div}(\mathbf{u}n)=-(\mathbf{u}\nabla)n-n\,\mathrm{div}\,\mathbf{u}.$$

For the temperature field $T(\mathbf{r},t)$ in an ideal gas[10] we have

$$\mathscr{L}T=-(\mathbf{u}\nabla)T-T[(C_P-C_V)/C_V]\,\mathrm{div}\,\mathbf{u},$$

where C_P and C_V are the specific heats of the gas at constant pressure and at constant volume, respectively. For the diffusion of a magnetic field $\mathbf{B}(\mathbf{r},t)$ we have $\mathscr{L}\mathbf{B}=\mathrm{curl}[\mathbf{uB}]$. The theory derived below is completely valid for any of these fields. For definiteness, we will restrict the discussion below to the field of impurity particles with a concentration $n(\mathbf{r},t)$ in certain specific calculations, we will consider only the case of incompressible turbulence (div $\mathbf{v}=0$):

$$\left(\frac{\partial}{\partial t}-D_m\nabla^2\right)n(\mathbf{r},t)=-u(\mathbf{r},t)\nabla n(\mathbf{r},t)\equiv\mathscr{L}n. \qquad (2)$$

We consider the diffusion of particles in an infinite, homogeneous, and isotropic turbulent medium. The two-point correlation function for the velocities of the liquid or gas in such a medium is[11]

$$\langle u_i(1)u_j(2)\rangle\equiv B_{ij}(\mathbf{R},\tau)=\delta_{ij}B_\perp(R,\tau)$$
$$+R_iR_jR^{-2}(B_\parallel(R,\tau)-B_\perp(R,\tau))+e_{ijp}R_pC(R,\tau). \qquad (3)$$

Here $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$ and $\tau=t_1-t_2$. The angle brackets $\langle\ ...\ \rangle$ mean an average over the ensemble of realizations of the turbulence velocity field $\mathbf{u}(\mathbf{r},t)$; and $B_\parallel$ and $B_\perp$ describe the correlation of the velocity components respectively along and perpendicular to $\mathbf{R}$. The function $C(R,\tau)$ describes the possible helicity of the medium $[\langle\mathbf{u}(1)\ \mathrm{rot}\ \mathbf{u}(1)\rangle=6C(0,0)]$. For an incompressible medium we have

$$B_\perp=B_\parallel+(R/2)\,\partial B_\parallel/\partial R.$$

We also assume $\langle\mathbf{u}\rangle=0$, i.e., that the medium as a whole is at rest. The Fourier transform of $B_{jq}(\mathbf{R},\tau)$ in the variable $\mathbf{R}$ has a simple form in the case of an incompressible medium:

$$\tilde{B}_{jq}(\mathbf{p},\tau)=\Pi_{jq}(\mathbf{p})f(p,\tau)+ie_{jqt}p_tD(p,\tau), \qquad (4)$$

where

$$\Pi_{jq}(\mathbf{p})=\delta_{jq}p^2-p_jp_q,\quad f(p,\tau)=-\frac{2}{p}\frac{\partial\tilde{B}_\parallel(p,\tau)}{\partial p},$$
$$D(p,\tau)=\frac{1}{p}\frac{\partial\tilde{C}(p,\tau)}{\partial p}. \qquad (5)$$

Below we will use the notation

$$d1=d\mathbf{r}_1dt_1,\quad d2=d\mathbf{r}_2dt_2,\quad f(n)=f(\mathbf{r}_n,t_n),$$
$$f(1-2)=f(\mathbf{r}_1-\mathbf{r}_2,\ t_1-t_2)$$

etc.

An integral equation for the Green's function of Eq. (2) [or of Eq. (1)] is

$$G(1,2)=G_m(1-2)+\int d3G_m(1-3)\mathscr{L}(3)G(3,2), \qquad (6)$$

where

$$G_m(1-2)\equiv G_m(R,\tau)=\theta(\tau)(4\pi D_m\tau)^{-3/2}\exp(-R^2/4D_m\tau)$$

is the Green's function of the operator $\mathscr{L}_0$, and $\theta(\tau)=1$ for $\tau>0$ and $\theta(\tau)=0$ for $\tau<0$. Using the method of successive approximations to find $G(1,2)$ and thus the diffusion coefficient D_T from (6) is exceedingly inefficient, since the expansion is carried out in terms of the molecular Green's functions G_m, which do not describe the convective nature of the particle transport. It becomes necessary to go over to a new, renormalized form of the equation for $G(1,2)$. In this new equation, the particle transport by the turbulence would be taken into account even in the free term. Direct substitution quickly verifies that Eq. (6) is equivalent to the following two equations:

$$M(1-2)=G_m(1-2)+\int d3\int d4G_m(1-3)K(3-4)M(4-2), \qquad (7)$$

$$G(1,2)=M(1-2)+\int d3M(1-3)\Big[\mathscr{L}(3)G(3,2)$$
$$-\int d4K(3-4)G(4,2)\Big]. \qquad (8)$$

Substitution of (7) into (8) leads to (6). The system of equations (7), (8) is useful in that Eq. (7) can be solved exactly, and explicitly, by going over to Fourier transforms in $\mathbf{R}$ and τ. On the other hand, it is possible to choose a kernel $K(3–4)$ in such a way that the function $M(R,\tau)$ would largely describe the convective nature of the particle transport. The following average kernel was adopted as $K(3–4)$ in Refs. 5, 9, and 10:

$$K(3-4)=\langle\mathscr{L}(3)G_m(3-4)\mathscr{L}(4)\rangle. \qquad (9)$$

Equation (7) in this case, for a turbulence with a "short memory,"

$$f(p,\tau)=2f(p)\exp(-\tau^2/\tau_0^2)/\pi^{1/2}\tau_0 \quad \text{as} \quad \tau_0\to 0$$

becomes the exact equation for the average Green's function $\langle G(1,2)\rangle$, and the function

$$M(R,\tau)=\langle G(1,2)\rangle$$

becomes the diffusion Green's function $G_T(R,\tau)$, with the limiting value of the diffusion coefficient:[13]

$$D_T=u_0^2\tau_0/3.$$

Equation (7) with kernel (9) is the same as the equation for the average Green's function $\langle G(R,\tau)\rangle$ if the parameter $\xi R_0/l$ is small, where l is the length scale of the substantial variations in the average quantities, in particular, the average Green's function $\langle G\rangle$ itself. The averaging is usually carried out over scales much larger than the length scale of the turbulence fluctuations, R_0, so the relation $R_0/l\ll 1$ holds, and the function $M(R,\tau)$ gives a good idea of the average Green's function up to $\xi\simeq 1$. This circumstance makes it possible, by taking successive iterations of (8), to calculate the turbulent diffusion coefficient D_T for turbulence spectra with values of the parameter ξ from zero to one.

3. EXPRESSION FOR THE TURBULENT DIFFUSION COEFFICIENT

Let us derive an expression for D_T in terms of the Green's function $G(1,2)$. We write all quantities as the sum of an average value and a fluctuating part: $n = \langle n \rangle + n_1$, $G = \langle G \rangle + G_1$, etc. Here $\langle n_1 \rangle = 0$ and $\langle G_1 \rangle = 0$. Taking the average of (2), we find

$$\mathscr{L}_0\langle n\rangle=-\langle(\mathbf{u}\nabla)n_1\rangle\equiv Q(\mathbf{r},t)=\langle\mathscr{L}n_1\rangle, \tag{10}$$

$$\mathscr{L}_0 n_1=\mathscr{L}\langle n\rangle-\langle\mathscr{L}n_1\rangle+\mathscr{L}n_1. \tag{11}$$

The equation for $n_1(\mathbf{r},t)$ differs from the original equation, (2), only in the presence of the term $\mathscr{L}\langle n\rangle - \langle \mathscr{L} n_1\rangle$. Treating this term as a source, and using the Green's function $G(1,2)$ of Eq. (2), we can therefore write

$$n_1(1)=\int d2G(1,2)\,[\mathscr{L}(2)\langle n(2)\rangle-Q(2)]. \tag{12}$$

Applying the operator $\mathscr{L}(1)$ to (12), and taking an average, we find an equation for the quantity $Q(1)$:

$$\begin{aligned}Q(1)=\int d2[\langle\mathscr{L}(1)G(1,2)\mathscr{L}(2)\rangle\langle n(2)\rangle\\ -\langle\mathscr{L}(1)G(1,2)\rangle Q(2)].\end{aligned} \tag{13}$$

Applying the operator $\mathscr{L}(1)$ to Eq. (6), written as an infinite series of iterations, we easily find the identity

$$\langle\mathscr{L}(1)G(1,2)\rangle=\int d3\langle\mathscr{L}(1)G(1,3)\mathscr{L}(3)\rangle G_m(3-2). \tag{14}$$

Using expression (14), we can write the equation for $Q(1)$ in its final form:

$$\begin{aligned}Q(1)=\int d2\langle\mathscr{L}(1)G(1,2)\mathscr{L}(2)\rangle\langle n(2)\rangle\\ -\int d2\int d3\langle\mathscr{L}(1)G(1,3)\mathscr{L}(3)\rangle G_m(3-2)Q(2).\end{aligned} \tag{15}$$

By virtue of the homogeneity and steady-state nature of the ensemble of turbulence velocities, the expressions inside the averaging symbols depend on the differences between the coordinates and the times. In other words, Eq. (15) can be solved explicitly by going over to Fourier transforms:

$$Q(\mathbf{r},t)=\frac{1}{(2\pi)^4}\int d\mathbf{p}\int_{-\infty}^{\infty} d\omega\tilde{Q}(p,\omega)\exp(-i\mathbf{p}\mathbf{r}-i\omega t). \tag{16}$$

For an incompressible medium, with $\mathscr{L}f= -(\mathbf{u}\nabla)f$, we find

$$\begin{aligned}\tilde{Q}(p,\omega)&=\frac{p_ip_j\tilde{R}_{ij}(\mathbf{p},\omega)\langle\tilde{n}(p,\omega)\rangle}{1+p_sp_r\tilde{R}_{sr}(\mathbf{p},\omega)\tilde{G}_m(p,\omega)}\\ &\equiv p_ip_j\tilde{D}_{ij}(\mathbf{p},\omega)\langle\tilde{n}(p,\omega)\rangle,\end{aligned} \tag{17}$$

where $\tilde{R}_{ij}(\mathbf{p},\omega)$ is the Fourier transform of the correlation function

$$\langle u_i(1)G(1,2)u_j(2)\rangle,$$

and

$$\tilde{G}_m(p,\omega)=(i\omega+D_mp^2)^{-1}.$$

Taking the inverse Fourier transforms, we find the kernel $D_{ij}(\mathbf{R},\tau)$ from (17). As a result, Eq. (10) becomes

$$\begin{aligned}\left(\frac{\partial}{\partial t}-D_m\nabla^2\right)\langle n(\mathbf{r},t)\rangle\\ =\nabla_i\int d\mathbf{R}\int_0^t d\tau D_{ij}(\mathbf{R},\tau)\nabla_j\langle n(\mathbf{r}-\mathbf{R},t-\tau)\rangle.\end{aligned} \tag{18}$$

It is easy to see that the tensor $\tilde{D}_{ij}(\mathbf{p},\omega)$ changes relatively little in magnitude, since the function $\tilde{R}_{ij}$ appears in both the numerator and the denominator of expression (17). This circumstance means that, in contrast, the tensor $D_{ij}(\mathbf{R},\tau)$ is a strong function of R and τ in the coordinate-time representation, with scale values $\sim R_0$ and scale times $\sim\tau_0$ or $\sim t_0$ for the case $\tau_0 \gg t_0 = R_0/u_0$. Assuming that $\langle n(\mathbf{r},t)\rangle$ is a sufficiently smooth function over these time and length scales, we can take $\langle n(\mathbf{r}-\mathbf{R},t-\tau)\rangle$ out of the integral in (18) at the points $r = 0$ and $\tau = 0$. As a result, Eq. (18) becomes the diffusion equation

$$\left(\frac{\partial}{\partial t}-(D_m+D_T)\nabla^2\right)\langle n(\mathbf{r},t)\rangle=0, \tag{19}$$

where the turbulent diffusion coefficient is

$$\begin{aligned}D_T&=\frac{1}{3}\int d\mathbf{R}\int_0^\infty d\tau D_{ii}(\mathbf{R},\tau)\equiv\frac{1}{3}\tilde{R}_{ii}(0,0)\\ &=\frac{1}{3}\int d\mathbf{R}\int_0^\infty d\tau\langle u_i(1)G(1,2)u_i(2)\rangle.\end{aligned} \tag{20}$$

Making use of the sharpness of the functional dependence $D_{ij}(\mathbf{R},\tau)$, we have extended the integration over τ to infinity. Substituting some expression for $G(1,2)$ into (20), we find specific formula for D_T.

The velocity field is usually assumed to be Gaussian. In other words, the velocity correlation function of odd order are assumed to be zero, and those of even order are assumed to be equal to the sum of all possible products of binary correlation functions. One can show by direct iterations of Eq. (15) that for a Gaussian ensemble of the velocity field the quantity $Q(1)$ is equal to the irreducible part of the free term in Eq. (15):

$$\begin{aligned}Q(1)=\langle\mathscr{L}n_1\rangle=\text{irreducible part of }\int d2\langle\mathscr{L}(1)G(1,2)\mathscr{L}(2)\rangle\\ \times\langle n(2)\rangle.\end{aligned} \tag{21}$$

We recall[12] that the reducible expressions (which are weakly coupled) have a structure

$$\int d2\int d3\int d4\langle A(1,3)\rangle G_m(3-4)\langle B(4,2)\rangle\langle n(2)\rangle.$$

In other words, the average blocks are separated by a molecular Green's function. The kernels of such expressions are not sharp functions of the time. The reducible expressions describe a gradual establishment of an average concentration $\langle n\rangle$. In the calculation of the source

$$Q(1)=-\operatorname{div}\langle\mathbf{F}\rangle,$$

which is equal to the divergence of the average particle flux $\langle \mathbf{F}(1)\rangle$, the nearest space-time neighborhood of the observation point, described by correlation functions of an irreducible (strongly coupled) type, is important. One can show that the set of irreducible correlation functions describes an elementary event in which the turbulence interacts with the impurity field (with length scales $\sim R_0$ and time scales $\sim \tau_0$ or t_0), while the reducible correlation functions incorporate the temporal contribution of these interactions in the shaping of the average concentration $\langle n\rangle$. This inspection of the Gaussian case clarifies the reason for the sharpness of the kernel $D_{ij}(\mathbf{R},\tau)$ and, in general, of non-Gaussian ensembles of turbulent velocities. To go over from expression (21), which is exact for a Gaussian ensemble, to an approximate, diffusion expression involves taking $\langle n(2)\rangle$ out of the integral at point 1. As a result, we find expression (20) for D_T, in which we can omit the words "irreducible part of," since the reducible terms disappear after an integration by parts.

4. HIERARCHY OF NONLINEAR EQUATIONS FOR $\langle G(1,2)\rangle$

Equations (10) and (11) show that the average concentration $\langle n\rangle$ depends on the contribution of fluctuations, which is determined in turn by the average concentration. The meaning here is that an individual equation for $\langle n\rangle$ should be nonlinear. Furthermore, incorporating contributions from all fluctuations—those which are remote and those which are close in space-time—leads to equations with progressively higher degrees of nonlinearity. In other words, the result is a hierarchy of nonlinear equations. We will illustrate the method for deriving such a hierarchy of equations for the average Green's function $\langle G(1,2)\rangle$.

Equations analogous to (10)–(12) can be written for $\langle G(1,2)\rangle$ and $G_1(1,2)$. In particular, the following expressions hold:

$$\langle G(1,2)\rangle = G_m(1-2) + \int d3\, G_m(1-3)\langle \mathscr{L}(3) G(3,2)\rangle, \tag{22}$$

$$G_1(1,2) = \int d3\, G(1,3)[\mathscr{L}(3)\langle G(3,2)\rangle - \langle \mathscr{L}(3) G(3,2)\rangle]. \tag{23}$$

Adding (23) to $\langle G(1,2)\rangle$, we find an equation for $G(1,2)$ which does not formally contain molecular Green's functions:

$$G(1,2) = \langle G(1,2)\rangle + \int d3\, G(1,3)[\mathscr{L}(3)\langle G(3,2)\rangle - \langle \mathscr{L}(3) G(3,2)\rangle]. \tag{24}$$

Equation (24) cannot, of course, be used to calculate the Green's function $G(1,2)$ directly, since the average Green's function $\langle G(1,2)\rangle$ itself depends on $G(1,2)$. Equation (24) makes it an extremely simple matter to derive a hierarchy of nonlinear equations for $\langle G(1,2)\rangle$.

Expanding $G(1,2)$ in a power series in the operator $\mathscr{L}$, and substituting the result into (24), we find a series of iterations for $G(1,2)$ in which the functions $\langle G(1,2)\rangle \equiv g(1-2)$ are used:

$$\begin{aligned} G(1,2) &= g(1-2) + \int d3\, g(1-3)\mathscr{L}(3) g(3-2) \\ &+ \int d3 \int d4\, g(1-3)[\mathscr{L}(3) g(3-4)\mathscr{L}(4) - \langle \mathscr{L}(3) g(3-4)\mathscr{L}(4)\rangle] g(4-2) \\ &+ \int d3 \int d4 \int d5\, g(1-3)[\mathscr{L}(3) g(3-4)\mathscr{L}(4) g(4-5)\mathscr{L}(5) \\ &- \langle \mathscr{L}(3) g(3-4)\mathscr{L}(4)\rangle g(4-5)\mathscr{L}(5) - \mathscr{L}(3) g(3-4) \\ &\times \langle \mathscr{L}(4) g(4-5)\mathscr{L}(5)\rangle] g(5-2) + \dots . \end{aligned} \tag{25}$$

It is simple to show that each integral term in (25) vanishes when the averaging is taken. If we apply the operator $\mathscr{L}(1)$ to (25), and take an average, we find several approximations for the quantity $\langle \mathscr{L}(1) G(1,2)\rangle$, which appears in the integral term of Eq. (22) for $\langle G(1,2)\rangle$. This series can be expressed as a power series in the average Green's function $g(1-2)$ itself:

$$\begin{aligned} \langle \mathscr{L}(1) G(1,2)\rangle &= \int d3 \langle \mathscr{L}(1) g(1-3)\mathscr{L}(3)\rangle g(3-2) \\ &+ \int d3 \int d4 \int d5 [\langle \mathscr{L}(1) g(1-3)\mathscr{L}(3) \\ &\times g(3-4)\mathscr{L}(4) g(4-5)\mathscr{L}(5)\rangle \\ &- \langle \mathscr{L}(1) g(1-3)\langle \mathscr{L}(3) g(3-4)\mathscr{L}(4)\rangle g(4-5)\mathscr{L}(5)\rangle \\ &- \langle \mathscr{L}(1) g(1-3)\mathscr{L}(3)\rangle g(3-4) \\ &\times \langle \mathscr{L}(4) g(4-5)\mathscr{L}(5)\rangle] g(5-2) + \dots . \end{aligned} \tag{26}$$

Substituting the terms of this series into Eq. (22), we find approximate nonlinear equations for $\langle G(1,2)\rangle = g(1-2)$. Retaining only the first term from (26), we find an extremely simple nonlinear equation with a quadratic nonlinearity:

$$g(1-2) = G_m(1-2) + \int d3 \int d4\, G_m(1-3) \times \langle \mathscr{L}(3) g(3-4)\mathscr{L}(4)\rangle g(4-2). \tag{27}$$

Substituting in the first and second terms from (26), we find an equation for $\langle G\rangle$ which also contains $\langle G\rangle$ raised to the fourth power, etc. For Gaussian ensembles of the velocity field, the hierarchy of nonlinear equations simplifies slightly. For example, the equation containing the second and fourth powers of $g(1-2)$ becomes

$$\begin{aligned} g(1-2) &= G_m(1-2) + \int d3 \int d4\, G_m(1-3) \\ &\times \langle \mathscr{L}(3) g(3-4)\mathscr{L}(4)\rangle g(4-2) \\ &+ \int d3 \int d4 \int d5 \int d6\, G_m(1-3)\mathscr{L}(3) g(3-4)\mathscr{L}(4) g(4-5) \\ &\times \mathscr{L}(5) g(5-6)\mathscr{L}(6) g(6-2). \end{aligned} \tag{28}$$

In the second integral term, we have averaged $\mathscr{L}(3)$ with $\mathscr{L}(5)$, and $\mathscr{L}(4)$ with $\mathscr{L}(6)$.

The extremely simple nonlinear equation in (27) is found directly from (22) and (23) by assuming, in the calculation of $G_1(1,2)$, that the average Green's function is much larger than the fluctuational part of $G_1(1,2)$. An estimate of the term with the fourth-degree nonlinearity in (28) shows that it generates corrections $\sim (\xi R_0/l)^2$ in the case $\xi \lesssim 1$ and a correction $\sim (R_0/l)^2$ in the case $\xi \gg 1$.

5. NUMERICAL SOLUTION OF THE SIMPLEST NONLINEAR EQUATION

Equation (27) is the simplest nonlinear equation for the average Green's function

$$\langle G(1,2)\rangle = \theta(\tau) g(R,\tau).$$

The easiest way to solve this equation is to take Fourier transforms in $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$ and Laplace transforms in $\tau=t_1-t_2$:

$$\tilde{g}(p,s)=\int d\mathbf{R}\int_0^\infty d\tau g(R,\tau)\exp(-i\mathbf{p}\mathbf{R}-s\tau). \tag{29}$$

In terms of these variables, Eq. (27) becomes

$$\tilde{g}(p,s)=\left[s+D_m p^2+\frac{1}{(2\pi)^3}\int d\mathbf{q}p_i\tilde{B}_{ij}(\mathbf{p}-\mathbf{q})\times q_j\tilde{g}\left(q,s+\frac{1}{\tau_0}\right)\right]^{-1}. \tag{30}$$

Here we have also introduced the restriction

$$B_{ij}(\mathbf{R},\tau)=B_{ij}(\mathbf{R})\exp(-|\tau|/\tau_0),$$

and we have used

$$\tilde{G}_m(p,s)=(s+D_m p^2)^{-1}.$$

Expression (30) is an equation of the continued fraction type. For an incompressible medium, Eq. (30) simplifies:

$$\tilde{g}(p,s)=\left[s+D_m p^2+\frac{p^2}{(2\pi)^3}\int d\mathbf{q}q^2(1-\mu^2)f(q)\times\tilde{g}\left(|\mathbf{p}-\mathbf{q}|,s+\frac{1}{\tau_0}\right)\right]^{-1}. \tag{31}$$

Here $\mathbf{p}\cdot\mathbf{q}=pq\mu$. Since the quantity $p^4f(p)$ is proportional to the energy spectrum of the turbulence fluctuations, $E(p)\gtrsim 0$, the kernel in (31) is a positive function. In other words, expression (31) is the analog of a continued fraction with positive terms. Consequently, expression (31) can easily be solved numerically by the "fork" method, with $g_>^{(0)}(p,s)=\tilde{G}_m(p,s)$ used as an initial approximation. All the odd approximations then represent the exact value of $\tilde{g}(p,s)$ with a deficiency, while the even ones do the same with an excess. Very quickly, the odd and even iterations become more nearly the same; ~ 30 iterations are sufficient for essentially perfect agreement. The first iteration of (31) leads to the function $\tilde{M}(p,s)$ which was used in Refs. 5, 9, and 10. Note also that the (30) and (31) do not depend on the helicity of the medium. Such a dependence does rise in the case of a scalar impurity field with nonlinear equations which are more complex, beginning with (28). For the case of magnetic-field diffusion, even the simplest nonlinear equation (more precisely, the simplest system of two equations), of the form (27), depends on the helicity.

For small values of p and s (i.e., for $p\ll p_0\sim R_0^{-1}$ and $s\tau_0\ll 1$) we find from (31)

$$\tilde{g}(p,s)\approx[s+(D_m+D_T^{(1)})p^2]^{-1}\equiv\tilde{G}_{m+T}(p,s). \tag{32}$$

In other words, the function $\tilde{g}(p,s)$ is the same as the Green's function of the diffusion equation (19) with a diffusion coefficient $D_m+D_T^{(1)}$ (an expression for $D_T^{(1)}$ is given in the following section of this paper). In the opposite limit $p\gg p_0$ we find

$$\tilde{g}(p,s)\approx\left[s+D_m p^2+\frac{1}{3}u_0^2p^2\tilde{g}\left(p,s+\frac{1}{\tau_0}\right)\right]^{-1}. \tag{33}$$

At sufficiently large values of $p((u_0\tau_0p)^2\gg s\tau_0, u_0^2\tau_0\gg D_m)$ we can ignore the first two terms in square brackets; we find

$$\tilde{g}(p,s)\approx(u_0^2p^2\tilde{g}(p,s+1/\tau_0)/3)^{-1}.$$

We then find an s-independent asymptotic expression

$$g_a^{(0)}(p,s)=3^{1/2}/u_0p. \tag{34}$$

Substituting this asymptotic expression into (33), we find the more accurate expression

$$\tilde{g}_a^{(1)}(p,s)=(s+D_mp^2+pu_0/3^{1/2})^{-1}. \tag{35}$$

An expression which is even more accurate is found by substituting (35) into (33):

$$\tilde{g}_a^{(2)}(p,s)=\left[s+D_mp^2+\frac{1}{3}u_0^2p^2\left(s+\frac{1}{\tau_0}+D_mp^2+\frac{pu_0}{3^{1/2}}\right)^{-1}\right]^{-1}. \tag{36}$$

At very large values of p, according to (34) and (35), the term with $\tilde{g}$ in (33) (this term is linear in p) becomes smaller than the molecular term D_mp^2, and the function $\tilde{g}(p,s)$ becomes the same as the molecular Green's function

$$\tilde{G}_m=(s+D_mp^2)^{-1}.$$

The condition $D_m\ll D_T$ holds in essentially all cases, and we can ignore the term with D_m in calculating D_T.

Let us compare the approximate expression (36) with the results of accurate numerical calculations of expression (31). Interestingly, expression (36) is independent of the turbulence length scale $R_0\sim 1/p_0$ and in practice depends only weakly on τ_0. In other words, $\tilde{g}(p,s)$ effectively depends on only the energy scale of the turbulent fluctuations, $\sim u_0^2$. Equations (34) and (35), which do not depend on τ_0, also hold for an arbitrary time dependence of the velocity correlation function. In numerical calculations it is convenient to use the dimensionless variables $x=p/p_0$ and $y=s\tau_0$ and to write $\tilde{g}(p,s)$ in the form

$$\tilde{g}(p,s)\equiv\tau_0\tilde{\tilde{g}}(x,y). \tag{37}$$

In this notation, expression (36) becomes $(D_m\to 0)$

$$\tilde{\tilde{g}}_a^{(2)}(x,y)=\frac{y+1+\xi_0x}{y(y+1)+\xi_0xy+x^2\xi_0^2}\to\frac{1}{\xi_0x}\quad\text{as}\quad\xi_0x\to\infty, \tag{38}$$

where

$$\xi_0^2=u_0^2\tau_0^2p_0^2/3,\ \text{a}\ u_0^2=\langle u^2(r,t)\rangle.$$

Tables I and II show the functions $\tilde{\tilde{g}}(x,1)$ for the correlation functions

$$f(p,\tau)=\pi^2u_0^2p_0^{-4}\delta(p-p_0)\exp(-|\tau|/\tau_0), \tag{39}$$

$$f(p,\tau)={}^2/_3\pi^2u_0^2p_0^{2/3}p^{-17/3}\theta(p-p_0)\exp(-|\tau|/\tau_0). \tag{40}$$

These results correspond to a δ-function spectrum

$$E(p)=u_0^2\delta(p-p_0)$$

and a Kolmogorov spectrum

TABLE I. Values of the Green's function $\tilde{\tilde{g}}(x,1)$ for the δ-function spectrum in (39). The values of $\tilde{\tilde{g}}_a^{(2)}$ and $\tilde{\tilde{g}}_{\mathrm{diff}}$ were calculated from (38) and (32), respectively.

x	$\xi_0=0{,}3$			$\xi_0=0{,}9$			$\xi_0=3$		
	$\tilde{\tilde{g}}(x,1)$	$\tilde{\tilde{g}}_a^{(2)}(x,1)$	$\tilde{\tilde{g}}_{diff}(x,1)$	$\tilde{\tilde{g}}(x,1)$	$\tilde{\tilde{g}}_a^{(2)}(x,1)$	$\tilde{\tilde{g}}_{diff}(x,1)$	$\tilde{\tilde{g}}(x,1)$	$\tilde{\tilde{g}}_a^{(2)}(x,1)$	$\tilde{\tilde{g}}_{diff}(x,1)$
0	1	1	1	1	1	1	1	1	1
0,1	0,9996	0,9995	0,9991	0,9964	0,9961	0,9940	0,9759	0,9623	0,9709
0,2	0,9982	0,9982	0,9965	0,9857	0,9854	0,9763	0,9105	0,8784	0,8930
0,3	0,9960	0,9961	0,9923	0,9686	0,9689	0,9482	0,8210	0,7817	0,7876
0,4	0,9930	0,9932	0,9864	0,9458	0,9479	0,9114	0,7227	0,6896	0,6760
0,5	0,9891	0,9896	0,9789	0,9184	0,9236	0,8682	0,6298	0,6087	0,5718
0,6	0,9843	,9853	0,9699	0,8876	0,8970	0,8206	0,5483	0,5398	0,4811
0,7	0,9789	0,9804	0,9595	0,8544	0,8689	0,7707	0,4786	0,4818	0,4052
0,8	0,9726	0,9749	0,9477	0,8197	0,8399	0,7201	0,4206	0,4343	0,3428
0,9	0,9657	0,9689	0,9347	0,7847	0,8107	0,6703	0,3730	0,3920	0,2918
1	0,9581	0,9623	0,9206	0,7496	0,7817	0,6222	0,3329	0,3571	0,2503
2	0,8559	0,8784	0,7435	0,4771	0,5398	0,2916	0,1621	0,1818	0,0770
10	–	0,3571	0,1039	–	0,1196	0,0162	–	0,0343	0,0033

TABLE II. Values of the green's function $\tilde{\tilde{g}}(x,1)$ for the Kolmogorov spectrum in (40). The values of $\tilde{\tilde{g}}_a^{(2)}$ and $\tilde{\tilde{g}}_{\text{diff}}$ were calculated from (38) and (32), respectively.

x	$\xi_0=0{,}1$			$\xi_0=0{,}3$			$\xi_0=1$		
	$\tilde{\tilde{g}}(x,1)$	$\tilde{\tilde{g}}_a^{(2)}(x,1)$	$\tilde{\tilde{g}}_{diff}(x,1)$	$\tilde{\tilde{g}}(x,1)$	$\tilde{\tilde{g}}_a^{(2)}(x,1)$	$\tilde{\tilde{g}}_{diff}(x,1)$	$\tilde{\tilde{g}}(x,1)$	$\tilde{\tilde{g}}_a^{(2)}(x,1)$	$\tilde{\tilde{g}}_{diff}(x,1)$
0	1	1	1	1	1	1	1	1	1
0,5	0,999	0,999	0,998	0,991	0,990	0,985	0,936	0,909	0,911
1	0,996	0,995	0,991	0,967	0,962	0,943	0,790	0,750	0,720
2	0,983	0,982	0,966	0,882	0,878	0,805	0,514	0,500	0,391
3	0,962	0,962	0,927	0,776	0,782	0,647	0,354	0,357	0,222
4	0,935	0,937	0,878	0,672	0,690	0,508	0,266	0,273	0,138
5	0,902	0,909	0,821	0,582	0,609	0,398	0,212	0,219	0,093
6	0,867	0,878	0,762	0,508	0,540	0,314	0,176	0,182	0,067
7	0,829	0,846	0,701	0,448	0,482	0,252	0,150	0,155	0,050
8	0,791	0,814	0,642	0,399	0,433	0,205	0,131	0,135	0,039
9	0,752	0,782	0,587	0,359	0,392	0,169	0,116	0,119	0,031
10	0,715	0,750	0,535	0,327	0,357	0,142	0,104	0,107	0,025
20	0,416	0,500	0,223	0,167	0,182	0,040	0,052	0,052	0,006

$$E(p)=(^2/_3)u_0^2p_0^{2/3}p^{-5/3}.$$

It can be seen from these tables that expression (38) gives a good picture of the exact values of $g(x,1)$ at essentially all values of x and ξ_0. At large values of the parameter ξ_0, expression (38) is more accurate. For comparison, the values of $\tilde{g}(x,1)$ in the diffusion approximation are

$$\tilde{\tilde{g}}_{diff}(x,y)=(y+\xi_0^2\tilde{D}_T^{(1)}x^2)^{-1},$$

where

$$\tilde{D}_T^{(1)}=D_T^{(1)}(u_0^2\tau_0/3)^{-1}.$$

It can be seen from these tables that $\tilde{\tilde{g}}_{\rm diff}$ gives a fairly good representation of $\tilde{\tilde{g}}(x,1)$ up to $x\simeq 1$ at $\xi_0\lesssim 1$. With increasing ξ_0, the accuracy of $\tilde{\tilde{g}}_{\rm diff}$ suffers. To calculate D_T for slowly decreasing spectra of the form (40), we would also need to know $\tilde{\tilde{g}}(x,1)$ for $x\gg 1$, where the diffusion approximation is totally inapplicable.

6. CALCULATION OF THE TURBULENT DIFFUSION COEFFICIENTS

Equation (27) is a particular case of general equation (7) with the kernel

$$K(3-4)=\langle\mathscr{L}(3)g(3-4)\mathscr{L}(4)\rangle$$

and

$$M(1-2)\equiv g(1-2).$$

Once we have solved (27), we can take iterations of the renormalized version of Eq. (8) and find a series of approximations for the Green's function $G(1,2)$. Substituting this series into (20), we find a series of approximations for the turbulent diffusion coefficient D_T. In the case of an incompressible medium, the first two terms of this series are found to be

$$D_T^{(1)}=\frac{1}{3}\int d\mathbf{R}\int_0^\infty d\tau B_{ii}(\mathbf{R},\tau)g(R,\tau)$$

$$=\frac{1}{3\pi^2}\int_0^\infty dp\int_0^\infty d\tau p^4f(p,\tau)\tilde{g}(p,\tau), \qquad (41)$$

$$D_T^{(2)}=\frac{1}{3}\int d\mathbf{R}_1\int d\mathbf{R}_2\int d\mathbf{R}_3\int_0^\infty d\tau_1\int_0^\infty d\tau_2\int_0^\infty d\tau_3 g(R_1,\tau_1)B_{ij}$$

$$\times(\mathbf{R}_1+\mathbf{R}_2,\tau_1+\tau_2). \qquad (42)$$

$$\nabla_i^{(2)}g(R_2,\tau_2)B_{ii}(\mathbf{R}_2+\mathbf{R}_3,\tau_2+\tau_3)\nabla_j^{(3)}g(R_3,\tau_3).$$

Under our restriction $B_{ij}(\mathbf{R},\tau)\sim\exp(-|\tau|/\tau_0)$, expressions (41) and (42) simplify markedly:

$$D_T^{(1)}=\frac{1}{3\pi^2}\int_0^\infty dp\,p^4f(p)\tilde{g}\left(p,\frac{1}{\tau_0}\right)\equiv\frac{1}{3}\int_0^\infty dpE(p)\tilde{g}\left(p,\frac{1}{\tau_0}\right), \qquad (43)$$

$$D_T^{(2)}=\frac{1}{24\pi^4}\int_0^\infty dp\,p^4\int_0^\infty dq\,q^4\int_{-1}^1 d\mu(1-\mu^2)[\mu pqf(p)f(q)$$

$$+D(p)D(q)]\tilde{g}\left(p,\frac{1}{\tau_0}\right)\tilde{g}\left(q,\frac{1}{\tau_0}\right)\tilde{g}\left((p^2+q^2+2pq\mu)^{1/2},\frac{2}{\tau_0}\right). \qquad (44)$$

Here $E(p)=p^4f(p)/\pi^2$ is the turbulence spectrum, i.e., a representation of the quantity $\langle u^2(\mathbf{r},t)\rangle\equiv u_0^2$ as an integral over all positive wave numbers, with a density $E(p)$.

Tables III and IV show values of D_T calculated from the exact solution of the nonlinear equations (31) and the asymptotic expression (38). Shown for comparison are values of D_T found by the so-called self-consistent method,[3] in which the exact value of $\tilde{g}(p,s)$ in (43) is replaced by

TABLE III. Values found for the diffusion coefficient $\tilde{D}_T=D_T(\frac{1}{3})u_0^2\tau_0)^{-1}$ in the case of correlation function (39) with $\xi=\xi_03=u_0\tau_0p_0/3^{3/2}$. The values of $\tilde{D}_T^{(c)}$ and $\tilde{D}_T^{(a)}$ were calculated from (45) and (38).

ξ	$\tilde{D}_T^{(1)}$	$-\tilde{D}_T^{(2)}$	$\tilde{D}_T^{(\gamma)}$	$\tilde{D}_T^{(c)}$	$\tilde{D}_T^{(a)}$
0	1	0	0	1	1
0,1	0,9581	0,0001	0,0201	0,9233	0,9623
0,2	0,8598	0,0011	0,0603	0,7806	0,8783
0,3	0,7496	0,0033	0,0938	0,6538	0,7817
0,4	0,6505	0,0057	0,1137	0,5555	0,6896
0,5	0,5681	0,0078	0,1227	0,4805	0,6087
0,6	0,5010	0,0094	0,1250	0,4223	0,5398
0,7	0,4463	0,0105	0,1236	0,3761	0,4818
0,8	0,4014	0,0112	0,1201	0,3388	0,4331
0,9	0,3642	0,0117	0,1158	0,3081	0,3920
ξ	$\tilde{D}_T^{(1)}$	$-\tilde{D}_T^{(2)}$	$\tilde{D}_T^{(\gamma)}$	$\tilde{D}_T^{(c)}$	$\tilde{D}_T^{(a)}$
1	0,3329	0,0119	0,1110	0,2824	0,3571
1,2	0,2834	0,0120	0,1015	0,2419	0,3017
1,4	0,2464	0,0117	0,0928	0,2114	0,2601
1,6	0,2176	0,0113	0,0851	0,1877	0,2279
1,8	0,1947	0,0109	0,0784	0,1688	0,2024
2	0,1760	0,0104	0,0725	0,1533	0,1818
3	0,1187	0,0084	0,0523	0,1051	0,1195
4	0,0893	0,0070	0,0407	0,0799	0,0886
5	0,0715	0,0059	0,0332	0,0645	0,0702

TABLE IV. Values of the turbulent diffusion coefficient $\widetilde{D}_T = D_T(\frac{1}{3}u_0^2\tau_0)^{-1}$ for the Kolmogorov spectrum (40). The values of $\widetilde{D}_T^{(c)}$ and $\widetilde{D}_T^{(a)}$ were calculated from (45) and (38).

ξ_0	$\widetilde{D}_T$	$\widetilde{D}_T^{(c)}$	$\widetilde{D}_T^{(a)}$	ξ_0	$\widetilde{D}_T$	$\widetilde{D}_T^{(c)}$	$\widetilde{D}_T^{(a)}$
0	1	1	1	1	0,389	0,317	0,370
0,1	0,869	0,765	0,829	1,2	0,346	0,281	0,324
0,2	0,754	0,654	0,734	1,4	0,310	0,253	0,288
0,3	0,674	0,575	0,660	1,6	0,282	0,230	0,258
0,4	0,610	0,514	0,598	1,8	0,257	0,210	0,233
0,5	0,557	0,465	0,546	2	0,237	0,194	0,212
0,6	0,514	0,425	0,501	3	0,169	0,140	0,144
0,7	0,476	0,392	0,461	4	0,130	0,109	0,108
0,8	0,443	0,363	0,427	5	0,105	0,090	0,086
0,9	0,414	0,339	0,397				

$\tilde{g}_{\text{diff}}(p,s)$ [see (32)], but with an unknown coefficient D_T. Expression (43) then becomes an equation for finding D_T.

$$D_T^{(c)}=\frac{\tau_0}{3}\int_0^\infty dpE(p)(1+\tau_0 D_T^{(c)}p^2)^{-1}. \qquad (45)$$

In calculations of D_T, the correlation functions in (39) and (40) were chosen as examples of an extremely narrow turbulence spectrum and of a broad spectrum.

Table III shows values of $D_T^{(1)}$ and $D_T^{(2)}$ calculated through the use of the exact solution of (31) for the correlation function (39), which represents turbulence with an extremely narrow spectrum. Here we allowed for the helicity of the medium, which we chose in the form

$$D(p)=\gamma pf(p),$$

where $|\gamma|\leqslant 1$ is a measure of the degree of helicity. The diffusion coefficient in this case can be written

$$\widetilde{D}_T\equiv D_T\left(\frac{1}{3}u_0{}^2\tau_0\right)^{-1}=\widetilde{D}_T^{(1)}+\widetilde{D}_T^{(2)}+\gamma^2\widetilde{D}_T^{(\gamma)}+\ldots, \qquad (46)$$

where $\widetilde{D}_T^{(1)}=\tilde{g}(1,1)$, and the corrections $\widetilde{D}_T^{(2)}$ and $\widetilde{D}_T^{(\gamma)}$ are

$$\widetilde{D}_T^{(2)}=\frac{3}{8}\xi_0{}^2\tilde{\tilde{g}}^2(1,1)\int_{-1}^{1}d\mu\mu(1-\mu^2)\tilde{\tilde{g}}((2(1+\mu))^{1/2},2), \qquad (47)$$

$$\widetilde{D}_T^{(\gamma)}=\frac{3}{8}\xi_0{}^2\tilde{\tilde{g}}^2(1,1)\int_{-1}^{1}d\mu(1-\mu^2)\tilde{\tilde{g}}((2(1+\mu))^{1/2},2). \qquad (48)$$

The case $|\gamma|=1$ corresponds to the maximum possible helicity [according to the Khinchin-Bochner theorem,[7] the inequality $pf(p)\geqslant|D(p)|$ holds]. It can be seen from Table III that the contribution of the helicity increases with increasing ξ_0. Using the asymptotic expression in (34), we find the following results in the limit $\xi_0\to\infty$:

$$\widetilde{D}_T^{(1)}\to\xi_0{}^{-1},\quad \widetilde{D}_T^{(2)}\to -2/(35\xi_0),\quad \widetilde{D}_T^{(\gamma)}\to 0{,}4/\xi_0.$$

These results show the relative contributions of the quantities $\widetilde{D}_T^{(1)}$, $\widetilde{D}_T^{(2)}$, and $\widetilde{D}_T^{(\gamma)}$ to the turbulent diffusion coefficient fairly accurately at values $\xi_0\gg 1$. Table III shows that in the absence of helicity we can use the simple expression (43) to calculate D_T in the limit $\xi_0\to\infty$, to within an error $\approx 8\%$. This error falls off monotonically with decreasing ξ_0, approaching zero as $\xi_0\to 0$. If several iterations of the original equation, (6), are used in calculating D_T, we obtain the series

$$D_T=\frac{1}{3}u_0{}^2\tau_0\left(1-\frac{1}{2}\xi_0{}^2+\frac{1}{4}\gamma^2\xi_0{}^2+\frac{9}{20}\xi_0{}^4+\ldots\right), \qquad (49)$$

which can be used to calculate D_T under the condition $\xi_0^2=u_0^2\tau_0^2p_0^2/3\ll 1$.

What does the method described above lead to in the case of turbulent media with a broad energy spectrum? For specific calculations, we adopt a Kolmogorov spectrum as in (40). Table IV shows the results of an accurate numerical calculation based on Eqs. (31) and (41), along with results calculated by the self-consistent method [see (45)] and through the use of approximate expression (36). Interestingly, when we use iterations of Eq. (6) to obtain a series of the form (49) in the case of a broad spectrum, we obtain only the first finite term, $u_0^2\tau/3$. The other terms of the series, beginning with the second, diverge. It can be seen from Table IV that the approximate expression (36) leads to results more accurate than those found by the self-consistent method. It can be seen from (44) that in the limit $\xi_0\to 0$ we have $D_T^{(2)}\sim\xi_0^2\to 0$. We therefore restrict the discussion to an estimate of the contribution $D_T^{(2)}$ to the diffusion coefficient in the limit $\xi_0\to 0$, in which this contribution is at a maximum with respect to the contribution of the leading term, $D_T^{(1)}$. In the limit $\xi_0\gg 1$, the asymptotic expression $\tilde{\tilde{g}}^{(0)}=(\xi_0 x)^{-1}$ is a good approximation of $\tilde{\tilde{g}}(x,1)$. It follows from the numerical calculations that this asymptotic expression becomes valid in practice at values $x\approx 1$. In other words, the expressions found for $D_T^{(2)}$ are quite accurate. The substitution

$$\tilde{g}^{(0)}(p,1/\tau_0)=3^{1/2}/u_0 p$$

in (44) leads to the result ($\xi_0^2=u_0^2p_0^2/3$):

$$D_T^{(2)}=\frac{1}{3}u_0{}^2\tau_0\frac{1}{\xi_0}(0{,}012+0{,}104\gamma^2)=\frac{u_0}{3^{1/2}p_0}(0{,}012+0{,}104\gamma^2). \qquad (50)$$

As before, we have assumed $D(p)=\gamma pf(p)$. We recall that in the limit $\xi_0\to\infty$ the leading term has the behavior $D_T^{(1)}\to 0.4u_0/3^{1/2}p_0$. In other words, in the absence of helicity, we can ignore the contribution $D_T^{(2)}$. The maximum helicity contribution is about 25% of $D_T^{(1)}$. A comparison with the previous example shows that the contribution from $D_T^{(2)}$ is smaller for broad spectra than for narrow spectra.

If the requirements on the accuracy with which D_T is calculated are not too stringent, it is quite sufficient to use the approximate analytic expression

$$D_T=\frac{\tau_0}{3}\int_0^\infty dpE(p)\frac{2+u_0\tau_0p/3^{1/2}}{2+u_0\tau_0p/3^{1/2}+(u_0\tau_0p/3^{1/2})^2}, \tag{51}$$

which is found from (43) by replacing the exact function $\tilde{g}(p,1/\tau_0)$ by the simple approximation (36). This expression has an error $\approx 10\%$ for spectra with $\xi_0 \gg 1$ and a far smaller error for spectra with $\xi_0 \lesssim 1$. An expression like (51) can easily be written for an arbitrary time dependence of the correlation function $f(p,\tau)$ if we replace the asymptotic expression (36) by

$$\tilde{g}_a^{(2)}(p,s)=\Big[s+D_mp^2+\frac{p^2}{3\pi^2}\int_0^\infty dq\,q^4\int_0^\infty d\tau f(q,\tau)$$

$$\times\exp\Big[-\Big(s+D_mp^2+\frac{u_0p}{3^{1/2}}\Big)\tau\Big]\Big]^{-1}. \tag{52}$$

The values of the turbulent diffusion coefficient D_T are determined by the joint dependence of the correlation function $f(p,\tau)$ on p and τ. In other words, both the spatial distribution of the turbulent motions among various scales and the average lifetime of these motions are important. For turbulent motions with $\xi_0=\tau_0/t_0 \ll 1$ (short-lived jets), motions of all scales can be realized (particles are transported over distances $\sim u\tau_0$ in a time $\sim\tau_0$, and we have $D_T\approx u_0^2\tau_0$). In this limit we have $g(R,\tau)\approx\delta(R)$, and the following expression holds [see (41)]:

$$D_T=\frac{1}{3}\int_0^\infty d\tau B_{ii}(0,\tau)=\frac{1}{3\pi^2}\int_0^\infty dp\int_0^\infty d\tau\,p^4f(p,\tau). \tag{53}$$

In other words, the contribution of a scale is proportional to the fraction of the energy of the turbulent motions which corresponds to this scale.

For turbulence with $\xi_0=\tau_0/t_0\gg 1$ ("frozen" turbulence), the largest-scale motions, with typical values $\sim u_0t_0\approx R_0$, are most important. These motions cause transport of impurity particles over the greatest distances. In this case we have

$$\tilde{g}(p,\tau)\approx\tilde{g}_a^{(1)}(p,\tau)=\exp(-u_0p\tau/3^{1/2})$$

and

$$D_T=\frac{1}{3\pi^2}\int_0^\infty dp\int_0^\infty d\tau p^4f(p,\tau)\exp(-u_0p\tau/3^{1/2}). \tag{54}$$

It can be seen from this expression that small-scale turbulent motions (larger than p) make a contribution to D_T which is much smaller than that of large-scale motions.

The spectrum which is actually observed in many cases of turbulence[12] has a maximum at small p (a region of large-scale mixing, which brings energy into the turbulent motion). Later, at intermediate scales (this is the inertial interval), the Kolmogorov mechanism operates to transfer energy to vortices of progressively smaller scale, down to scales at which the energy of turbulent motions is converted into heat by viscous dissipation. The "frozen" nature of turbulent motions which is observed experimentally at scales in the inertial interval means that the condition $\xi_0\gg 1$ holds for motions in this interval, although the velocity correlations of large-scale motions can have an arbitrary dependence on the time difference $\tau=t_1-t_2$. The evolution of the correlation function $B_{ij}(\mathbf{R},\tau)$ as a function of $\mathbf{R}$ and τ has received essentially no study before now. It is customary to examine only the evolution of the energy spectrum, i.e., to examine simultaneous correlations. That approach is clearly inadequate for calculating turbulent diffusion coefficients.

The contribution to the turbulent diffusion coefficient D_T from the non-Kolmogorov part of the spectrum may turn out to be comparable to or even greater than the contribution of the Kolmogorov part. For example, if we adopt the spectrum

$$E(p)=\frac{10}{17}u_0^2\begin{cases}p_0^{-5}p^4, & p<p_0,\\ p_0^{2/3}p^{-5/3}, & p>p_0,\end{cases} \tag{55}$$

and if we assume that all the turbulence is "frozen" (i.e., $\tau_0\to\infty$), then by using (54) we obtain $D_T=0.5u_0/p_03^{1/2}$, where the contribution of the non-Kolmogorov part of the spectrum is $\approx 30\%$. It is thus not possible to select any single diffusion coefficient D_T for a Kolmogorov spectrum, since large-scale motions make an important contribution to the diffusion in the region in which energy is "pumped" into the turbulence. Such motions may be quite diverse.

Equation (27), the simplest nonlinear equation, depends only on the binary correlation function of the turbulent velocities, so it is the same for Gaussian and non-Gaussian ensembles of velocities. The non-Gaussian nature of an ensemble could thus affect only the correction terms $D_T^{(2)}$, $D_T^{(3)}$, etc. The values calculated for $D_T^{(2)}$ in the Gaussian approximation can be apparently serve, to some extent, as estimates of the contribution of the non-Gaussian nature of the ensemble to D_T.

In the case of magnetic-field diffusion, the general method presented above leads to two coupled nonlinear equations, instead of Eq. (31). In the absence of a helicity, this system of equations degenerates to (31). In this case the diffusion coefficients for a magnetic field and for a scalar impurity are the same.

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