Electromagnetic shock wave with trapped relativistic electrons in a collisional plasma

V. B. Krasovitskiĭ and G. V. Fomin

Rostov State University

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A steady-state solution in the form of an electromagnetic shock wave exists in a dense, collisional plasma with a bunched relativistic electron beam. The damping of the field in the plasma due to Coulomb collisions is accompanied by a decrease in the depth of the potential wells for the electrons trapped by the electromagnetic radiation. As a result, the longitudinal dimensions of the bunches increase. The shock wave is thus an infinite set of periodic equilibrium solutions for a collisionless plasma.

A relativistic electron beam in the form of distinct bunches is known[1,2] to be at equilibrium with electromagnetic radiation in a plasma if the modulation frequency $\omega_m = 2\pi v/l$ (v is the beam velocity, and l is the distance between bunches) is lower than the plasma frequency of the plasma and if the dielectric constant $\varepsilon = 1 - \omega_p^2/\omega_m^2$ is negative. The electromagnetic radiation emitted by the beam in the plasma is a superposition of the vacuum field of bunches moving at a constant velocity and of the polarization wave excited by this field in the plasma. If $\varepsilon < 0$, this wave, whose field is stronger than the Coulomb field of the bunches under resonance conditions, is out of phase with the charge density wave of the beam, and it focuses the bunches. In other words, the polarization wave induced by the beam displaces plasma electrons from the volume occupied by the beam, and the field of the net ion charge cancels the Coulomb repulsive force and the gradient of the kinetic pressure in the bunches. The beam is thereby brought to equilibrium with the radiation.

In a real dense plasma, a perturbation caused by a beam is damped by Coulomb collisions, and the lifetime of a focused beam is limited by the time scale of this effect. For this reason, the one-dimensional periodic solution found in Ref. 3 is time-varying. It describes a slowing of the nonrelativistic beam as a whole. If the solution in a collisional plasma is written formally as a monochromatic plane wave, the dielectric constant of the plasma acquires an imaginary increment $\varepsilon_c = \varepsilon + i\nu_c/\omega_p$ (ν_c is the collision rate). This increment implies the appearance of a phase difference between the polarization wave and the charge density wave of the beam. A numerical integration of the equation for the radial equilibrium of a bunch under the condition $\nu_c > 0$ shows that there is no steady-state periodic solution in a collisional plasma with a modulated electron beam.[4]

In the present paper we establish a new result: A steady-state solution in the form of an electromagnetic shock wave exists in a collisional plasma with relativistic electron bunches. This solution differs in a fundamental way from those of Refs. 1–4, in that there is no periodicity in terms of the variable $z' = z - vt$, because of the temporal damping of the waves in the plasma. Since the longitudinal dimensions of the bunches increase with decreasing strength of the focusing field,[1] an observer in the frame of reference of the plasma sees bunches moving at a constant velocity and becoming longer as time elapses. There is a corresponding increase in the amount of Coulomb charge which they must have in order to offset the energy loss and to maintain a steady-state wave in the plasma.

Allowance for collisions in the form of a friction force $-m\nu_c \mathbf{v}_p$ in the equations of motion of the plasma electrons leads to additional terms in the equation for the effective potential (cf. Ref. 1):

$$\left(\frac{\partial^2}{\partial t\,\partial\tau}+\omega_p^2\right)\left[\frac{\partial}{\partial\tau}\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)-\frac{\omega_p^2}{c^2}\frac{\partial}{\partial t}\right]\psi =4\pi e\frac{\partial}{\partial\tau}\left(\frac{1}{\gamma^2}\frac{\partial^2}{\partial t\,\partial\tau}-\frac{\omega_p^2v^2}{c^2}\right)\rho, \quad (1)$$

where ρ and v are the density and velocity of the beam, $\gamma = (1 - v^2/c^2)^{-1/2}$ is the relativistic factor, $\partial/\partial\tau = \partial/\partial t + \nu_c$, $\psi = \varphi - vA_z/c$, and φ and A_z are the scalar and vector potentials. These potentials satisfy the gauge condition

$$\frac{\partial^2 A_z}{\partial t\,\partial z}+\frac{1}{c}\left(\frac{\partial^2}{\partial t\,\partial\tau}+\omega_p^2\right)\varphi=0.$$

Since the mean free path $l_c = v/\nu_c$, of an electron in the plasma, is considerably larger than the characteristic size $\pi v/\omega_p$, of a bunch, $\nu_c\partial/\partial t$ is the most important term near the plasma resonance. Incorporating this term only in the first operator on the left side of Eq. (1), discarding the terms on the order of $\gamma^{-2} \ll 1$, and switching to the variable $z' = z - vt$, we find the equation

$$\left(\frac{\partial^2}{\partial\zeta^2}-\nu\frac{\partial}{\partial\zeta}+1\right)\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right)-1\right]\psi=-\psi_0\left(\frac{\psi}{\psi_m}\right)^2\theta(\psi), \quad (2)$$

where we have retained the notation of Ref. 1:

$$\xi=\frac{\omega_p r}{c},\quad \zeta=\frac{\omega_p z'}{c},\quad \psi_0=\frac{4\pi e\rho_0c^2}{\omega_p^2},\quad \nu=\frac{\nu_c}{\omega_p}\ll 1,$$

$$\rho=\rho_0\left(\frac{\psi}{\psi_m}\right)^2\theta(\psi),\quad \theta(\psi)=\begin{cases}1 & \psi>0,\\ 0, & \psi<0,\end{cases}$$

ρ_0 is the density of the electrons trapped by the wave, and ψ_m is the potential at the center of a bunch.

Variables can be separated in Eq. (2):

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$$\psi(\xi,\zeta)=\psi_0 R(\xi)W(\zeta).$$

The ν-independent equation for the radial function,

$$\xi^{-1}(\xi R')'-R+R^2=0, \qquad R(\infty)=R'(0)=0 \tag{3}$$

is the same as that in Ref. 1. It has at $R_0 = R(0) = 2.39$ the soliton solution which is necessary for the occurrence of a self-focusing.

The longitudinal equilibrium of the bunches is described by the equation of a nonlinear oscillator with a small negative friction:

$$W''-\nu W'+W-\left(\frac{W}{R_0 W_m}\right)^2\theta(W)=0, \qquad W'(0)=0, \quad \psi_m=\psi_0 R_0 W_m. \tag{4}$$

In a collisionless plasma,[1] the change of variables

$$W=(R_0 W_m)^2 Z$$

makes it possible to eliminate the constant factor W_m from the equation for the auxiliary function $Z(\zeta)$. When collisions are taken into account ($\nu>0$), on the other hand, the wave amplitude $W_m(\zeta)$ varies slowly along the beam, $W'_m \simeq \nu W_m \ll W_m$, and an additional term (a nonlinear friction) appears in the equation:

$$Z''-\left(\nu+4\frac{Z_0'}{Z_0}\right)Z'+Z-Z^2\theta(Z)=0 \tag{5}$$

(the point $\zeta=0$ corresponds to the center of a bunch). In going from (4) to (5) we made use of the obvious relation $W_m=(R_0^2 Z_0)^{-1}$, where $Z_0(\zeta)$ is the maximum value of the function $Z(\zeta)$ at the centers of the bunches.

Equation (5) can be used for a qualitative analysis, by analogy with problems in mechanics.[5] In the absence of dissipation ($\nu=0$), the integral of (5) is

$$\frac{Z'^2}{2}+U(Z)=E, \quad U(Z)=\frac{Z^2}{2}-\frac{Z^3}{3}\theta(Z), \quad E=U(Z_0). \tag{6}$$

From the plot of the potential energy $U(Z)$ in Fig. 1 we see that in the region

$$Z_{min}=-(2E)^{1/2}\leqslant Z\leqslant Z_0<1$$

the motion of a particle is periodic, with turning points Z_{min}

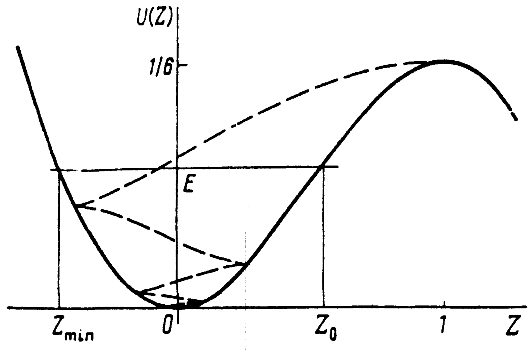

FIG. 1. Plot of the potential energy. The dashed line shows the transformation of a soliton into a shock wave at $\nu = 0.3$.

and Z_0. As $Z_0\to 1$, the solution transforms into a soliton.

The soliton solution $Z_0 = 1$ is a separatrix in the phase plane. It describes two semi-infinite electron beams, separated by a gap $\pi c/\omega_p$ (Ref. 1). The physical meaning of this solution becomes apparent when collisions are taken into account, since under the condition $\nu_{\rm eff}=\nu+4Z_0'/Z_0<0$ Eq. (5) describes slightly damped nonlinear oscillations corresponding to a slow descent of the particle from the point $Z_0 = 1$ to the bottom of the potential well, $Z_0 = 0$ (Fig. 1), at which point the soliton transforms into a shock wave.[5]

The small parameter $\nu\ll 1$ in Eq. (5) can be utilized to find an analytic solution in the adiabatic approximation. Since this solution describes weakly damped oscillations, we can take the unperturbed solution to be the periodic solution found for a collisionless plasma ($\nu=0$) in Ref. 1:

$$Z(\zeta)=\begin{cases}\left[1-\dfrac{\mathrm{sn}^2(\varkappa\zeta,k)}{\sin^2\varphi}\right]dn^{-2}(\varkappa\zeta,k), & |\zeta|\leqslant\zeta_0,\\ -\left(1-\dfrac{2}{3}Z_0\right)^{1/2}\cos\left(\dfrac{L}{2}-\zeta\right), & \zeta_0\leqslant|\zeta|\leqslant\dfrac{L}{2}.\end{cases} \tag{7}$$

In Eq. (7) we have retained the notation of Ref. 1:

$$(2\varkappa)^4=\left(1-\frac{2}{3}Z_0\right)(1+2Z_0), \quad 2k^2=1+(2Z_0-1)(2\varkappa)^{-2},$$

$$\sin^2\varphi=2\left[1+\frac{3}{4\varkappa^2}\left(1-\frac{2}{3}Z_0\right)\right]^{-1}.$$

The points $\zeta=0$ and $\zeta_0=(L-\pi)/2$ correspond to the center and boundary of a bunch, and $\mathrm{sn}(\varkappa\zeta,k)$ and $\mathrm{dn}(\varkappa\zeta,k)$ are elliptic functions.

The nonlinear spatial period of the beam is expressed in terms of an incomplete elliptic integral of the first kind:

$$L=\pi+2\varkappa^{-1}F(\varphi,k) \tag{8}$$

It increases without bound in accordance with $L\sim\ln(1-Z_0)^{-1}$ as the periodic solution transforms into a soliton.[1]

We now move on to the next approximation in the small parameter ν. Working from Eqs. (5) and (7), and taking an average over the fast oscillations, we find an equation for the slowing varying function $E(\zeta)$:

$$\frac{dE}{d\zeta}=\left(\nu+4\frac{Z_0'}{Z_0}\right)\frac{1}{L}\int_0^L\left[\frac{dZ(\zeta',E)}{d\zeta'}\right]^2 d\zeta'. \tag{9}$$

The integral on the right side of (9) is the following adiabatic invariant from mechanics:[6]

$$I=2^{3/2}\int_{Z_{min}}^{Z_0}[E(\zeta)-U(Z)]^{1/2}dZ. \tag{10}$$

Using the relation $I'(E)=L$, and integrating (9), we find

$$Z_0^{-4}I(Z_0)=I_\infty\exp[\nu(\zeta-\zeta_\infty)], \tag{11}$$

where ζ_∞ corresponds to the soliton solution $Z_0 = I$, $E=U(Z_0)$.

The integral in (10) can be expressed in terms of incomplete elliptic integrals of the first and second kinds:

$$I(Z_0)=\frac{\pi Z_0^2}{2}\left(1-\frac{2}{3}Z_0\right)+\frac{3}{5}\left\{(2\varkappa)^5[2(1-k^2+k^4)E(\varphi,k)\right.$$

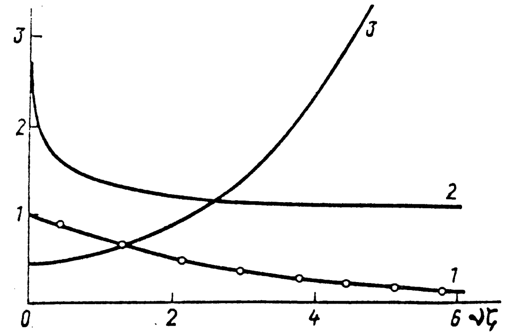

FIG. 2. The slowly varying functions (*1*) Z_0, (*2*) $L/2\pi$, and (*3*) W_m. The points on line 1 correspond to the maxima of the function $Z(\nu\zeta, \zeta)$.

$$-(2-k^2)(1-k^2)F(\varphi,k)]-2Z_0\left(Z_+^{-1}-\frac{1}{3}\right)\left(1-\frac{2}{3}Z_0\right)^{1/2}\Bigg\},\tag{12}$$

$$Z_+=\frac{3}{4}\left\{1-\frac{2}{3}Z_0+\left[\left(1-\frac{2}{3}Z_0\right)(1+2Z_0)\right]^{1/2}\right\}.$$

For highly compressed bunches, with $Z_0 \ll 1$, and with a nearly monochromatic wave,[1,2] Eqs. (8) and (12) simplify:

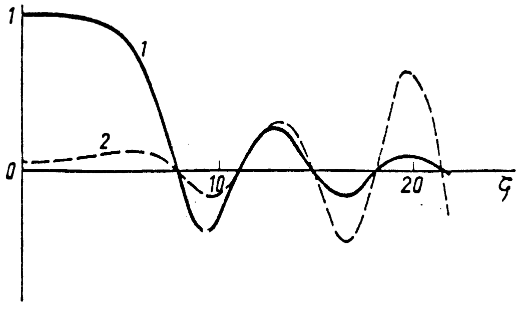

FIG. 3. Shock wave at $\nu = 0.3$. *1*—The function $Z(\nu\zeta, \zeta)$; *2*—potential at the beam axis, $\psi(0,\zeta)/(20\psi_\infty)$.

$$L=2\pi+\frac{4}{3}Z_0+\frac{Z_0^2}{3}\left(\frac{5\pi}{4}-\frac{4}{3}\right),\quad I=\pi Z_0^2.\tag{13}$$

For a near-solution a soliton solution (with $\Delta = 1 - Z_0 \ll 1$), we find the following results from (8) and (12):

$$L=\pi+2\ln\left[\frac{12}{\Delta}(2-3^{1/2})\right],$$
$$I=I_\infty-\frac{\Delta^2}{2}(1+L),\quad I_\infty=\frac{\pi}{6}+\frac{6}{5}\left(1-\frac{2}{3^{1/2}}\right).\tag{14}$$

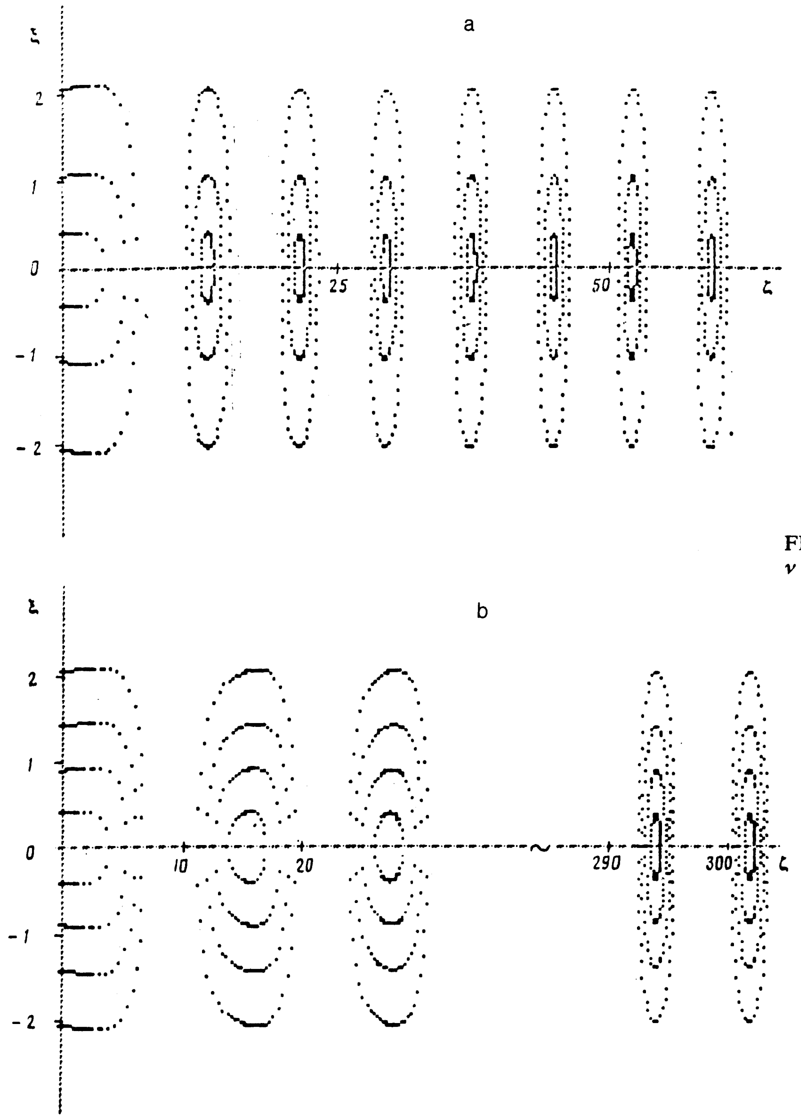

FIG. 4. Curves of constant electron density in the bunches. a: $\nu = 0.1$, $C = 0.9, 0.5, 0.1$. b: $\nu = 0.01$; $C = 0.9, 0.6, 0.3, 0.1$.

Equations (11), (12), and (14) determine the function $Z_0(\nu\zeta)$ and thus the slowly varying coefficients $k(\nu\zeta)$, $\varkappa(\nu\zeta)$, and $\varphi(\nu\zeta)$ in Eqs. (7) and (8). The results of the adiabatic theory are shown in Fig. 2, where line *1* is a plot of $Z_0(\nu\zeta)$. Lines *2* and *3* show the slow spatial evolution of the beam modulation period and of the function $W_m = (R_0^2 Z_0)^{-1}$. These curves are in qualitative agreement with the asymptotic expressions in (13) and (14). The two-scale solution $Z(\nu\zeta, \zeta)$ which follows from Eqs. (7), (11), and (12) is shown in Fig. 3.

The auxiliary function $Z(\nu\zeta, \zeta)$, which has the form of a classical shock wave, determines the profile of the potential along the longitudinal coordinate:

$$\psi(\xi, \zeta) = \psi_0 [R_0 Z_0(\nu\zeta)]^{-2} R(\xi) Z(\nu\zeta, \zeta),$$

A simple physical explanation can be found by introducing the time $T = -\zeta$. The time $T \to -\infty$ then corresponds to the creation of a infinitely deep potential well for the bunches whose longitudinal dimension is equal to half the wavelength, $\pi c/\omega_p$ (cf. Refs. 1 and 2). At later times, Coulomb collisions in the plasma reduce the field amplitude. As $T \to \infty$, this amplitude tends toward the asymptotic value[1] $\psi_\infty = \psi_0/R_0$. Figure 3 shows the evolution of the potential at the beam axis.

In the adiabatic approximation $(\nu \to 0)$ the particle descends infinitely slowly to the bottom of the potential well (Fig. 1), and the shock wave is an infinite set of periodic solutions. At nonzero values of the parameter ν, discreteness appears in this system. However, since the period varies only slightly over the length of an individual bunch, the constant-density surfaces

$$\frac{\rho(\xi,\zeta)}{\rho_0} = \left[\frac{R(\xi)}{R_0} \frac{Z(\nu\zeta,\zeta)}{Z_0(\nu\zeta)}\right]^2 = C \qquad (15)$$

are approximately collisionless. Figure 4 shows the results of some numerical calculations based on Eqs. (3) and (5) and those of some calculations based on Eqs. (11) and (12). The curves here correspond to the intersection of the surfaces in (15) with a plane passing through the beam axis. As the parameter ν decreases, the spatial evolution of the longitudinal dimensions of the bunches becomes more noticeable (Fig. 4b).

This study was a continuation and a development of the study reported in Ref. 1 for a collisional plasma. Consequently, the physical mechanism for self-focusing which was discussed in the collisionless approximation in Ref. 1 remains in force when collisions are infrequent, i.e., under the condition $\nu \ll 1$. At the same time, the absence of periodic solutions in a collisional plasma means that the beam modulation regime must be corrected if the steady-state wave required for transporting relativistic electron bunches over large distances is to be maintained in the plasma.

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