Cellular nonlinear dynamo

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A random flow of a conducting fluid is represented as consisting of cells within each of which an independent strengthening or weakening of the magnetic field occurs. The cells are coupled with each other by one-dimensional magnetic diffusion. An increasing magnetic field suppresses strengthening through a nonlinear inverse effect. The distribution function of the steady-state magnetic field is non-Gaussian. The field is intermittent in time and space; i.e., it is clumped in distinct strong concentrations. A comparison is made with the distribution of a scalar property in the same flow.

1. INTRODUCTION

An initially weak magnetic field frozen in a random flow of a turbulent conducting fluid increases exponentially with time.[1] When the energy of this field becomes comparable to the energy of hydrodynamic motions, a dynamic equilibrium is reached (a stage of nonlinear saturation).

The flow of the fluid in the first stage of evolution of the magnetic field, in which the inverse effect of this field on the motion can be ignored, is called a "kinematic dynamo." When the inverse effect is taken into account, one speaks in terms of a "nonlinear dynamo." A kinematic dynamo is characterized by intermittence: the magnetic field is concentrated in structures which occupy only a small fraction of the volume but which determine the overall magnetic energy. The contrast between the maximum concentration and the background magnetic field increases exponentially.[2]

In the nonlinear stage, this intermittence can of course not be as catastrophic as in the linear stage. On the other hand, one might expect that intermittence effects would still be manifested, in a slightly altered form.

A nonlinear dynamo in a turbulent flow has been studied through a direct joint numerical solution of the Navier-Stokes equations and the induction equation.[3] These studies have shown that the magnetic field does indeed concentrate in distinct formations which resemble braids. The equilibrium magnetic-field distribution was studied by analytic methods in Ref. 4. A simple model of the nonlinearity was used there. An expression of the form $H^{-\beta}$ was derived for the distribution of the magnetic field, although the index β could not be calculated analytically. Under the condition $\beta < 0$, the magnetic field is typically very low, and the maximum value is close to the mean value. This shows the intermittent nature of the magnetic field distribution.

Direct numerical simulations are not the only way to study the generation of magnetic fields in random flows. A magnetic field apparently generated by this process can be observed directly in, for example, galactic clusters.[5,6] Several properties of growing solutions can be identified through an asymptotic study of the induction equation in a turbulent medium (Ref. 7, for example). Analysis of the results found by this approach leads to the conclusion that the generation process is governed by more than the customary universal characteristics of turbulence such as the index of the energy spectrum. The process apparently also depends substantially on quantities which have not been studied extensively, either theoretically or by astronomical observation. Under these circumstances, an additional approach to the theory of a turbulent dynamo emerged in the late 1980s. In that approach, the velocity field is not specified; instead, the effect of this field on the magnetic field is effectively described. This approach goes back to Zel'dovich's classical "figure-eight," which describes the elementary doubling of a magnetic-field loop in a flow with a frozen-in magnetic field. This figure-eight is usually invoked in qualitative discussions of the mechanism for the dynamo. This approach was developed in Ref. 8 into a qualitative model of magnetic-field generation in a single turbulence cell. It was later suggested in Refs. 9 and 10 that the flow be represented as consisting of distinct cells, within which a nontrivial three-dimensional strengthening or weakening of the magnetic field occurs. For example, there may be a conversion of a Zel'dovich figure-eight type in a generalized form. The cells communicate with each other by magnetic diffusion.

We recently took this approach to study the evolution of the magnetic field in the kinematic stage.[11] Our purpose in the present paper is to study the process in the nonlinear stage.

2. NONLINEAR MODEL FOR THE EVOLUTION OF THE MAGNETIC FIELD

Following Ref. 11, we consider a set of cells in which an initial magnetic field H_0 is given. Over the time scale which is characteristic of the magnetic-field conversion in cell i, and which does not depend on magnetic diffusion, the magnetic field vector may increase by a factor of $p(i)$ as a result of the dynamo mechanism, and it may reverse direction. In each discrete step of this sort, the field distribution is averaged with the help of a one-dimensional diffusion operator with a magnetic diffusion coefficient ε. As a result we have

$$H_{n+1}(i) = S_\varepsilon p_n(i) q_n(i) H_n(i), \tag{1}$$

where the random numbers $p_n(i)$ and $q_n(i)$ are independent for different values of the discrete time n and for different cells. The random number $q_n(i)$ describes a reorientation of the magnetic field, taking on values of ± 1 with equal probabilities. The numbers p are written in the form $\exp\xi$, where ξ is a quantity with a normal distribution, a zero mean, and a standard deviation of 2. This representation corresponds to a

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velocity field with a normal distribution in the cells. The magnetic diffusion operator is

$$\hat{S}_\varepsilon H_n(i)=\frac{1}{\|\ldots\|}\sum_{i=1}^{N}\exp\left[-\frac{(i-j)^2}{2\varepsilon^2}\right]H_n(j), \qquad (2)$$

where the normalization factor is equal to the area under the gaussoid curve used for the diffusion-induced smearing:

$$\|\ldots\|=\sum_{i=1}^{N}\exp\left[-\frac{(i-N/2)^2}{2\varepsilon^2}\right].$$

A region of 300 cells was adopted for a numerical simulation. The numerical simulation shows that this number is sufficient for achieving stable results. The magnetic field is assumed to vanish at the boundary of this region.

We introduce a nonlinearity, assuming that the coefficient by which the magnetic field is strengthened (or weakened, depending on the sign of ξ) depends on the value of the field at the given point. For example, we have

$$p_n(i)=(1+H^2/H_*^2)^{-1}\exp\xi_n(i), \qquad (3)$$

where H_* is a characteristic value of the magnetic field. It can be assumed that this value corresponds to an approximate equality of the magnetic energy and the kinetic energy of the random motions.

A problem[12,13] of the evolution of a scalar field closely related to the problem at hand can be found by setting $q_n = 1$.

3. EQUILIBRIUM DISTRIBUTION OF THE MAGNETIC FIELD

A numerical solution of problem (1) shows that any initial magnetic field distribution quickly becomes nonuniform (Fig. 1). A steady-state distribution is reached in 20–40 steps of the characteristic time (which is on the order of the reciprocal of the field growth rate in the kinematic problem with a standard deviation of 2; Ref. 11). The steady-state probability distribution of the magnetic field (Fig. 2) is stable, changing only slightly if a change occurs in the form of the nonlinearity [if, say, the quadratic dependence on the magnetic field in (3) is replaced by a cubic dependence] or upon variation of the parameters of the problem. It can be seen from this figure that the maximum magnetic field is well below H_* (by a factor 2 or 3). The apparent reason for this result is that the rate of the exponential growth in a random

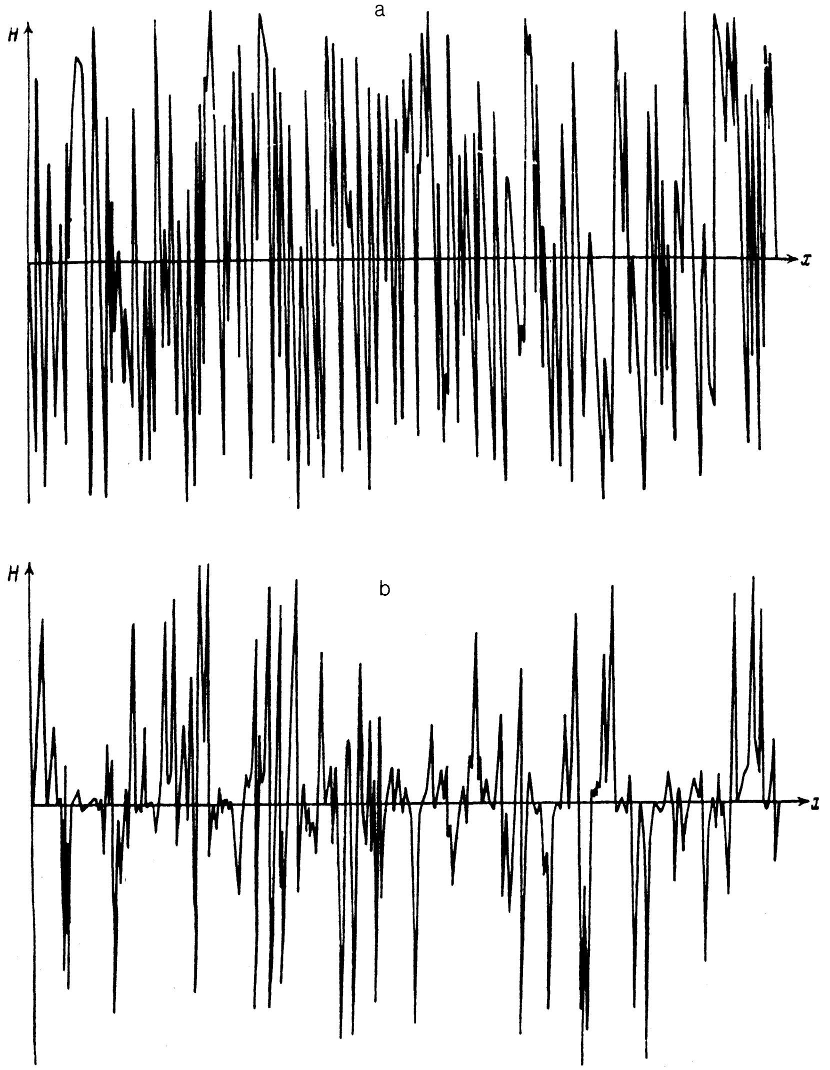

FIG. 1. Snapshots of the spatial distribution of the magnetic field (over the cells) after the first time step (a) and after the 41st (b). The initial distribution is rectangular, with zero boundary values.

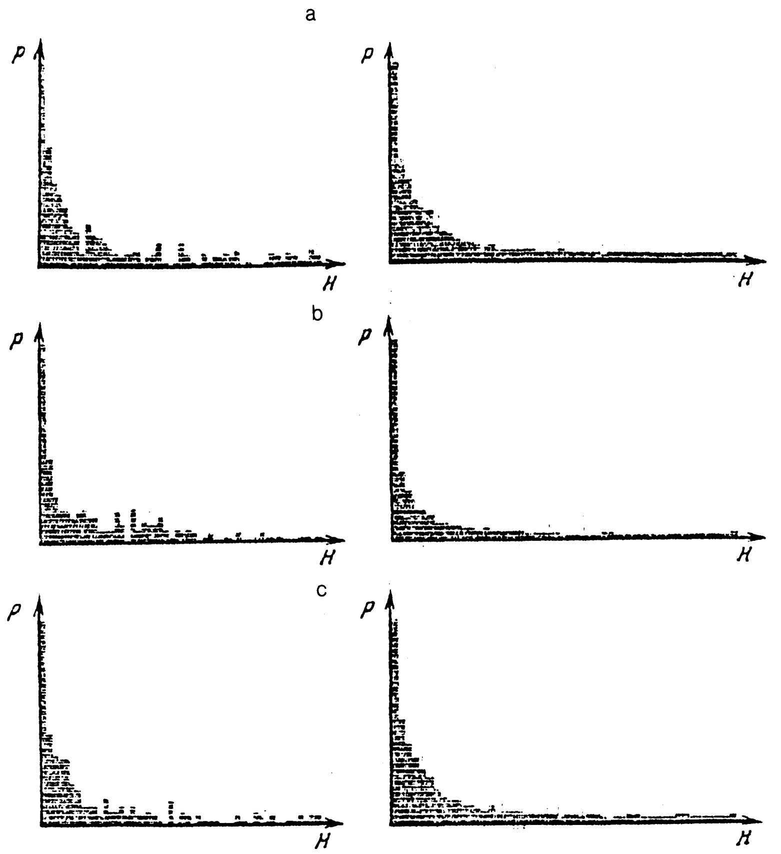

FIG. 2. Probability density of the magnetic field distribution. Left: Snapshot at the time $n = 39$. Right: Average over 20 time steps (from the 22nd to the 41st). The magnetic diffusion coefficient ε is (a) 0.5 or (b) 3. The nonlinearity in (3) is quadratic in the magnetic field. c—Probability density for a cubic nonlinearity with a diffusion coefficient 0.5.

flow is low, and even a slight weakening is sufficient to suppress generation.

The most probable value of the magnetic field is close to zero, so the distribution function can be approximated by a δ-function near the origin and by some decreasing function at $H \neq 0$. The latter is not Gaussian; it is more nearly a power law (Fig. 3). A calculation of the moment coefficient of kurtosis $F = \langle H^4 \rangle / \langle H^2 \rangle^2$, where $\langle ... \rangle$ means an average over

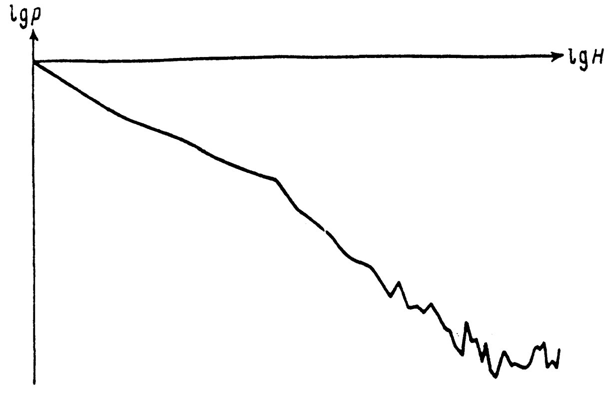

FIG. 3. Probability density versus the magnetic field, in logarithmic scale. The approximately linear shape of this distribution supports the contention that the distribution is not Gaussian. The parameter values are the same as for Fig. 2a.

cells, shows that this quantity, which is random in time, varies from 7 to 9. We recall that the moment coefficient of kurtosis of a Gaussian distribution would be 3. A value higher than 3 implies that the tail of the distribution is playing an important role, i.e., that there is an increased probability for large deviations.

The behavior of the probability density thus implies an intermittent distribution of the magnetic field. A snapshot of the magnetic field distribution in the stage of nonlinear saturation (Fig. 1b) shows that the magnetic field is clumpy. Table I shows calculated evolutions of several statistical moments of the magnetic field. The growth rates increase exponentially with increasing index of the moment, providing further support for the idea that the magnetic field distribution is intermittent in both the initial kinematic stage and the nonlinear stage.

The solution found here for the nonlinear magnetic-field problem is quite different from that for a scalar field. A transition to the case of a scalar field can be made by setting $q_n = 1$. In the scalar case, there cannot be a well-defined peak in the distribution at the origin, and the distribution itself will be bell-shaped, indicating that the scalar field is concentrated in "columns" with a height determined by the maximum of the distribution. The tail of the distribution, particularly at small diffusion coefficients, is quite substantial. The intermittence is thus seen in the scalar case also. Interestingly, the magnetic field distribution (this is a vector

TABLE I. Time evolution of several statistical moments of the magnetic field for two values of the magnetic diffusion coefficient.

n	$\ln H_{max}$	$\ln \langle H\rangle^{1/2}$	$\ln \langle H\rangle^2$	$\ln \langle H\rangle^4$
		$\varepsilon=0$		
10	−0,347	−0.594	−1,299	−0,919
20	−0,347	−2,861	−1,471	−1,012
30	−0,3517	−3,229	−1,617	−1,129
40	−0,3473	−3,727	−1,812	−1,239
50	−0,4366	−3,861	−1,724	−1,153
		$\varepsilon=0,5$		
10	−0,511	−2,46	−1,59	−1,19
20	−0,459	−2,37	−1,54	−1,16
30	−0,491	−2,36	−1,57	−1,18
40	−0.483	−2,61	−1,64	−1,20
50	−0,504	−2,52	−1,61	−1,21

problem in which reconnection of magnetic field lines is taken into account) is quite stable. When the parameter values are varied, the only change is in the right-hand end of the field distribution. In contrast, the distribution of a scalar changes substantially when the parameters of the problem (primarily the diffusion coefficient) are varied.

We also show here (Fig. 4) a two-point correlation function of the magnetic field. This function was calculated for an average cell (the 150th) and then averaged over time (1600 steps). The correlation function falls off to zero over roughly the diffusion length; i.e., the "large" structures are of a diffusive nature. The oscillations in the tail of the correlation function reflect fluctuations associated with the discrete nature (i.e., the cellular nature) of the space.

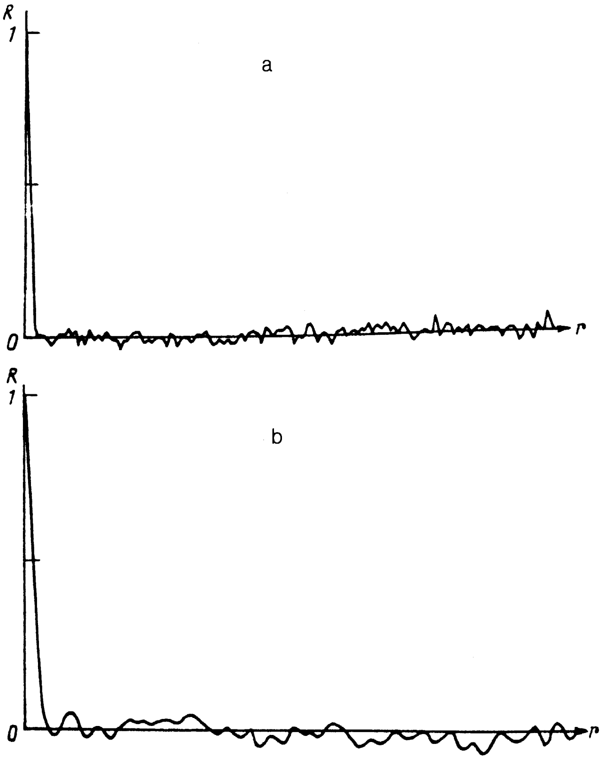

FIG. 4. Spatial correlation function of the magnetic field at $\varepsilon = 3$. The nonlinearity is quadratic. a—The magnetic diffusion coefficient is 0.5; b—3.

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