Effect of dissipative processes on the evolution of a density perturbation in an expanding isotropic universe

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Gauge-invariant equations are derived for a theory of gravitational stability, describing the evolution of irrotational perturbations in an isotropic universe of critical density filled with a dissipative gaseous medium. These equations reduce to a singularly perturbed third-order equation which is nonlinear in several parameters. The solutions of this third-order equation thus have bifurcation properties. This equation has a singular point, whose position on the time axis, $t = t_s$, is determined by the dissipation characteristics of the system. Near the singular point, the system loses its dynamic stability, and condensations with comparatively low peculiar velocities undergo an anomalous growth. This effect is interpreted as the formation of dissipative structure in the expanding universe. One possible critical point which appears to be pertinent to the theory of galaxy formation is at $t \approx 10^{16}$ s, where most of the mass of the baryon subsystem of the universe is in neutral hydrogen and helium. The effect of cold dark matter on the nature of a nonequilibrium phase transition in the baryon subsystem is analyzed.

INTRODUCTION

It is presently believed that the large-scale structure of the hot universe arises as the result of gravitational instability.[1-3] It is usually assumed that small fluctuations in the density of matter, whose evolution subsequently leads to the appearance of structure (star clusters, galaxies, and galactic clusters), themselves originate from quantum fluctuations of the space-time metric. Later, according to certain hypotheses, these fluctuations can grow to a macroscopic size in the inflationary epoch.[4,5] After this epoch, they can grow as the result of accretion of surrounding matter. Models of dark matter are often invoked in explanations of the observed large-scale structure.[4,6] In particular, entities which have been proposed as candidates for the role of a nonbaryonic dark matter are particles predicted by various gauge theories[7,8] and exotic entities such as superconducting strings in a primordial magnetic field.[9-11]

When this picture of continuous evolution of density perturbations from an initial cosmological singularity to galaxies and galactic clusters is compared with experimental data, however, a hot model runs into certain experimental difficulties. These difficulties are not easily overcome. Specifically, the amplitude of density perturbations in the prerecombination epoch is known to be related to the anisotropy of the relic background microwave radiation. However, observations impose some severe limitations on this anisotropy.[1-4] According to the observational data and their theoretical interpretation,[12-16] the quantity $\Delta T/T$ must be less than 10^{-4}. Since a long time has now passed without any reliable reports of a discovery of such an anisotropy, we believe it is necessary to seek some new, alternative hypotheses regarding possible scenarios for the generation of the large-scale structure of the universe. One possibility might be the hypothesis of a strong, superadiabatic growth of density perturbations as a result of nonlinear effects associated with dissipative processes in the cold viscous gas in the later stages of cosmological evolution. Explosive effects of this sort might occur if the system is sufficiently complex for the appearance of dissipative structure. A correct approach to the formulation of this problem and to its solution requires the simultaneous consideration of numerous factors which influence the evolution of perturbations of the density of matter in the epoch following the time of recombination.

In this paper we focus on the dynamics of density perturbations near singular points at which the evolution time is comparable in order of magnitude to the time scale of the relaxation process. The reasoning here is that a viscous gaseous medium filling a time-varying universe is in a varying gravitational field and in this sense is an open dissipative system. Under certain conditions, dissipative structure can appear in such a system, in the form of density inhomogeneities. We know that dissipative structure arises because the open system searches for new equilibrium states, different from the homogeneous state of local thermodynamic equilibrium.[17-19] We would therefore expect that the age of the universe at the time at which the dissipative structure forms is comparable to the time scale of the relaxation process.

Current ideas regarding the gravitational stability of an expanding universe are based on the linear theory of gravitational stability (LTGS) derived by E. M. Lifshitz.[20,21] We will be using the mathematical tools of that theory, specifically stipulating that those tools are valid for solving the problems formulated above.

The authors of the various papers on the LTGS have used a variety of methods for solving its equations, but all employ approximations. Our major goal here is to construct a method for solving the LTGS equations which would make it possible to analyze the asymptotic behavior of the solutions only after the nature and properties of the equations of the theory have been determined.[1)] It would then become possible to numerically integrate the exact equations which are constructed. We have done this in the present study.

We believe that ignoring any terms in the initial equations of the theory may lead to a theory with a seriously impoverished physical content. For example, if we ignore the viscosity, we are ignoring relaxation processes in the system.[2,20] If we instead take viscosity into account, we must take account of the nonzero sound velocity ($u^2 \neq 0$) for any relaxing system (our approach here stands in contrast with common practice in solving the LTGS equations in the late stages of evolution of the universe[1,2,22]). The reason is that, since this system is open and time-varying, both the relaxation time and the velocity of sound in the medium depend on the world time t. There can thus be a situation in which terms of the energy-momentum tensor associated with the pressure are comparable in order of magnitude to the dissipative terms, $p \simeq \xi \dot{a}/a$, at large values of the viscosity coefficient. (Large values of the bulk viscosity coefficient occur for molecular hydrogen, for example, since the large size of the rotational quantum of the H_2 molecule, $T_r = 85$ K, makes the time scale for rotational relaxation of the gas molecules long.) These times may be singular for momentum transport in the system. Near these points on the time axis, a hydrodynamic description is of course incorrect. In the LTGS, however, the interaction of the spatially uniform mode $a(t)$ and a spatially nonuniform mode $h_{ik}(t)$ is taken into account exactly (ignoring the interaction of the spatially nonuniform modes with each other). However, since the open nature of a dissipative system results specifically from this interaction, it is manifested considerably earlier, within the range of applicability of the linear approximation.

Yet another distinctive feature of the LTGS on an expanding background is the existence of two independent degrees of freedom (isotropic and anisotropic)[20] for the perturbed gravitational field. While momentum transport for one degree of freedom has a singularity (the momentum flux in the density wave is comparable in order of magnitude to the momentum sink due to dissipation), the system tends to maintain the existence of this degree of freedom. The behavior of this system thus becomes anomalous: An initial density perturbation breaks up into growing condensations with relatively small peculiar velocities. In other words, dissipative processes give rise to dissipative structure.

Obviously, if an approximate solution method is used, and the LTGS equations are reduced to the post-Newtonian equations of the general theory of relativity,[3] this effect will be lost.

A point which deserves particular emphasis is that taking the viscosity and the compressibility of the medium into account simultaneously leads to an effect of the thermal regime on the course of the gravitational instability. As we will show below, this system is described by a third-order singularly perturbed equation in the parameter k^2u^2/a^2, where k is the wave number.[2)] This equation contains logarithmic derivatives of the sound velocity, $\dot{u}/u$ (as was first shown in Ref. 25). Since the thermal regime of this continuous medium is not adiabatic, and it depends primarily on the effect of the viscosity on the temperature of the medium, the behavior of all modes of the solution is distorted by thermal effects.

We therefore assume that the only approximation procedure which is legitimate for the Einstein equations describing the expanding universe is the standard linearization procedure.[1-3,20-23,25-32] This procedure allows one to correctly formulate the problem of the evolution of normal hydrodynamic modes superposed on an expanding background.[32]

This Introduction to the paper is followed by five numbered sections, a Conclusion, and an Appendix. In Sec. 1 we derive a system of LTGS equations in terms of linear invariants, i.e., in a gauge-invariant form which does not depend on the state of an observer following the evolution of inhomogeneities in the expanding universe. The asymptotic solutions of the gauge-invariant equations of the LTGS far from these singular points, where the evolution time is equal in order of magnitude to the time scale of a relaxation process, are studied in Sec. 2. In Sec. 3 we carry out an asymptotic analysis of the equations of the theory near singular points. We also offer a qualitative interpretation of some new physical effects which appear. Section 4 is an analysis of numerical solutions of the LTGS equations which have been derived. In Sec. 5 we estimate the effect of cold dark matter on the nature of the phase transition to an inhomogeneous state in the baryon subsystem. The Appendix contains certain details of the derivation of the LTGS solutions near singular points.

1. EQUATIONS OF THE LINEAR THEORY OF GRAVITATIONAL STABILITY IN TERMS OF LINEAR INVARIANTS IN THE LATE STAGE OF EVOLUTION OF THE UNIVERSE

We describe the evolution of normal hydrodynamic modes in the expanding universe by means of the system of Einstein equations with dissipative effects[33,34] (we are using a rational system of units with $\hbar = c = 1$):

$$R_i^k - {}^1/_2 \delta_i^l R = 8\pi G T_i^k = 8\pi G[(p+\varepsilon)u_i u^k - \delta_i^k p + \tau_i^k], \tag{1.1}$$

$$(nu^i + \nu^i)_{;i} = 0, \tag{1.2}$$

$$(\sigma u^i)_{;i} = \frac{\tau_i^k}{T} u^i_{;k} + \frac{\mu}{T} \nu^i_{;i}\,. \tag{1.3}$$

Here

$$\tau_i^k = \eta D_i^k + \xi d_i^k$$

is a dissipation tensor, where

$$D_i^k = (\delta_l^k - u_l u^k) u^l_{;i} + (\delta_i^l - u_i u^l) u^k_{;l} - {}^2/_3 (\delta_i^k - u_i u^k) u^l_{;l},$$
$$d_i^k = (\delta_i^k - u_i u^k) u^l_{;l}$$

are the tensors of shear and bulk deformations of the velocity field of the matter. The quantity

$$\nu_i = \varkappa \left(\frac{nT}{p+\varepsilon}\right)^2 \left[\left(\frac{\mu}{T}\right)_{;i} - u_i u^k \left(\frac{\mu}{T}\right)_{;k}\right]$$

is the heat-flux vector; η, ξ, and $\varkappa$ are the coefficient of shear viscosity, the coefficient of bulk viscosity, and the thermal conductivity; n, ε, and σ are the number density of particles, the energy, and the entropy; p is the pressure; T is the temperature; and μ is the chemical potential of the continuous medium.

We write these equations out explicitly, noting that the real medium is described by the energy-momentum tensor of an ideal gas, distorted by dissipative effects:[35]

$$\varepsilon = mn + \frac{3}{2} nT + n\left[f(T) - T\frac{df}{dT}\right], \qquad p = nT,$$
$$\mu = \frac{p+\varepsilon-\sigma T}{n},$$
$$\sigma = n \ln\left[\frac{e}{n}\left(\frac{mT}{2\pi}\right)^{3/2}\right] + n\left(\xi_c + \frac{3}{2}\right) - n\frac{df}{dT},$$
$$\frac{d^2 f}{dT^2} = -\frac{c_r(T)}{T},$$

$$f(T) = -T\ln(Z_r), \qquad Z_r = \sum_{K=0}^{\infty} (2K+1)\exp\left[-\frac{T_r}{T}K(K+1)\right]. \tag{1.4}$$

Here m is the mass of a gas molecule; $c_r(T) = c_r$ is the specific heat of the rotational degrees of freedom; and ξ_c is the chemical constant of the translational degrees of freedom of the gas.

The initial stage of the process is described by Eqs. (1.1)–(1.3), linearized about an isotropic cosmological background:

$$ds^2=dt^2-a^2(t)(dx^2+dy^2+dz^2). \tag{1.5}$$

We now write the physical quantities describing the matter and the gravitational field in the form $A = \bar{A} + \delta A$ (background part plus a perturbation).

An unperturbed Friedmann solution for the isotropic universe with dissipation is found from the equations

$$8\pi G\bar{\varepsilon}=3\frac{\dot{a}^2}{a^2},\quad 8\pi G\bar{p}=-2\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}+24\pi G\bar{\xi}\frac{\dot{a}}{a}, \tag{1.6}$$

$$\bar{u}^0=\bar{u}_0=1,\quad \bar{u}^\alpha=0,\quad (\sigma a^3)^\cdot=9\frac{\dot{a}^2}{a^2}\frac{\bar{\xi}}{T},\quad n=\frac{n_0}{a^3},$$

where

$$\dot{a}=\frac{da}{dt},\quad (\sigma a^3)^\cdot=\frac{d}{dt}(\sigma a^3).$$

Equations (1.1)–(1.3) will now be put in the standard form for the theory of gravitational stability. All perturbed quantities are found as tensors in the background space, $\bar{g}_{ik}$. Specifically, the metric tensor, the 4-velocity of the matter, the number density of the particles, the energy, the entropy, the temperature, the pressure, and the chemical potential of the continuous medium are written in the form

$$g_{ik}=\bar{g}_{ik}+h_{ik},\ u_i=\bar{u}_i+v_i,\ \varepsilon=\bar{\varepsilon}+\delta\varepsilon,$$
$$n=n+\delta n,\ \sigma=\bar{\sigma}+\delta\sigma,\ T=\bar{T}+\delta T,$$
$$p=\bar{p}+\delta p,\ \mu=\bar{\mu}+\delta\mu.$$

The linearized system of Einstein equations is

$$P_i^k-h_m^k\bar{R}_i^m-\frac{\delta_i^k}{2}(P-h_m^l\bar{R}_l^m)=8\pi G(\theta_i^k-h_m^k\bar{T}_i^m), \tag{1.7}$$

$$\bar{u}_i\bar{u}^kh_k^i-2\bar{u}^iv_i=0, \tag{1.8}$$

$$(\delta n\bar{u}^i+\bar{n}v^i)\left[\ln\left(\frac{e}{\bar{n}}\left(\frac{mT}{2\pi}\right)^{3/2}\right)-\frac{df}{d\bar{T}}\right]_{,i}$$
$$+\bar{n}\bar{u}^i\delta\left[\ln\left(\frac{e}{\bar{n}}\left(\frac{mT}{2\pi}\right)^{3/2}\right)-\frac{df}{dT}\right]_{;i}=\frac{\delta\tau_i^ku^i_{;k}}{\bar{T}}+\frac{\tau_i^k\delta u^i_{;k}}{\bar{T}}$$
$$-\frac{\bar{\tau}_i^ku^i_{;k}}{\bar{T}^2}\delta T+\left[\frac{\bar{\mu}}{\bar{T}}+\ln\left(\frac{e}{\bar{n}}\left(\frac{m\bar{T}}{2\pi}\right)^{3/2}+\left(\xi_c+\frac{3}{2}\right)-\frac{df}{d\bar{T}}\right]\delta v^i_{;i}. \tag{1.9}$$

The unperturbed Ricci tensor $\bar{R}_l^m$, the energy-momentum tensor $\bar{T}_l^m$, and the dissipation tensor $\bar{\tau}_l^m$ are constructed from the quantities $\bar{g}_{ik}$, $\bar{\varepsilon}$, $\bar{p}$, $\bar{\eta}$, $\bar{\xi}$, $\bar{\varkappa}$, $\bar{\mu}$, and $\bar{T}$ in the usual way:

$$P_i^k=\frac{1}{2}(h_l^{m;k}{}_{;m}+h_m^{k;m}{}_{;i}-h_l^{k;m}{}_{;m}-h_m^{m;k}{}_{;i}), \tag{1.10}$$

$$\theta_i^k=(\bar{p}+\bar{\varepsilon})(\bar{u}_iv^k+v_i\bar{u}^k)+(\delta p+\delta\varepsilon)\bar{u}_i\bar{u}^k-\delta_i^k\delta p-h_i^k\bar{p}+\delta\tau_i^k, \tag{1.11}$$

$$\delta v_{;i}^i=\bar{\varkappa}\left(\frac{\bar{n}\bar{T}}{\bar{p}+\bar{\varepsilon}}\right)^2\bar{g}^{ip}\left[\delta\left(\frac{\mu}{T}\right)_{,i}\right.$$
$$\left.-\delta(u_iu^k)\left(\frac{\mu}{T}\right)_{,k}-\bar{u}_i\bar{u}^k\delta\left(\frac{\mu}{T}\right)_{,k}\right]_{,p} \tag{1.12}$$

$$\delta\tau_i^k=\bar{\eta}(\delta D_i^k-h_p^kD_i^p)+\bar{\xi}(\delta d_i^k-h_p^k\bar{d}_i^p)+\delta\eta\bar{D}_i^k+\delta\xi d_i^k. \tag{1.13}$$

In Eqs. (1.7)–(1.13) and in the equations below, the semicolon means a covariant derivative in the space $\bar{g}_{ik}$.

In order to solve Eqs. (1.7)–(1.13) superposed on (1.5), we need to rewrite these equations in a (3 + 1)-dimensional form; in other words, we need to separate the spatial and temporal coordinates. To do this, we define the operation of (3 + 1)-dimensional covariant differentiation[25] in a space with the metric

$$\overset{(0)}{\gamma}_{\alpha\beta}=\mathrm{diag}(+1,+1,+1)$$

and the three-dimensional covariant quantities

$${}^{(3)}h^\alpha=-h_0^\alpha,\quad {}^{(3)}h^{\alpha\beta}=-h^{\alpha\beta},\quad {}^{(3)}v_\alpha=v_\alpha$$

(the Greek-letter indices take on the values 1, 2, 3). In the discussion below, we will be dealing exclusively with three-dimensional quantities and operations, so we will omit the superscript (3) from the corresponding tensors.

With these definitions in place, and after some simple calculations, we can rewrite the dissipative part of the energy-momentum tensor in the explicit form

$$\delta\tau_\alpha^\beta=\bar{\eta}\left\{-(v^\beta_{,\alpha}+v_\alpha^{,\beta})+(h^\beta_{\alpha,0}+h_\alpha^{,\beta}+h^\beta_{,\alpha})\right.$$
$$\left.-\frac{2}{3}\delta_\alpha^\beta\left[-v^\gamma_{,\gamma}+\frac{1}{2}(h_{,0}+2h^\gamma_{,\gamma})\right]\right\}$$
$$+\bar{\xi}\left[-v^\gamma_{,\gamma}+\frac{1}{2}(h_{,0}+2h^\alpha_{,\alpha})\right]\delta_\alpha^\beta+\delta_\alpha^\beta\frac{\dot{a}}{a}\delta\xi, \tag{1.14}$$

$$\delta\tau_\alpha^0=-3\frac{\dot{a}}{a}\bar{\xi}v_\alpha,\qquad \delta\tau_0^0=0.$$

To construct explicit equations for the linearized theory, we make the substitution

$$\delta p(\bar{n},\bar{T})=u^2(t)\delta\varepsilon(\bar{n},\bar{T})+\delta p_1(\bar{n},\bar{T}), \tag{1.15}$$

where the function

$$u^2(t)=\frac{d\bar{p}}{d\bar{\varepsilon}}=\frac{\dot{\bar{p}}}{\dot{\bar{\varepsilon}}}=\left[-\frac{\dddot{a}}{a}+\frac{\dot{a}^3}{a^3}+12\pi G\dot{\bar{\xi}}\frac{\dot{a}}{a}\right.$$
$$\left.+12\pi G\bar{\xi}\left(\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)\right]\Big/\left(3\frac{\dot{a}\ddot{a}}{a^2}-3\frac{\dot{a}^3}{a^3}\right)$$

is determined for the thermal regime set by the properties of the unperturbed solution (the dots mean derivatives with respect to t). Here δp_1 is the deviation of the momentum flux density of the matter from the local equilibrium values as a result of dissipation.

Here and below, $a(t)$ is a self-consistent solution of unperturbed equations (1.1)–(1.3) with a fixed kinetic coefficient $\bar{\xi}$.

The perturbation of the bulk viscosity coefficient, $\delta\xi$, is found from

$$\delta\xi=\left(\left\{u^2T/n\left[1+\frac{T}{m}(ci-c_V)\right]\right\}\frac{\partial\bar{\xi}}{\partial T}\right.$$
$$+\left.\left\{(1-c_Vu^2)/m\left[1+\frac{T}{m}(ci-c_V)\right]\right\}\frac{\partial\bar{\xi}}{\partial n}\right)\delta\varepsilon$$
$$+\left(\left\{\left(1+ci\frac{\bar{T}}{m}\right)\Big/\bar{n}\left[1+\frac{\bar{T}}{m}(ci-c_V)\right]\right\}\frac{\partial\bar{\xi}}{\partial T}\right.$$
$$\left.-\left\{c_V/m\left[1+\frac{\bar{T}}{m}(ci-c_V)\right]\right\}\frac{\partial\bar{\xi}}{\partial\bar{n}}\right)\delta p_1, \tag{1.16}$$

where

$$ci={}^3/_2+\frac{f(\bar T)}{\bar T}-\frac{df}{d\bar T},\qquad c_V={}^3/_2+c_r,$$

$$F=1-\frac{\bar T}{mu^2}+ci\frac{\bar T}{m}.$$

Using these relations, we can rewrite Eqs. (1.7)–(1.9) in the form

$$\frac{1}{a^2}[(1+3u^2)(h_{\delta,\gamma}^{\gamma,\delta}-h_{,\gamma}^{\gamma})-2h_{0,\gamma}^{0,\gamma}]+2(h_{,0,0}+2h_{,\gamma,0}^{\gamma})$$
$$+6(u^2+1)\frac{\dot a}{a}(h_{,0}+2h_{,\gamma}^{\gamma})-6h_{0,0}^{0}$$
$$+6\left[-2\left(\frac{\dot a}{a}\right)^{\cdot}-3\frac{\dot a^2}{a^2}-3u^2\frac{\dot a^2}{a^2}\right]h_0^0$$
$$-16\pi G\left\{3\bar\xi\left[-\frac{v_{,\gamma}^{\gamma}}{a^2}+\frac{1}{2}\left(h_{,0}+2h_{,\gamma}^{\gamma}-3\frac{\dot a}{a}h_0^0\right)\right]\right.$$
$$+9\frac{\dot a}{a}\left[\frac{F}{\bar n}\frac{\partial\bar\xi}{\partial\bar T}+\frac{(1-c_Vu^2)}{mu^2}\frac{\partial\bar\xi}{\partial\bar n}\right]$$
$$\left.\times\left[1+\frac{\bar T}{m}(ci-c_V)\right]^{-1}\delta\varepsilon\right\}=48\pi G\Gamma\delta p_1,\tag{1.17}$$

$$\frac{1}{a^2}(-h_{\beta,\gamma}^{\gamma\alpha}-h_{\gamma,\beta}^{\alpha,\gamma}+h_{\beta,\gamma}^{\alpha,\gamma}+h_{0,\beta}^{0,\alpha}+h_{,\beta}^{,\alpha})-h_{\beta,0,0}^{\alpha}-(h_{,\beta}^{\alpha}+h_{\beta}^{,\alpha})_{,0}$$
$$-3\frac{\dot a}{a}(h_{\beta,0}^{\alpha}+h_{\beta}^{,\alpha}+h_{,\beta}^{\alpha})$$
$$=16\pi G\bar\eta\left[-\frac{1}{a^2}(v_{,\beta}^{\alpha}+v_{\beta}^{,\alpha})+(h_{\beta,0}^{\alpha}+h_{\beta}^{,\alpha}+h_{,\beta}^{\alpha})\right],\quad\alpha\neq\beta,\tag{1.18}$$

$$8\pi G\delta\varepsilon=\frac{1}{2}\left\{\frac{1}{a^2}(h_{\delta,\gamma}^{\gamma,\delta}-h_{,\gamma}^{,\gamma})+2\frac{\dot a}{a}(h_{,0}+2h_{,\gamma}^{\gamma})-6\frac{\dot a^2}{a^2}h_0^0\right\},\tag{1.19}$$

$$v_\alpha=\frac{1}{4(\dot a_2/a^2-\ddot a/a)}\left\{-(h_{\alpha,\gamma}^{\gamma}-h,\alpha)_{,0}+h_{\gamma,\alpha}^{,\gamma}-h_{\alpha,\gamma}^{,\gamma}-2\frac{\dot a}{a}h_{0,\alpha}^{0}\right\},\tag{1.20}$$

$$\left(\frac{\delta n}{\bar n}-\frac{h_0^0}{2}\right)9\frac{\dot a^2}{a^2}\frac{\bar\xi}{\bar T}+\bar n\delta\left[\ln\left(\frac{e}{n}\left(\frac{mT}{2\pi}\right)^{3/2}\right)-\frac{df}{dT}\right]_{,0}$$
$$=\left[\frac{\bar\mu}{\bar T}+\ln\left(\frac{e}{\bar n}\left(\frac{m\bar T}{2\pi}\right)^{3/2}\right)+\left(\xi_c+\frac{3}{2}\right)-\frac{df}{d\bar T}\right]\frac{\bar\varkappa}{a^2}\left(\frac{\bar n\bar T}{\bar p+\bar\varepsilon}\right)^2$$
$$\times\left[-\delta\left(\frac{\mu}{T}\right)_{,\alpha}^{,\alpha}+v_{,\gamma}^{\gamma}\left(\frac{\bar\mu}{\bar T}\right)_{,0}\right]$$
$$+\frac{6\bar\xi}{\bar T}\frac{\dot a}{a}\left[-\frac{v_{,\gamma}^{\gamma}}{a^2}+\frac{1}{2}(h_{,0}+2h_{,\alpha}^{\alpha}-3h_0^0)\right]$$
$$-9\frac{\bar\xi}{\bar T^2}\frac{\dot a^2}{a^2}\delta T+\frac{9}{\bar T}\frac{\dot a^2}{a^2}\delta\xi,\tag{1.21}$$

where

$$\Gamma=3\frac{\dot a}{a}\left[\frac{1+ci\frac{\bar T}{m}}{\bar n}\frac{\partial\bar\xi}{\partial\bar T}-\frac{c_V}{m}\frac{\partial\bar\xi}{\partial\bar n}\right]\left[1+\frac{\bar T}{m}(ci-c_V)\right]^{-1}-1.$$

Perturbations of type I (in the classification of Lifshitz[20]) about an isotropic cosmological background can be expanded in a series in plane waves,

$$Q_k=\exp(ik_\alpha x^\alpha),$$

and the expansion coefficients can be expressed in terms of scalar functions.

For the perturbations of the metric h_i^k and of the physical quantities $\delta\varepsilon$ and δu_α, these expansions are

$$h_0^0=\sum_{\mathbf k}\varphi_{\mathbf k}Q_{\mathbf k},\quad h_\alpha=\sum_{\mathbf k}w_{\mathbf k}Q_{\mathbf k,\alpha},\quad h_\alpha^\beta=\sum_{\mathbf k}\left\{\frac{1}{3}\delta_\alpha^\beta(\rho_{\mathbf k}+\gamma_{\mathbf k})Q_{\mathbf k}+\frac{\gamma_{\mathbf k}}{k^2}Q_{\mathbf k,\alpha}^{,\beta}\right\},$$

$$\delta\varepsilon=\sum_{\mathbf k}\delta\varepsilon_{\mathbf k}Q_{\mathbf k},\quad\delta u_\alpha=v_\alpha=\sum_{\mathbf k}v_{\mathbf k}Q_{\mathbf k,\alpha},\quad\delta u^\alpha=\sum_{\mathbf k}V_{\mathbf k}Q_{\mathbf k}^{,\alpha}.$$

For the coefficients of the Fourier expansion we have the following system of equations (we will omit the scalar-function index k from this point on):

$$-\dot M-3\frac{\dot a}{a}M+\dot N+3\frac{\dot a}{a}\dot N-\frac{1}{a^2}\left[\frac{k^2}{3}N+k^2\varphi\right]$$
$$+16\pi G\bar\eta\left\{\left[k^2\left(\dot N-3\frac{\dot a}{a}\varphi\right)\Big/3a^2\left(-\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\right)\right]\right.$$
$$\left.+\dot N-M\right\}=0,\tag{1.22}$$

$$\dot M+\left[3\frac{\dot a}{a}(1+u^2)\right]M+\frac{1}{a^2}\left[(1+3u^2)\frac{k^2}{3}N+k^2\varphi\right]$$
$$-3\frac{\dot a}{a}\dot\varphi-\left[2\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}+3u^2\frac{\dot a^2}{a^2}\right]3\varphi+4\pi G\bar\xi k^2$$
$$\times\left(\dot N-3\frac{\dot a}{a}\varphi\right)\Big/a^2\left[-\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\right]-12\pi G\bar\xi\left(M-3\frac{\dot a}{a}\varphi\right)$$
$$-72\pi Gu^2\frac{\dot a}{a}\left[\frac{F}{\bar n}\frac{\partial\bar\xi}{\partial\bar T}+\frac{(1-c_Vu^2)}{mu^2}\frac{\partial\bar\xi}{\partial\bar n}\right]$$
$$\times\left[1+\frac{\bar T}{m}(ci-c_V)\right]^{-1}\delta\varepsilon=24\pi G\Gamma\delta p_1,\tag{1.23}$$

$$8\pi G\delta\varepsilon=\frac{k^2}{3a^2}N+\frac{\dot a}{a}M-3\frac{\dot a^2}{a^2}\varphi,\tag{1.24}$$

$$V=-\frac{(v-w)}{a^2}=-\left\{1/6a^2\left[\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}\right]\right\}\left(\dot N-3\frac{\dot a}{a}\varphi\right)+\frac{w}{a^2},\tag{1.25}$$

$$-\varphi+\frac{2}{9}\frac{a^2}{\dot a^2}\frac{\bar n\bar T}{\bar\xi}\delta\left[\ln\left(\frac{e}{\bar n}\left(\frac{m\bar T}{2\pi}\right)^{3/2}\right)\right.$$
$$\left.-\frac{df}{dT}\right]_{,0}-\frac{4}{3}\frac{a}{\dot a}\left[\frac{k^2v}{a^2}+\frac{1}{2}\left(M-3\frac{\dot a}{a}\varphi\right)\right]$$
$$+2\left(-\frac{\delta\xi}{\bar\xi}+\frac{\delta n}{\bar n}+\frac{\delta T}{\bar T}\right)$$
$$-\frac{2}{9}\bar\varkappa k^2\left[\frac{\bar\mu}{\bar T}+\ln\left(\frac{e}{\bar n}\left(\frac{m\bar T}{2\pi}\right)^{3/2}\right)+\left(\xi_c+\frac{3}{2}\right)-\frac{df}{d\bar T}\right]$$
$$\times\left[\delta\left(\frac{\mu}{T}\right)-v\left(\frac{\bar\mu}{\bar T}\right)_{,0}\right]\frac{\bar n^2\bar T^3}{\dot a^2(\bar p+\bar\varepsilon)^2\bar\xi}=0.\tag{1.26}$$

Here $N=\rho+\gamma$ and

$$M=\dot\rho-2k^2w/a^2$$

are functions constructed from the perturbations of the metric. With $h_0^0=h_\alpha^0=0$; and $\bar\eta,\bar\xi,\bar\varkappa=0$, Eqs. (1.22)–(1.25) are the same as the familiar equations of the Lifshitz theory.[20] There obviously exists a particular solution of the system of equations (1.22)–(1.25): $N=\dot a/a$, $\varphi=0$, $\delta p_1=0$. It is thus logical to make the substitution $N=(\dot a/a)\psi$. Adding (1.22) and (1.23), we find the expression $M=M(\psi,\dot\psi,\ddot\psi,\varphi,\dot\varphi,\delta p_1,\delta\dot p_1)$. Substituting M into (1.22), we find a system of equations for the invariant

$J=[(a/\dot a)N]^*-3\varphi \qquad (J'=J \quad \text{for} \quad x^{i'}=x^i+\eta^i$, $h_i^{k'}=h_i^k+\eta^k_{;i}+\eta_i^{;k})$ and for the quantity[3)] δp_1:

$$\delta\dot p_1+l\delta p_1=Y, \tag{1.27}$$

$$\dot J+AJ+BJ=\frac{24\pi Ga}{\dot a}\Gamma(\delta\dot p_1-C\delta p_1). \tag{1.28}$$

Here

$$A=\frac{4}{3}\pi G\frac{k^2(4\bar\eta+3\bar\xi)}{a^2(-\ddot a/a+\dot a^2/a^2)}-\frac{\dot x}{x}$$

$$+2\frac{\ddot a}{\dot a}+4\frac{\dot a}{a}-2\frac{\dot u}{u}+32\pi G\bar\eta+3u^2x\frac{\dot a}{a},$$

$$B=-\Omega\frac{\dot x}{x}+2\left(\frac{\dddot a}{\dot a}-\frac{\ddot a^2}{\dot a^2}\right)+7\frac{\ddot a}{a}+2\frac{\dot a^2}{a^2}-2\frac{\dot u}{u}\left(\frac{\dot a}{a}+2\frac{\ddot a}{\dot a}\right)$$

$$+16\pi G\bar\eta\left(3\frac{\dot a}{a}+2\frac{\ddot a}{\dot a}-2\frac{\dot u}{u}\right)+16\pi G\dot{\bar\eta}$$

$$+\frac{4}{3}\pi Gk^2\left[\frac{4\bar\eta+3\bar\xi}{a^2(-\ddot a/a+\dot a^2/a^2)}\right]^{\cdot}$$

$$+\left(3\frac{\dot a}{a}-2\frac{\dot u}{u}\right)\frac{4\pi Gk^2(4\bar\eta+3\bar\xi)}{3a^2(-\ddot a/a+\dot a^2/a^2)}$$

$$+u^2x\left[\frac{k^2}{a^2}+3\left(\frac{\ddot a}{a}+2\frac{\dot a^2}{a^2}+16\pi G\bar\eta\frac{\dot a}{a}\right)\right.$$

$$\left.+16\frac{\dot a}{a}\frac{\pi Gk^2\bar\eta}{a^2(-\ddot a/a+\dot a^2/a^2)}\right],$$

$$l=-\frac{(\bar n\bar T)^{\cdot}}{(\bar n\bar T)}+\frac{\dot c_V}{c_V}-\frac{8\pi G\bar\xi a}{u^2F\dot a}\left(-\frac{\dot{\bar\xi}}{\bar\xi}+\frac{\dot{\bar T}}{\bar T}\right)-\frac{\bar\varkappa k^2}{c_Va^2\bar n}\left(\frac{F}{x}-1\right).$$

$$Y=-\frac{k^2\bar\xi}{3a^2c_V}J+\frac{\bar\varkappa k^2\dot a}{24a^3c_V\bar n}\frac{(\dot J+\Omega J)}{\pi Gx},$$

$$x=1-\frac{16\pi G\bar\eta a}{3u^2\dot a}-\frac{4\pi G\bar\xi a}{u^2\dot a}-3\frac{\dot a}{a}\left[\frac{F}{\bar n}\frac{\partial\bar\xi}{\partial\bar T}+\frac{(1-c_Vu^2)}{mu^2}\frac{\partial\bar\xi}{\partial\bar n}\right],$$

$$\Omega=2\frac{\ddot a}{\dot a}+\frac{\dot a}{a}+16\pi G\bar\eta+\frac{4\pi Gk^2(4\bar\eta+3\bar\xi)}{3a^2(-\ddot a/a+\dot a^2/a^2)},$$

$$C=2\frac{\dot u}{u}+\frac{\dot x}{x}-4\frac{\dot a}{a}+\frac{\ddot a}{\dot a}-\frac{\dot\Gamma}{\Gamma}-16\pi G\bar\eta.$$

The general solution of Eq. (1.27) is

$$\delta p_1=\delta p_{10}\exp\left(-\int_{t_0}^{t}l\,dt\right)+\exp\left(-\int_{t_0}^{t}l\,dt\right)\int_{t_0}^{t}Y\exp\left(\int_{t_0}^{t'}l\,dt'\right)dt, \tag{1.29}$$

where the constant δp_{10} corresponds to the solution of the Cauchy problem with the initial conditions $\delta p_1(t_0)=\delta p_{10}$. This constant is nonzero if the dissipation coefficients $\varkappa$, η, and ξ are nonzero.

After substituting (1.28) into (1.27), we find a third-order scalar equation for the invariant:

$$\dot Z+\left(l-\frac{\dot{\mathfrak{C}}}{\mathfrak{C}}\right)Z+\mathfrak{C}Y=0, \tag{1.30}$$

where

$$Z=\ddot J+\left(A+\frac{\bar\varkappa k^2F}{a^2c_V\bar nx}\right)\dot J+\left[B+\frac{\bar\varkappa k^2}{a^2}\frac{F}{c_V\bar n}\frac{\Omega}{x}-\frac{8\pi Gk^2\bar\xi}{a\dot ac_V}\right]J,$$

$$\mathfrak{C}=\frac{24\pi Ga}{\dot a}(l+C)\Gamma.$$

Equation (1.30), a third-order differential equation, contains a singular perturbation. It describes the time evolution of three hydrodynamic normal modes of collective motion of the continuous medium: the acoustic modes and the thermal mode.[36]

We wish to stress that, in the version of the LTGS which we are proposing here, the analysis of the perturbations is carried out in a generally covariant form, without the imposition of any additional gauge conditions on the perturbed quantities.

The invariant characteristics of the density and velocity of the matter are found by subtracting from (1.24) and (1.25) functions generated by varying the proper time

$$\tau=\int g_{00}^{1/2}\,dx^0$$

in the unperturbed solution:

$$(8\pi G)\,\delta\bar\varepsilon_{\rm inv}=8\pi G\left[\delta\varepsilon-\dot\varepsilon\left(t_0+\frac{1}{2}\int_{t_1}^{t}\varphi\,dt'\right)\right]$$

$$=-\frac{1}{u^2x}\left[\frac{8\pi G\Gamma Z}{\mathfrak{C}}+\frac{\dot a}{3a}(\dot J+\Omega J)\right]+\frac{4}{3}\pi G\dot{\bar\varepsilon}\int_{t_1}^{t}J\,dt', \tag{1.31}$$

$$V_{\rm inv}=V+\frac{1}{a^2}\left(t_0+\frac{1}{2}\int_{t_1}^{t}\varphi\,dt'\right)-\frac{w}{a^2}$$

$$=\frac{1}{6a^2}\left\{\frac{\dot a}{a}\frac{J}{(-\ddot a/a+\dot a^2/a^2)}-\int_{t_1}^{t}J\,dt'\right\}. \tag{1.32}$$

Here

$$t_0=\int_{t_1}^{t_0}J\,dt;$$

t_1 is an arbitrary constant, and t_0 is a time which we use to specify the initial conditions.

This procedure for eliminating fictitious solutions in the LTGS has a completely clear working interpretation. A difference between the reading of the clock of the local observer and the cosmological time t can be detected by comparing the proper time τ ($d\tau=g_{00}^{1/2}dt$) with the time of an observer moving along with remote matter which is expanding according to the Friedmann law (i.e., by observing a universe which is homogeneous and isotropic at length scales larger than those of the perturbations). This procedure does not impose any restriction on the nature of the motion of the frame of reference (as is easily verified by working from the equations of relativistic kinematics). The working interpretation assumed for the construction of the functions $\delta\varepsilon_{\rm inv}$ and $V_{\rm inv}$ means that observable physical quantities can be found directly from experiment, and the invariant increments in them can be eliminated in a simple way, by making a comparison with experimental results. It can be seen from (1.31) and (1.32) that the fictitious "gauge modes" do not "sense the dissipation" (just as we would expect).

2. SOLUTIONS FAR FROM A SINGULAR POINT

The three gauge-invariant physical modes of collective motions of a continuous medium which are described by Eq. (1.30) can be classified in an obvious way as acoustic modes and a thermal mode. [It is easy to show that the general solution of system (1.27), (1.28) depends on the same three constants that the solution of Eq. (1.30) depends on; specifically, there exists a functional relationship $\delta p_{10} = \delta p_{10}(c_1, c_2, c_3)$.] The hydrodynamic description of the continuous medium is valid under the condition $t \gg t_r$. This condition is violated near the critical points $x = 0$, where

$$x=1-\frac{16\pi G\bar{\eta}}{3u^2}\frac{a}{\dot a}-\frac{4\pi G\bar{\xi}}{u^2}\frac{a}{\dot a}-3\frac{\dot a}{a}\left(\frac{F}{\bar n}\frac{\partial\bar\xi}{\partial \bar T}+\frac{1}{mu^2}\frac{\partial\bar\xi}{\partial\bar n}\right). \tag{2.1}$$

In this section of the paper we consider the system far from such points, assuming $x \approx 1$. Let us consider perturbations whose size is smaller than the distance to the horizon:

$$L=\frac{a}{k}\ll t,\qquad k=\frac{2\pi a}{\lambda}.$$

We do not impose any *a priori* restrictions on the phase of the perturbations:

$$\int\frac{ku}{a}\gtrless 1.$$

Using the conditions $kt/a \gg 1$ and $u^2 \ll 1$, we see that we need retain only k^2u^2/a^2 among the various coefficients of Eq. (1.30) which contain u^2; we can set $u^2 = 0$ in the other coefficients. We thus use an unperturbed solution for dust:[33]

$$a=(a_0 t^{1/3})^2,\qquad \bar\varepsilon=\frac{1}{6\pi G t^2}.$$

For the sound velocity $u^2(t)$ it is then sufficient to use the expression

$$u^2=u_{\rm ad}{}^2=\frac{c_p}{c_V}\frac{\bar T}{m},$$

and the temperature can be found from the law describing the increase in entropy of unperturbed solution (1.6):

$$\dot{\bar T}+\frac{2}{c_V t}\bar T=\frac{24\pi G m\bar\xi}{c_V}. \tag{2.2}$$

We consider the standard problem of the dissipation of hydrodynamic normal modes of collective motions of the continuous medium, which we represent as an ideal gas with an arbitrary specific heat. The solution of this problem depends on the relations among three independent time scales of this system: the dissipation time

$$t_{r(\xi,\eta,\varkappa)}=\frac{l_{(\xi,\eta,\varkappa)}}{u}=\left(\frac{\bar\xi}{m\bar n u^2},\ \frac{\bar\eta}{m\bar n u^2},\ \frac{\bar\varkappa}{\bar n u^2}\right);$$

the reciprocal of the particle oscillation frequency in a sound wave,

$$t_h=\omega^{-1}=\frac{a}{ku}=\frac{\lambda}{2\pi u};$$

and the cosmological time t.

Under our assumptions, the conditions $t,\ t_h \gg t_r$ always hold. The relation between t_h and t fixes the wavelength region under consideration for the acoustic hydrodynamic modes.

We construct solutions of Eqs. (1.30) in the limiting cases of short waves, $ut/\lambda \ll 1$, and long waves, $ut/\lambda \gg 1$, using for this purpose asymptotic methods of the theory of ordinary differential equations.[37]

a) Short waves, $ut/\lambda \gg 1$. The fundamental system of solutions is

$$J=\sum_{i=1}^{3}c_iJ_i,$$

$$J_{1,2}=\exp\left(\pm i\int\frac{2\pi u}{\lambda}dt\right)\frac{u^{1/2}}{\dot a a^{3/2}}$$
$$\times\exp\left\{-\frac{1}{2m}\int\left(\frac{2\pi}{\lambda}\right)^2\frac{1}{\bar n}\left[\left(\frac{4}{3}\bar\eta+\bar\xi\right)+\bar\varkappa m\left(\frac{1}{c_V}-\frac{1}{c_p}\right)\right]dt\right\}, \tag{2.3}$$

$$J_3=\frac{1}{a}\left(1-\frac{2}{3}\frac{\dot c_p}{c_p}\frac{a}{\dot a}\right)\exp\left\{-\int\left(\frac{2\pi}{\lambda}\right)^2\frac{\bar\varkappa}{\bar n c_p}dt\right\}.$$

For the relative changes in the density and velocity of the medium we find the expressions

$$\frac{\delta\varepsilon_{inv}}{\bar\varepsilon}=\mathfrak{A}(t)\exp\left\{-\frac{1}{2m}\int\left(\frac{2\pi}{\lambda}\right)^2\frac{1}{\bar n}\left[\left(\frac{4}{3}\bar\eta+\bar\xi\right)+\bar\varkappa m\left(\frac{1}{c_V}-\frac{1}{c_p}\right)\right]dt\right\}\left\{c_1\exp\left[i\int\left(\frac{2\pi}{\lambda}\right)\frac{c_p}{c_V}\frac{\bar T}{m}dt\right]\right.$$
$$\left.+c_2\exp\left[i\int\left(\frac{2\pi}{\lambda}\right)\frac{c_p}{c_V}\frac{\bar T}{m}dt\right\}+c_3\mathfrak{B}(t)\exp\left\{-\left(\frac{2\pi}{\lambda}\right)^2\frac{\bar\varkappa}{\bar n c_p}dt\right\}, \tag{2.4}$$

where

$$\mathfrak{A}(t)=-3ikt^{-1/3}/4a_0{}^7\left(\frac{c_p}{c_V}\frac{\bar T}{m}\right),\qquad \mathfrak{B}(t)=\frac{\dot{}k^2}{18a_0{}^6c_V},$$

$$V_{inv}=\mathfrak{D}(t)\exp\left\{-\frac{1}{2m}\int\left(\frac{2\pi}{\lambda}\right)^2\frac{1}{\bar n}\left[\left(\frac{4}{3}\bar\eta+\bar\xi\right)+\bar\varkappa m\left(\frac{1}{c_V}-\frac{1}{c_p}\right)\right]dt\right\}\left\{c_1\exp\left[i\int\left(\frac{2\pi}{\lambda}\right)\frac{c_p}{c_V}\frac{\bar T}{m}dt\right]\right.$$
$$\left.+c_2\exp\left[-i\int\left(\frac{2\pi}{\lambda}\right)\frac{c_p}{c_V}\frac{\bar T}{m}dt\right]\right\}$$
$$+c_3\mathfrak{C}(t)\exp\left\{-\int\left(\frac{2\pi}{\lambda}\right)^2\frac{\bar\varkappa}{\bar n c_p}dt\right\}. \tag{2.5}$$

Here

$$\mathfrak{D}(t)={}^3/{}_4\left(\frac{c_p}{c_V}\frac{\bar T}{m}\right)^{1/4}\Big/\ a_0{}^9t,$$

$$\mathfrak{C}(t)=t_1{}^{1/3}\left(1-\frac{\dot c_p}{c_p}t\right)t^{-4/3}/2a_0{}^4c_V.$$

The exponential laws in (2.4) and (2.5) agree with the known damping of the acoustic modes and of the thermal mode in gaseous media.[36] (This result can of course be thought of as a test of the validity of the system of equations being used.) Cosmological expansion leads to the appearance of a nontrivial dependence of the coefficients on the time t. We also wish to stress that the results found here are valid to the extent that the absorption rates given by (2.3)–(2.5) are small. In gases, this condition holds for a certain interval of wavelengths λ. As an example, let us estimate the first term in the absorption coefficient for a sound wave,

$$\mathfrak{I}=\frac{k^2\bar\eta t}{a^2m\bar n}\approx\frac{(2\pi)^2ult}{\lambda^2}\approx\left(\frac{2\pi l}{\lambda}\right)^2\frac{t}{t_r}\ll 1 \tag{2.6}$$

under the conditions

$$2\pi l\left(\frac{t}{t_r}\right)^{1/2} \ll\lambda\ll 2\pi ut.$$

Corresponding estimates for $\bar{\varkappa}$ and $\bar{\xi}$ are valid. There is no difficulty in writing solutions for a constant value of c_V, by using (2.4) and (2.6).

b) Model solutions at a constant specific heat. Analytical solutions of Eq. (1.30) can be constructed for a constant specific heat, $c_V = \text{const}$, only if we ignore small terms of order $t_r/t \ll 1$ (i.e., only if we ignore the relatively weak damping due to dissipative processes). We can then integrate (1.30) over t once; as a result we find the equation

$$\ddot{J}+\left(2\frac{\ddot{a}}{a}+4\frac{\dot{a}}{a}-2\frac{\dot{u}}{u}\right)\dot{J}+\frac{k^2u^2}{a^2}J=\frac{c_3u^2}{c_Va^3}\left(1-\frac{2}{3}\frac{a}{\dot{a}}\frac{\dot{c}_p}{c_p}\right). \tag{2.7}$$

Let us rewrite (2.7) in an explicit form, making use of the dependence of the coefficients on the temperature, the time, and the specific heat. We must set the constants c_3 equal to zero, since we are ignoring dissipative processes in this approximation. The entropy of the system therefore does not increase, and we have $\delta p_1 = 0$. Consequently, under the same approximations as were used in the derivation of Eq. (2.7), we have

$$\left(\frac{t}{t_\gamma}\right)J''+2\frac{c_p}{c_V}\left(\frac{t}{t_\gamma}\right)J'+K_\gamma^2\frac{c_p}{c_V}\frac{\bar{T}}{m}\left(\frac{t}{t_\gamma}\right)^{2/3}J=0, \tag{2.8}$$

where

$$K_\gamma^2=\left(\frac{2\pi t_\gamma}{\lambda_\gamma}\right)^2.$$

Here the prime means a derivative with respect to t/t_γ, and λ_γ is the wavelength of the perturbation at the time t_γ (e.g., at the recombination time $t_\gamma = 5\cdot 10^{12}$ s).

Analytical solutions of Eq. (2.8) can be derived only for a constant specific heat, $c_V = \text{const}$, in which case the equation of interest is a homogeneous Bessel equation, and the sound velocity is equal to

$$u^2=\frac{c_p}{c_V}\frac{\bar{T}}{m}=\text{const}/a^{3/c_V}.$$

(The value $c_V = 5/2$ corresponds to a molecular gaseous medium of H_2 molecules at temperatures high in comparison with the rotational quantum of the gas molecule: $\bar{T} \gg \bar{T}_r = 85$ K. The value $c_V = 3/2$ corresponds to a gas with unexcited internal degrees of freedom.) The solution of Eq. (2.8) is then

$$J=\left(\frac{t}{t_\gamma}\right)^\rho\left\{c_1J_\nu\left[\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right]+c_2Y_\nu\left[\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right]\right\}, \tag{2.9}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of the first and second kinds,[38]

$$u_\gamma^2=\frac{c_p}{c_V}\frac{T_\gamma}{m},\qquad \nu=-\frac{1/2+1/c_V}{1/3+1/c_V},$$

$$\rho=-(1/2+1/c_V),\qquad \beta=1/3-1/c_V.$$

For the half-integer values of the specific heat, $c_V = 5/2(\nu = 27/2)$ and $c_V = 3/2(\nu = 7/2)$, the relative change in energy density is

$$\begin{aligned}
\frac{\delta\varepsilon_{inv}}{\bar{\varepsilon}}=&-\frac{K_\gamma}{6u_\gamma}\left(\frac{t}{t_\gamma}\right)^{-\rho+\beta-1}\left[c_1J_{\nu-1}\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)\right.\\
&\left.-c_2J_{-\nu+1}\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)\right]\\
&-\frac{1}{3\beta}\left(\frac{K_\gamma u_\gamma}{\beta}\right)^{-\alpha}\left(\frac{t}{t_\gamma}\right)^{-1}\left\{c_1\left(\frac{t}{t_\gamma}\right)^\beta\left[(\nu+\alpha-1)J_\nu\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)\right.\right.\\
&\times s_{\alpha-1,\nu-1}\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)\\
&\left.-J_{\nu-1}\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)s_{\alpha,\nu}\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)\right]\\
&-c_1\left[(\nu+\alpha-1)J_\nu\left(\frac{K_\gamma u_\gamma}{\beta}\right)s_{\alpha-1,\nu-1}\left(\frac{K_\gamma u_\gamma}{\beta}\right)\right.\\
&\left.-J_{\nu-1}\left(\frac{K_\gamma u_\gamma}{\beta}\right)s_{\alpha,\nu}\left(\frac{K_\gamma K_\gamma}{\beta}\right)\right]\\
&+c_2\left(\frac{t}{t_\gamma}\right)^\beta\left[(-\nu+\alpha-1)J_{-\nu}\right.\\
&\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)s_{\alpha-1,-\nu-1}\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)\\
&\left.-J_{-\nu-1}\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)s_{\alpha,-\nu}\left(\frac{K_\gamma u_\gamma}{\beta}\left(\frac{t}{t_\gamma}\right)^\beta\right)\right]\\
&-c_2\left[(-\nu+\alpha-1)J_{-\nu}\left(\frac{K_\gamma u_\gamma}{\beta}\right)s_{\alpha-1,-\nu-1}\left(\frac{K_\gamma u_\gamma}{\beta}\right)\right.\\
&\left.\left.-J_{-\nu-1}\left(\frac{K_\gamma u_\gamma}{\beta}\right)s_{\alpha,-\nu}\left(\frac{K_\gamma u_\gamma}{\beta}\right)\right]\right\},
\end{aligned} \tag{2.10}$$

where

$$\alpha=(\rho-\beta+1)/\beta,$$

and $s_{\mu,\nu}(z)$ is the Lommel function.[38]

The solutions (2.10) describe the evolution of hydrodynamic normal modes of collective motions of the continuous medium under the conditions $c_V = \text{const}$ and $\bar{\eta}=\bar{\xi}=\bar{\varkappa}=0$.

The physical meaning of the solutions which we have constructed can be seen easily in some limiting cases:

1) Short waves, $K_\gamma u_\gamma/\beta \gg 1$: The solutions are the same as (2.3)–(2.4) with $c_3 = 0$. In other words, Eq. (2.7) describes the evolution of acoustic modes superposed on an expanding background.

2) Long waves, $K_\gamma u_\gamma/\beta \ll 1$: In this limiting case, (2.10) becomes

$$\frac{\delta\varepsilon_{inv}}{\bar{\varepsilon}}\approx-\frac{K_\gamma}{6u_\gamma}\left\{c_1\left(\frac{K_\gamma u_\gamma}{2\beta}\right)^{2\nu}\frac{1}{\Gamma(\nu)}\left(\frac{t}{t_\gamma}\right)^{-1}\right.$$
$$\left.-c_2\left(\frac{K_\gamma u_\gamma}{2\beta}\right)^{-2\nu}\frac{1}{\Gamma(-\nu+2)}\left(\frac{t}{t_\gamma}\right)^{2/3}\right\}. \tag{2.11}$$

The solutions found here for acoustic modes in the long-wave approximation are the same as the standard LTGS solutions.[1,2,33] The time evolution of long-wave density perturbations does not depend on the specific heat c_V, since in this approximation the system under consideration here is a dusty medium with an equation of state $p \approx 0$. Using (1.32) and (2.9), we easily verify that perturbations of the velocity V_{inv} decay in time according to a power law in the long-wave approximation. We will not reproduce the corresponding expressions here, since they are of no particular interest for LTGS.

3. DENSITY PERTURBATIONS IN AN EXPANDING UNIVERSE NEAR A SINGULAR POINT

As we know, a local thermodynamic equilibrium is maintained in an unsteady system if the relaxation times are sufficiently short in comparison with the evolution time: $t_r \ll t$. If this inequality is violated for any process, we must go over to a kinetic description of the system. Corresponding to a time-varying gaseous phase in the late stages of cosmological evolution are several time scales for relaxation of the system to an equilibrium state.

Translational degrees of freedom of the system relax over a time $t_{pr} = \bar{\eta}/u^2 m\bar{n}$; rotational degrees of freedom correspond to the relaxation time $t_{\rm rot} = \bar{\xi}/u^2 m\bar{n}$; the heat flux in the medium relaxes over a time $t_{\rm th} = \bar{\varkappa}/u^2\bar{n}$. Near critical points $t_r \approx t$, for any process, the system is no longer able to adjust to energy-momentum transport in the state of local thermodynamic equilibrium. If a time-dependent system is nonlinear, it may break up into fragments, whose relaxation time is $t_{rf} \ll t_f$, where t_f is the evolution time of the fragment. In this case, a dissipative structure forms, because the system as a whole loses its thermodynamic stability.[17,18]

Fairly general considerations lead us to expect such an effect in the LTGS. In the first place, this is not a truly linear theory but instead takes exact account of the interaction of a spatially uniform mode $\bar{g}_{ik}(t)$ with a spatially nonuniform mode $h_{ik}(t,x^\alpha)$. Since it is this interaction which makes the system open, however, effects of this sort can arise in the linear theory. One of the critical points, which appears to be pertinent to relativistic astrophysics, is in a late stage of the evolution, at which most of the mass of the baryon component of the universe is in neutral hydrogen and helium. A gas of neutral H_2 is a special case because the following condition holds in it:

$$\bar{\xi} = \frac{2}{3}\bar{n}T\frac{c_r}{{}^3/_2 + c_r}\tau_r \gg \bar{\eta}.$$

The reason is that the time scale τ_r for the excitation of rotational degrees of freedom associated with collisions is much longer than the collision time, even at high temperatures.[39–41] Consequently, the position of the singular point, $t = t_r$ (i.e., $x = 0$), is determined primarily by the magnitude of the second viscosity.

On the basis of the discussion above, we "extend" the hydrodynamic equations to the applicability limit of the hydrodynamic approximation and examine the trends in the behavior of the system as $t_r \to t$. The results found in this manner suggest a tentative conclusion; a final conclusion can be reached on the basis of the relativistic kinetics (through the use of equations analogous to the relativistic generalization of the equations of relaxation hydrodynamics[42–44]). Accordingly, the analysis in this section of the paper is only illustrative. The appearance of this illustration is quite natural, since the equations themselves "do not know about" their limit of applicability. The formation of a dissipative structure (or the appearance of other nonlinear effects) in this case is a consequence of the general properties of nonlinear systems.

The coefficients of Eq. (1.30) are singular at the critical points $x(t_s) = 0$. As follows from the definition of the kinetic coefficients for a neutral gas, the time t_s is on the order of the time scale of the slowest relaxation process in the system (for neutral gaseous H_2, this time is $t_s = 2.3\cdot 10^3 t_\gamma$). Strictly speaking, all the equations which are involved in this paper are valid in the region $0 < x \leqslant 1$ ($t \leqslant t_s$). However, as we have already mentioned, it is interesting to examine the behavior of density perturbations in the region $x(t_s) \to 0$, which includes the vicinity of the critical points $x(t_s) = 0$, where the arguments about thermodynamic quantities retain their meaning.

As was shown in Sec. 2, at $x \approx 1$ the viscous forces and heat-transfer processes lead to the ordinary damping of hydrodynamic normal modes. The asymptotic behavior of the invariant J near the critical points can be studied by expanding the coefficients of Eq. (1.30) in Laurent series in the quantity x. The general solution of (1.30) can thus be written as a series in x in two opposite limits in terms of the parameter

$$\tilde{K} = -\bar{\varkappa}k^2/a^2 c_p c_V \dot{x}\bar{n}$$

(see the Appendix).

a) Short waves, $\tilde{K} \gg 1$. This case corresponds to extremely small length scales, on the order of stellar scales and below. A general solution is constructed through a direct integration. Its asymptotic behavior in terms of x (as $x \to 0$) is

$$\begin{aligned} J = &-c_1\{b_{02}(0,0)\tilde{R}^2x^2 + O(x^3)\} \\ &+c_2\{b_{11}(0,1)\tilde{R}x\ln(\tilde{R}x) + O[x^2(\ln(\tilde{R}x))]\} \\ &-c_3\{b_{20}(0,2)[\ln(\tilde{R}x)]^2 + O[x(\ln(\tilde{R}x))^2]\}, \end{aligned} \tag{3.1}$$

where c_1, c_2, and c_3 are arbitrary constants; $\tilde{R} = -(\dot{\Omega}/\dot{x})\tilde{K}^2$,

$$b_{l\nu}(0,h) = \frac{1}{(h-l)!}\pi^{-\hbar-1}\sum_{\substack{\mu=0\\ \mu\neq 1,\ l\leqslant h}}^{l}\frac{(2\pi i)^\mu}{\mu!(l-\mu)!}B_\mu^{(h+1)}f_0^{(l)}(\nu),$$

$B_\mu^{(h+1)}$ are the Bernoulli numbers ($h = 0,1,2$); and

$$f_0(s) = \left\{\prod_{k=0}^{2}\Gamma(1+s-k)\right\}^{-1}.$$

The relative changes in the density of the medium and the velocity are

$$\begin{aligned} \frac{\delta\varepsilon_{inv}}{\bar{\varepsilon}} = &-\frac{\dot{x}a}{9u^2\dot{a}}\Big\{c_1[-2\tilde{R}^2b_{02}(0,0) + O(x^2)] + c_2\tilde{R}b_{11}(0,1) \\ &\times\left[\frac{4}{x} + O(\ln(\tilde{R}x))\right] + c_3b_{20}(0,2)\left[\frac{4}{x^2}\ln(\tilde{R}x) + O\left(\frac{1}{x}\right)\right]\Big\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} V_{inv} = &\frac{\dot{a}}{6a^3(-\ddot{a}/a + \dot{a}^2/a^2)}\Big\{-c_1[b_{02}(0,0)\tilde{R}^2x^2 + O(x)] \\ &+ c_2(b_{11}(0,1)\tilde{R}x\ln(\tilde{R}x) + O(x^2\ln(\tilde{R}x))] \\ &- c_3[b_{20}(0,2)(\ln(\tilde{R}x))^2 + O(x(\ln(\tilde{R}x))^2)]\Big\}. \end{aligned} \tag{3.3}$$

For two modes of collective motions of a continuous medium, the relative change in density has a power-law singularity, and we have

$$\lim_{x\to 0}\left(V_{inv}\Big/\frac{\delta\varepsilon_{inv}}{\bar{\varepsilon}}\right) = 0.$$

b) Long waves, $\tilde{K} \ll 1$. This case corresponds to the length scales of most physical interest: from galaxies to galactic clusters. Again in this case, a general solution can be constructed through direct integration of the third-order equation (1.30) near $x = 0$ (see the Appendix).

As $x \to 0$ we have an expansion of the solution in an asymptotic series:

$$J = c_1\left\{\frac{x^2}{2}\psi + O(x^3)\right\} + c_2\left\{\frac{x^2\psi}{2\pi}\ln(\psi x) - \frac{1}{\pi\psi} - \frac{x}{\pi} + O(x^2)\right\} + c_3\{x + O(x^2)\}, \tag{3.4}$$

where $\psi = -\Omega/\dot{x}$.

For the relative density change we find two stable solutions, corresponding to the integration constants c_1 and c_3. The energy density for one of the acoustic modes has a weak logarithmic singularity as $x \to 0$:

$$\frac{\delta\varepsilon_{inv}}{\bar{\varepsilon}} = -\frac{\dot{x}a}{9u^2\dot{a}}\left\{c_1\psi + \frac{c_2\psi}{\pi}\ln(\psi x) + c_3(O(1))\right\}, \tag{3.5}$$

but

$$\lim_{x\to 0}\left(V_{inv}\Big/\frac{\delta\varepsilon_{inv}}{\bar{\varepsilon}}\right) = 0.$$

Formally, the convergence region of series (3.1)–(3.5) is a segment along the time axis bounded on the left and right by those singularities of the coefficients of Eq. (1.30) which are closest to the point under consideration.[37] Consequently, perturbations with fairly arbitrary initial conditions, specified at $x(t_0) \sim 1$, go into the regime (3.1)–(3.5) as the critical point is approached. The short-wave region of the density perturbation spectrum, in which the solution for $\delta\varepsilon_{\text{inv}}/\bar{\varepsilon}$ has a power-law singularity, is a special region by virtue of the dynamics. Consequently, as $t \to t_s$, we see a tendency toward the formation of a dissipative structure whose elements have comparatively small peculiar velocities. In the course of this process, the restructuring of the open system should occur in such a way that the conditions for local thermodynamic equilibrium are preserved within the elements of the structure which arises.

We have a few words about the physical interpretation of the behavior of the density perturbations within the framework of this system of LTGS equations. A density wave which exists in a continuous medium superposed on a homogeneous and isotropic universe excites oscillations of two degrees of freedom of the gravitational field (in the linear approximation, these two degrees of freedom are independent). At the formal mathematical level, this excitation causes the field equations to "become close in the linear invariants"[25] M (which corresponds to the rate of change of the isotropic part of the field, $\dot{\rho}$) and $N = \rho + \gamma$. The momentum conservation law ($T^p_{\alpha;p} = 0$) of the system consisting of the matter plus the gravitational field then becomes

$$M\left\{3\frac{\dot{a}}{a}u^2 - 16\pi G\bar{\eta} - 12\pi G\bar{\xi} - 9u^2\frac{\dot{a}^2}{a^2}\left[\frac{F}{\bar{n}}\frac{\partial\bar{\xi}}{\partial\bar{T}} + \frac{(1-c_V u^2)}{mu^2}\frac{\partial\bar{\xi}}{\partial\bar{n}}\right]\left[1 + \frac{\bar{T}}{m}(ci - c_V)\right]^{-1}\right\} + \frac{k^2u^2}{3a^2}N$$

$$+ \ddot{N} + 3\frac{\dot{a}}{a}\dot{N} + 4\pi G k^2(4\bar{\eta} + 3\bar{\xi})\times\left(\dot{N} - 3\frac{\dot{a}}{a}\varphi\right)\Big/3a^2\left(-\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$$

$$+ 16\pi G\bar{\eta}\dot{N} - \frac{3k^2u^2}{a^2}\frac{\dot{a}}{a}\left[\frac{F}{\bar{n}}\frac{\partial\bar{\xi}}{\partial\bar{T}} + \frac{(1-c_V u^2)}{mu^2}\frac{\partial\bar{\xi}}{\partial\bar{n}}\right]\left[1 + \frac{\bar{T}}{m}(ci - c_V)\right]^{-1}N$$

$$- 3\frac{\dot{a}}{a}\dot{\varphi} - \left[2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 3u^2\frac{\dot{a}^2}{a^2}\right]3\varphi + 9\frac{\dot{a}}{a}\left\{3u^2\frac{\dot{a}^2}{a^2}\left[\frac{F}{\bar{n}}\frac{\partial\bar{\xi}}{\partial\bar{T}} + \frac{(1-c_V u^2)}{mu^2}\frac{\partial\bar{\xi}}{\partial\bar{n}}\right]\left[1 + \frac{\bar{T}}{m}(ci - c_V)\right]^{-1} + 4\pi G\bar{\xi}\right\}\varphi = 24\pi G\Gamma\delta p_1. \tag{3.6}$$

The general structure of this conservation law does not change, regardless of any gauge condition imposed on the components of the metric of the space-time. To put Eq. (3.6) in a form which is "invariant" in the LTGS (e.g., if we put aside the problems associated with a variation of the proper time), we impose the one gauge condition $h^0_0 = 0$, without rigidly fixing the frame of reference.

It can be seen from (3.6) that near the critical points $x = 0$ the momentum flux, transported exclusively by the isotropic part M of the gravitational field of the sound wave, is completely offset by the sink due to viscosity. For the dynamics of the second degree of freedom of the gravitational field, described by the "linear invariant" N, however, this point is not critical. The effective "turning off" of one of the degrees of freedom of the field as a result of momentum transport means that the normal energy-momentum transport dictated by the symmetry properties of the gravitational field becomes impossible under the conditions of local thermodynamic equilibrium. When transport processes are partially turned off, the particle momentum fluxes which previously had been carried off instead accumulate, superposed on the unperturbed solution which is formed. The buildup of particles leads to an increase in the gravitational field (the component M increases). The geometry based on the balance between the relativistic momentum fluxes becomes unstable. As a result, this is a thermodynamically nonequilibrium system at the critical points $t = t_s$, and it exhibits a tendency to form a dissipative structure consisting of growing condensations of matter.

The critical factor

$$x = 1 - \frac{16\pi G\bar{\eta}}{3u^2}\frac{a}{\dot{a}} - \frac{4\pi G\bar{\xi}}{u^2}\frac{a}{\dot{a}} - 3\frac{\dot{a}}{a}\left(\frac{F}{\bar{n}}\frac{\partial\bar{\xi}}{\partial\bar{T}} + \frac{1}{mu^2}\frac{\partial\bar{\xi}}{\partial\bar{n}}\right), \tag{3.7}$$

which (with $x = 0$) determines the time of the nonequilibrium phase transition of the system to a nonuniform state, can be written

$$x = 1 - \frac{t_r}{t}, \tag{3.8}$$

where the following crude estimate of t_r is valid:

$$t_r \simeq \frac{l_{dis}}{u} \approx \frac{\bar{\xi}}{\bar{\varepsilon}u^2}. \tag{3.9}$$

Since the dissipative structure forms near t_r, the same quantity is the time scale for the onset of the instability, t_s. The specific value $t_s \approx t_r$ is related to the choice of model for describing the internal degrees of freedom of the gas. It will be estimated in the following section of this paper (where we find $t_r \sim 10^{16}$ s).

A distinctive feature of this effect is an instability of perturbations for all possible wavelengths. However, the nature of the catastrophic growth of these perturbations near the singular point differs from one wavelength λ to another. For long waves, the perturbations of the energy density near this point grow as $\ln x$ according to (3.5). For short waves, as can be seen from (3.2), the instability develops more rapidly, in a power-law fashion. As we will see in the following section of this paper, the picture far from the singular point is just the opposite: Substantial growth of long-wave perturbations begins as early as $t \sim 100 t_\gamma$, and that of short-wave perturbations at $t \sim 1000 t_\gamma$ (see Figs. 1–5 below).

This circumstance has physical consequences: The condensation of small masses of matter occurs right at the transition point. In other words, the seeds of stars and star clusters arise within elements of the large-scale structure which is forming in the universe.

To draw a complete picture of the development of density perturbations throughout cosmological evolution, we would have to take systematic account of all the effects, thermal and dissipative, which influence the rate of the gravitational instability. The only way to solve that problem is to numerically integrate Eq. (1.30). This numerical integration is the subject of the following section of this paper.

4. NUMERICAL SOLUTIONS

We first note an important aspect of Eq. (1.30), which follows directly from its form: Its coefficients and thus its solutions depend on the specific heat of the medium, $c_V = c_V(T)$. Because this equation (the main equation of the theory) contains the logarithmic derivatives $\dot{u}/u$, the derivative of the specific heat with respect to the temperature, $dc_V/d\overline{T}$, should appear in (1.30). This derivative vanishes only if the specific heat is constant: $c_V = \text{const}$. (The condition $c_V = \text{const}$ is a necessary condition, but not a sufficient one, for maintaining an adiabatic thermal regime in the continuous medium.) Only if the continuous medium evolves adiabatically (with $u^2 = c_p \overline{T}/c_V m$ and $\overline{\xi} = 0$) is the solution of Eq. (1.30) exactly the same as the corresponding solution of Lifshitz' equations.[20] However, first the specific heat of this system, consisting of a gas of H_2 molecules, changes from $c_V = 5/2$ to $c_V = 3/2$ as the temperature of the medium decreases. Second, the value $c_V = 3/2$ is not reached in this system, even asymptotically, since the $\overline{T}(t)$ dependence is not adiabatic (the medium warms up as a result of the energy dissipation due to the viscosity). That this system is not thermally isolated is clear simply from the existence of a dimensionless numerical parameter which is a measure of the deviation of the temperature from its adiabatic value:

$$\chi = \frac{a}{\dot{a}} \frac{\dot{\overline{T}}}{\overline{T}} \approx \frac{24\pi G c_p \xi_a}{c_V{}^2 u^2 \dot{a}}. \qquad (4.1)$$

It is therefore clear that there is the possibility that energy will be pumped from molecular degrees of freedom into the energy of the varying gravitational field, $h_{ik}(x^\alpha, t)$. The thermal instability does not, on the other hand, give rise to new parameters (which depend on the wavelengths of the perturbations) to supplement the Jeans parameter. One might therefore say that the Jeans instability in an expanding universe is distorted by thermal effects. Clearly, a study of this distortion must be accompanied by a study of the influence of dissipative effects on the evolution of perturbations, and self-consistent solutions of Eq. (1.30) must be found. Since this is an extremely complicated problem, the only way to solve it is to integrate this equation numerically.

From the equations of motion for the background physical variables, (1.6), we find an explicit expression for the function $u^2 = u^2(t)$:

$$u^2 = \frac{c_p}{c_V} \frac{\overline{T}}{m} - \frac{8\pi G \overline{\xi} a}{c_V \dot{a}}.$$

The temperature $\overline{T}$ can be found from the law describing the increase in entropy of the unperturbed solution:

$$\dot{\overline{T}} + \frac{3}{c_V} \frac{\dot{a}}{a} \overline{T} = \frac{24\pi G m \overline{\xi}}{c_V}. \qquad (4.2)$$

In order to construct numerical solutions of (1.30) over the entire range of the physical time t, we need to specify some specific values of the kinetic coefficients $\overline{\eta}$, $\overline{\xi}$, and $\overline{\varkappa}$. The kinetic coefficients of a gas consisting of rotationally excited H_2 molecules can be calculated only through a rigorous quantum-mechanical analysis of the inelastic scattering of molecules. This complicated problem has yet to be finally resolved.[42] There are, on the other hand, some experimental data and some qualitative theoretical ideas[45–49] which indicate that the rotational relaxation time of a gas of such molecules (which is long in comparison with the relaxation time of the translational degrees of freedom, $t_{\text{rot}}/t_{pr} = \mathfrak{R} \sim 300$, even at comparatively high temperatures) at the very least does not decrease as the temperature decreases to $\overline{T}_r \sim 85$ K ($\overline{T}_r = 85$ K is the rotational quantum of the H_2 molecule).

We therefore use a model approximation of the viscosity coefficients, working from the results for the model of colliding smooth spheres for $t_{\text{rot}} \ll t$, i.e., within the range of applicability of classical hydrodynamics[50]

$$\overline{\eta} = \eta_0 (m\overline{T})^{1/2}/\sigma_c, \qquad \overline{\xi} = \mathfrak{R}\overline{\eta}, \qquad \overline{\varkappa} = \varkappa_0 \left(\frac{\overline{T}}{m}\right)^{1/2} /\sigma_c,$$

where m is the mass of the gas molecule, σ_c is the kinetic cross section, and η_0 and $\varkappa_0$ are numerical coefficients of order unity.

We can now find a rough estimate of the rotational relaxation time of the system:

$$t_{rot} \simeq \frac{l_{dis}}{u} \approx \frac{\overline{\xi}}{\overline{\varepsilon} u^2} \approx \frac{\mathfrak{R} \overline{T}^{1/2}}{\sigma_c m^{1/2} \overline{n} u^2} \approx \frac{3\mathfrak{R} \overline{T}^{1/2} (8\pi G) m^{1/2} t^2}{4\sigma_c u^2}, \qquad (4.3)$$

where l_{dis} is a length scale for the dissipation in the system.

Substituting the standard values of the physical constants into (4.3), and substituting in the value ($\overline{T}_\gamma = 3800$ K) of the temperature at the time of recombination ($t_\gamma = 5 \cdot 10^{12}$ s), we find an estimate of the time t_{rot} (in cgs units):

$$t_{rot} \sim t_s \sim \frac{5 \cdot 10^{18}}{c} \left(\frac{t_s}{t_\gamma}\right)^{2,4} \sim 10^{16} \text{ s}.$$

We can now move on to the numerical integration of (1.30) for various wavelengths λ, specifying the initial conditions at the recombination time t_γ, and using the solutions (2.4)–(2.5) and (2.10)–(2.11) to classify the hydrodynamic normal modes.

We determine the mass of the perturbation in the stan-

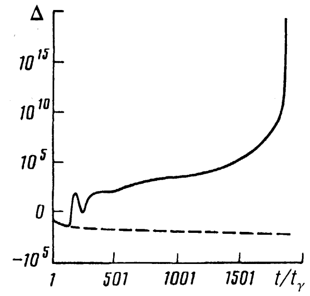

FIG. 1. Behavior of the reduced amplitude $\Delta = (t)(\delta\varepsilon_{\text{inv}}/\bar{\varepsilon}) / (t_\gamma)(\delta\varepsilon_{\text{inv}}/\bar{\varepsilon})$ for a growing acoustic mode in the region of long perturbation waves ($\mathfrak{M} = 2\cdot 10^{10}\mathfrak{M}_\odot$). Everywhere, the dashed line shows the Lifshitz solution, found without consideration of dissipation.

dard way,

$$\mathfrak{M} = \frac{4}{3}\pi m\bar{n}\left(\frac{\lambda}{2}\right)^3,$$

and we construct numerical solutions under the condition $\mathfrak{M} = \text{const}$.

Figure 1 shows $\delta\varepsilon_{\text{inv}}/\bar{\varepsilon}$ as a function of t for the growing Lifshitz solution at wavelengths shorter than the distance to the horizon: $ut \ll \lambda \ll t$ ($\mathfrak{M} = 2\cdot 10^{10}\mathfrak{M}_\odot$ is the scale of the galaxy). The superadiabatic growth of a relative density perturbation in the early stages of evolution in this model is a consequence of a thermal effect, the decrease in the specific heat of the rotational degrees of freedom of the gas at low temperatures. The reason for this effect is that the temperature of the medium decreases, and the energy which is released goes into collective excitations. This initial growth intensifies smoothly as the system approaches the critical point $t = t_s \sim 10^{16}$ s. The evolution of the system thereafter can be described only by a nonlinear theory.

Figures 2–5 show $\delta\varepsilon_{\text{inv}}/\bar{\varepsilon}$ as a function of t for an acoustic mode at short wavelengths, $\lambda \ll ut$ ($\mathfrak{M} = 7\mathfrak{M}_\odot$ is the stellar scale). We can trace the breakup of the density wave with increasing perturbation wavelength in the course of the evolution. We can also trace the anomalous behavior of the solutions near the critical point $t = t_s$.

Figures 6–7 show the function

$$\delta\varepsilon_{inv}/\bar{\varepsilon}$$

as a function of t for a relaxation mode at short wavelengths, $\lambda \ll ut$ ($\mathfrak{M} = 7\mathfrak{M}_\odot$ is the stellar scale). The initial evolution regime, with $\delta\varepsilon_{\text{inv}}/\bar{\varepsilon} = \text{const}$, is disrupted by the thermal

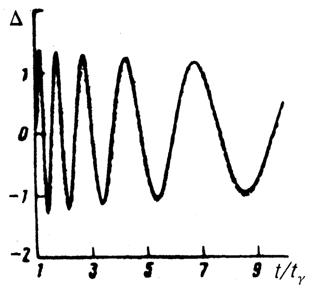

FIG. 2. The reduced amplitude for an acoustic mode at short perturbation wavelengths ($\mathfrak{M} = 7\mathfrak{M}_\odot$), for $t/t_\gamma \in (1, 10)$.

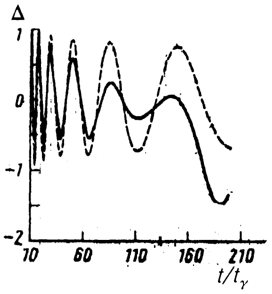

FIG. 3. The reduced amplitude for an acoustic mode at short perturbation wavelengths ($\mathfrak{M} = 7\mathfrak{M}_\odot$), for $t/t_\gamma \in (10, 200)$.

effect and by the damping due to the thermal conductivity. An anomalous growth of the perturbations then occurs near $t = t_s$.

A general feature of these numerical solutions is that the singular point at which the anomalous growth of the density perturbations occurs is shifted in the physical direction away from the value predicted by the asymptotic solution as $x \to 0$.

Specifically, the singularity in the spectrum of perturbations $\delta\varepsilon_{\text{inv}}/\bar{\varepsilon} \to \infty$ is reached at the parameter value $x \approx 0.123$ ($t_s' \approx 1856 t_\gamma$). Here we find a cooperative effect: the appearance of bifurcation in terms of the parameters χ [see (4.1)] (for the unperturbed solution) and x.

5. EFFECT OF COLD DARK MATTER ON A NONEQUILIBRIUM PHASE TRANSITION IN THE BARYON SUBSYSTEM

The astrophysical observational data available on the dynamics of large masses of matter in the local galaxy indicate that the universe contains a substantial amount of diffuse dark matter. Data on the abundances of the light elements, interpreted in accordance with the present understanding of the primordial nucleosynthesis, suggest that most of the mass is of a nonbaryonic nature and is nondissipative.[4)] Dark–matter models usually invoke various types of neutrinos and other particles predicted by gauge theories or supergauge theories.

Our purpose here is to determine how cold dark matter, i.e., matter consisting of heavy particles, would affect the nonequilibrium phase transition in the baryon subsystem. These particles have a small velocity spread in the post-recombination epoch, so their temperatures and pressures are low. The cold dark matter and the baryons evidently form independent subsystems, which are coupled only by the gen-

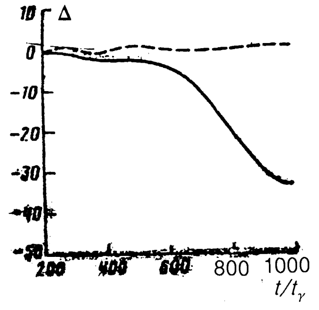

FIG. 4. The reduced amplitude for an acoustic mode in the region of short perturbation wavelengths ($\mathfrak{M} = 7\mathfrak{M}_\odot$), for $t/t_\gamma \in (200, 1000)$.

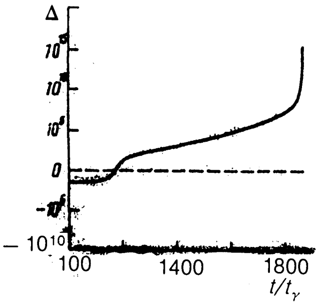

FIG. 5. The reduced amplitude for an acoustic mode in the region of short perturbation wavelengths ($\mathfrak{M} = 7\mathfrak{M}_\odot$), $t/t_\gamma \in (1000, 1856)$.

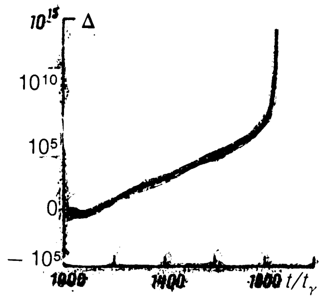

FIG. 7. The reduced amplitude for the thermal mode in the region of short perturbation wavelengths ($\mathfrak{M} = 7\mathfrak{M}_\odot$), $t/t_\gamma \in (1000, 1856)$.

eral gravitational field. In other words, this is a heterogeneous system. The gravitational field of the dark matter of course has a dominant effect on the system.

Even before we carry out any calculations, we can see certain qualitative features of the nature of a nonequilibrium phase transition in a heterogeneous cosmological medium. In the first place, a density wave propagating through the baryon subsystem changes the overall gravitational field of the system. This change in the field in turn will cause a leakage of momentum into the dark-matter subsystem, and the density of the matter of the latter subsystem will be redistributed. In other words, a new sink appears for the momentum carried by density perturbations propagating in the baryon subsystem. The critical condition for the degree of freedom M should therefore become more stringent than (3.6), and the numerical factor Ω_b will appear in the condition for a transition. This factor fixes the ratio of the flux density of momentum which has remained in the subsystem to the total momentum flux density. This factor is determined by the unperturbed solution and is equal to the ratio of the density of the baryon subsystem to the total density of matter in the universe in the comoving frame of reference (or it is equal to the mass ratio $\Omega_b = \mathfrak{M}_b/\mathfrak{M}$). Consequently, the following is a highly accurate condition for the occurrence of the phase transition:

$$1 - \frac{t_r}{\Omega_b t} = 0, \tag{5.1}$$

where t_r is the time scale (which we saw earlier) for the relaxation of a homogeneous universe whose density is at the critical level and whose matter consists entirely of baryons.

Equation (5.1) gives a good description of two possible limiting cases: the case in which there is no dark matter and the case in which dark matter is preponderant.

1) If there is no dark matter, we have $\Omega_b = 1$, and the physics of the transition is as described in detail earlier.

2) If the dark matter dominates, we have $\Omega_b \to 0$, and no transition occurs, since the gravitational field of the dark matter simply annihilates any perturbations in the baryon subsystem.

If, on the other hand, Ω_b is nonzero but small, we can say that the critical condition (5.1) holds and that the transition time shifts toward the recombination time, because the condition for a transition becomes more stringent than that in a model without dark matter.

Equation (5.1) can be put in the standard form by taking account of the change in the relaxation time of the baryon subsystem as a result of the decrease in the density of that subsystem:

$$t_{br} \approx \frac{\bar{\xi}}{\bar{\varepsilon}_b u^2_{(b)}} = \frac{\bar{\xi}}{\bar{\varepsilon}\Omega_b u^2_{(b)}}. \tag{5.2}$$

However, it is better to work with the transition condition as in (5.1), since the time t_{br} is determined by the parameters of the background solution.

As a mathematical model describing the heterogeneous cosmological system, we consider two-fluid hydrodynamics:

$$(\varepsilon_{(dm)} u^i_{(dm)})_{;i}=0, \tag{5.3}$$

$$(\varepsilon_{(dm)} u^i_{(dm)} u^k_{(dm)})_{;k}=0, \tag{5.4}$$

$$R_i^k - \frac{\delta_i^k}{2} R = 8\pi G (T^k_{i(b)} + \varepsilon_{(dm)} u_{i(dm)} u^k_{(dm)}), \tag{5.5}$$

$$(\sigma_{(b)} u^i_{(b)})_{;i} = \frac{\tau^k_{i(b)} u^i_{(b);k}}{T_{(b)}} + \left(\frac{\mu_{(b)}}{T_{(b)}}\right) \nu^i_{(b);i}. \tag{5.6}$$

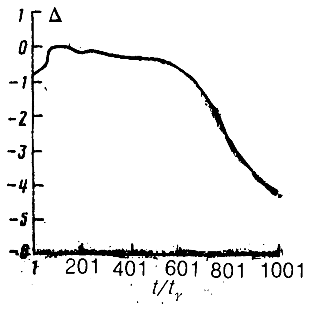

FIG. 6. The reduced amplitude for the thermal mode in the region of short perturbation wavelengths ($\mathfrak{M} = 7\mathfrak{M}_\odot$), $t/t_\gamma \in (1, 1000)$.

The subscript (dm) stands for the dark matter, while the (b) stands for the baryon subsystem. Equations (5.2)–(5.5) have two new equations, not present in (1.1)–(1.3). These new equations are the conservation law of the number of particles making up the dark-matter subsystem [Eq. (5.3)] and hydrodynamic equations for this subsystem [Eq. (5.4)].

Let us linearize system (5.3)–(5.6) about (1.5). As before, we refrain from choosing a frame of reference for the perturbations. The entire discussion will be conducted in a class of frames of reference having a common proper time, under the one additional condition $h_0^0 = 0$. For the perturbation of the pressure of the baryon subsystem, we use the ansatz described earlier, (1.15):

$$\delta p_{(b)}=u_{(b)}^2\delta\varepsilon_{(b)}+\delta p_1,$$

where

$$u_{(b)}^2\approx c_p\bar{T}/c_V m.$$

Going through this procedure, we find a system of equations which is closed in terms of the "quasi-invariants" M and N:

$$\dot{v}_{(dm)}=0, \tag{5.7}$$

$$(\delta\varepsilon_{(dm)})^{\cdot}+3\frac{\dot{a}}{a}\delta\varepsilon_{(dm)}+\bar{\varepsilon}_{(dm)}\left[\frac{k^2v_{(dm)}}{a^2}+\frac{M}{2}\right]=0, \tag{5.8}$$

$$\begin{aligned}\dot{M}&+3\frac{\dot{a}}{a}(1+u_{(b)}^2)M+\frac{1}{a^2}(1+3u_{(b)}^2)\frac{k^2}{3}N-24\pi Gu_{(b)}^2\\&\times\left\{1-3\frac{\dot{a}}{a}\left[\frac{F}{\bar{n}}\frac{\partial\bar{\xi}}{\partial\bar{T}}+\frac{1}{mu_{(b)}{}^2}\frac{\partial\bar{\xi}}{\partial\bar{n}}\right]\delta\varepsilon_{(dm)}\right.\\&\left.+12\pi G\bar{\xi}\left[\frac{k^2\dot{N}}{3a^2(-\ddot{a}/a+\dot{a}^2/a^2)}-M\right]\right.\\&\left.-72\pi G\frac{\dot{a}}{a}u_{(b)}^2\left[\frac{F}{\bar{n}}\frac{\partial\bar{\xi}}{\partial\bar{T}}+\frac{1}{mu_{(b)}{}^2}\frac{\partial\bar{\xi}}{\partial\bar{n}}\right]\delta\varepsilon\right\}=24\pi G\Gamma\delta p_1,\end{aligned} \tag{5.9}$$

$$\begin{aligned}&-\dot{M}-3\frac{\dot{a}}{a}M+\ddot{N}+3\frac{\dot{a}}{a}\dot{N}-\frac{k^2}{3a^2}N\\&+16\pi G\bar{\eta}\left\{\frac{k^2\dot{N}}{3a^2(-\ddot{a}/a+\dot{a}^2/a^2)}+\dot{N}-M\right\}=0,\end{aligned} \tag{5.10}$$

$$8\pi G\delta\varepsilon=8\pi G(\delta\varepsilon_{(b)}+\delta\varepsilon_{(dm)})=\left\{\frac{k^2}{3a^2}N+\frac{\dot{a}}{a}M\right\}. \tag{5.11}$$

Equation (5.7) of this system can be integrated. It gives us the well known hydrodynamic solution for dust,[20,25] $v_{(dm)}$ = const. This result leads to an inconsequential trivial contribution from the velocities of the dark matter to Eq. (5.8).

Using the algorithm described in Sec. 2 for solving a system of equations of the form (5.8)–(5.11), we reduce the latter system to an equation which can be used to study the dynamics of degree of freedom M. We single out the critical factor X, which is responsible for the anomalous dynamics near the point of the phase transition:

$$XM+L(\dot{N},\ \ddot{N},\ \dot{N},\ N,\ \delta\dot{p}_1,\ \delta p_1)=0, \tag{5.12}$$

$$X=(1-\Gamma_1)(1-\Omega_b)+\frac{2}{3}\frac{a^2}{\dot{a}^2}x\left[\left(\frac{\dot{a}}{a}\right)^{\cdot}+\frac{\dot{a}}{a}\frac{\dot{x}}{x}+(1-\Gamma_1)^{-1}\dot{\Gamma}_1\right], \tag{5.13}$$

where $x=1-t_r/t$ is the critical factor of the purely baryon problem, which was defined earlier [see (1.28)], and

$$\Gamma_1=3\frac{\dot{a}}{a}\left[\frac{F}{\bar{n}}\frac{\partial\bar{\xi}}{\partial\bar{T}}+\frac{1}{mu_{(b)}{}^2}\frac{\partial\bar{\xi}}{\partial\bar{n}}\right].$$

Substituting (5.12) into (5.10), we find a fourth-order equation for N. Its solution gives us exhaustive information about the system. However, obvious physical considerations based on the correspondence principle, along with the comments back in Sec. 3 regarding equations of the form in (5.12), indicate that the only way to establish that this system has critical properties is to study the resonant denominator X.

Substituting the cosmological solution for a into X, we find the following expression, which holds far from the point $t\approx t_r$:

$$X=-\Omega_b\left(1-\frac{t_r}{\Omega_b t}\right).$$

This expression is good for a qualitative analysis and rough estimates. Comparison of this expression with the exact expression (5.13) shows that the simple power law will in general be slightly distorted by the viscosity. (Actually, this distortion is slight, since the transition condition becomes more stringent, and its shifts are closer to the recombination time.)

To estimate the time scale of the evolution of the gravitational field of the system up to the phase-transition point, we note that the time evolution of the temperature can, roughly speaking, be regarded as adiabatic ($T\propto t^{-4/5}$), because the conditions for the occurrence of the phase transition become more stringent. This circumstance leads in turn to the estimate $t_{s1}\approx(\Omega_b)^{5/7}t_s$ of the transition time. The transition time thus moves closer to the recombination time (by a factor of order 5 for the value $\Omega_b=10$) than in the purely baryon problem ($t_s\simeq10^{16}$ s). This shift in time is not very important quantitatively, in view of our rough estimate of t_s.

CONCLUSION

It follows from the form of the solutions constructed here that in the late stages of cosmological evolution, after the recombination of the hydrogen plasma (between $t=10^{14}$ s and $t=10^{16}$ s), the amplitude of the density perturbations increases 1.5 orders of magnitude more than in the standard linear theory of gravitational stability (or in a model with the Jeans instability superposed on an expanding background[1]), even if the density fluctuations start from a low level. Consequently, at $\delta\varepsilon_{inv}/\bar{\varepsilon}\sim1$ nonlinear processes may come into play in the system. They should lead to the formation of growing condensations of matter, in a process accompanied by heating of the medium and the formation of structure. While the initial level of density fluctuations in the universe is low, and structure does not manage to form in the universe before the stage in which cold neutral hydrogen becomes predominant, at $t\approx10^{16}$ s there is a tendency toward the formation of dissipative structure, as the result of an explosive instability.

The theory of the branching of solutions of differential equations within the framework of standard hydrodynamics is inadequate for studying this process and for confirming the existence of a singular regime of the instability of the solutions constructed here.[51] Since the effect under consideration here lies right at the boundary of the range of applicability of relativistic hydrodynamics, the methods of relativistic kinetics must be used even in the linear theory. Specifically, we know that many important aspects of molecular systems with relaxing internal degrees of freedom can be analyzed on the basis of a relaxation hydrodynamics,[42] in which case the physics of the dissipative processes is substantially enriched by relaxation effects.[40,52,53] The effect of the dark matter of the universe on the growth of perturbations in the baryon subsystem of the universe also requires further research.

In summary, the physics of the processes which occur in the hot universe after recombination is rather complex. The results which have been found in turn pose new problems. A study of these new problems should shed some light on the origin of structure in the universe. These problems can be solved within the framework of a more general and more comprehensive theory.

APPENDIX. SOLUTIONS OF THE EQUATIONS OF THE LINEAR THEORY OF GRAVITATIONAL STABILITY NEAR A CRITICAL POINT

We expand the coefficients in (1.30) in Laurent series in the parameter x near the singular point $x=0$. We introduce the change of variables $t=t(x)$. We then obtain

$$Z'+\frac{(\tilde{K}+1)}{x}Z+\frac{\dot{x}^2\tilde{K}(\tilde{K}+1)}{x}\left(\frac{J'}{x}+\frac{\Omega}{\dot{x}}\right)=0, \tag{A1}$$

where

$$Z=\dot{x}^2J''-\frac{\dot{x}^2}{x}(\tilde{K}+1)J'-\frac{\Omega\dot{x}}{x}(\tilde{K}+1)J,$$

$$\tilde{K}=-\frac{\bar{\varkappa}k^2}{a^2c_pc_V\bar{n}\dot{x}}.$$

Here the prime means the derivative with respect to x.

The invariant characteristics of the perturbations of the density and velocity of matter near the point $x=0$ are

$$\varkappa\delta\varepsilon_{inv}=-\frac{1}{u^2x}\left\{\frac{\Gamma Z(8\pi G)}{\mathfrak{C}}+\frac{\dot{a}}{3a}(J'\dot{x}+\Omega J)\right\}, \tag{A2}$$

$$V_{inv}=\frac{\dot{a}}{6a^3}\frac{J}{(-\ddot{a}/a+\dot{a}^2/a^2)}, \tag{A3}$$

where

$$\mathfrak{C}=\frac{24\pi Ga\dot{x}}{\dot{a}}\frac{\Gamma}{x}(\tilde{K}+1).$$

Equation (A1) can be written in the following form near the point $x=0$:

$$J'''+\frac{\Omega}{\dot{x}}(\tilde{K}+1)\frac{J'}{x}-\frac{\Omega}{\dot{x}x^2}\tilde{K}(\tilde{K}+1)J=0. \tag{A4}$$

The point $x=0$ is an irregular singular point of Eq. (A4), so we need to make some further approximations in order to find a solution of this equation near this point.

We construct a solution of (A4) for two limiting cases in terms of the parameter $\tilde{K}$.

a) Short waves, $\tilde{K}\gg 1$. In this case, equation (A4) becomes

$$x^3J'''+x\tilde{R}J=0, \tag{A5}$$

where

$$\tilde{R}=-\frac{\Omega}{\dot{x}}\tilde{K}^2.$$

A solution of (A5) is the Mayer G function:[37]

$$J_{0h}=\sum_{l=0}^{h}[\ln(\tilde{R}x)]^{h-l}\sum_{\nu=0}^{\infty}b_{l\nu}(0,h)(\tilde{R}x)^{\nu}, \tag{A6}$$

where

$$b_{l\nu}(0,h)=\frac{1}{(h-l)!}\pi^{-h-1}\sum_{\mu=0}^{l}\frac{(2\pi i)^{\mu}}{\mu!(l-\mu)!}B_{\mu}^{(h+1)}f_0^{(l)}(\nu).$$

Here $B_\mu^{(h+1)}$ are the Bernoulli numbers of order $h+1$ and $\mu\neq 1$, and

$$f_0(s)=\left\{\prod_{k=0}^{2}\Gamma(1+s-k)\right\}^{-1},\qquad h=0,1,2.$$

The asymptotic behavior of the solution (A6) near the singular point can be written in the form

$$J=-c_1\{b_{02}(0,0)\tilde{R}^2x^2+O(x^3)\}+c_2\{b_{11}(0,1)\tilde{R}x\ln(\tilde{R}x)+O[x^2(\ln(\tilde{R}x))]\}-c_3\{b_{20}(0,2)(\ln(\tilde{R}x))^2+O[x(\ln(\tilde{R}x))^2]\}. \tag{A7}$$

Substituting (A6) into (A2) and (A3), we find $\delta\varepsilon_{\rm inv}/\bar{\varepsilon}, V_{\rm inv}$ in the form

$$\frac{\delta\varepsilon_{inv}}{\bar{\varepsilon}}=-\frac{\dot{x}a}{9u^2\dot{a}}\left\{c_1[-2\tilde{R}^2b_{02}(0,0)+O(x^2)]+c_2\tilde{R}b_{11}(0,1)\left[\frac{4}{x}+O(\ln(\tilde{R}x))\right]+c_3b_{20}(0,2)\left[\frac{4}{x^2}\ln(\tilde{R}x)+O\left(\frac{1}{x}\right)\right]\right\}, \tag{A8}$$

$$V_{inv}=\frac{\dot{a}}{6a^3(-\ddot{a}/a+\dot{a}^2/a^2)}\times\{-c_1[b_{02}(0,0)\tilde{R}^2x^2+O(x)]+c_2[b_{11}(0,1)\tilde{R}x\ln(\tilde{R}x)+O(x^2\ln(\tilde{R}x))]-c_3[b_{20}(0,2)(\ln(\tilde{R}x))^2+O(x(\ln(\tilde{R}x))^2)]\}. \tag{A9}$$

b) Long waves, $\tilde{K}\ll 1$. The asymptotic form of (A4) in this case is conveniently constructed directly from (A1). We expand (A1) in a series in $\tilde{K}$ for $\tilde{K}\ll 1$. The resulting equation is of the exactly integrable type

$$Z'+\frac{1}{x}Z=0. \tag{A10}$$

Equation (A10) has the general solution

$$J=x\{c_1J_2(2\psi^{1/2}x^{1/2})+c_2Y_2(2\psi^{1/2}x^{1/2})+c_3s_{-1,2}(x)\}, \tag{A11}$$

where $J_2(z)$, $Y_2(z)$, and $s_{-1,2}(z)$ are Bessel, Weber, and Lommel functions, respectively, and $\psi=-\Omega/\dot{x}$.

In the limit $x\to 0$ we have the following expansion in an asymptotic series:

$$J=c_1\left[\frac{x^2\psi}{2}+O(x^3)\right]+c_2\left[\frac{x^2\psi}{2\pi}\ln(\psi x)-\frac{1}{\pi\psi}-\frac{x}{\pi}+O(x^2)\right]+c_3[x+O(x^2)]. \tag{A12}$$

The invariant physical quantities are therefore

$$\frac{\delta\varepsilon_{inv}}{\bar{\varepsilon}}=-\frac{a}{9u^2\dot{a}}\times\left\{c_1[\psi+O(x)]+c_2\left[\frac{\psi}{\pi}\ln(\psi x)+O(x)\right]+c_3O(1)\right\}, \tag{A13}$$

$$V_{inv}=\frac{\dot{a}}{6a^3(-\ddot{a}/a+\dot{a}^2/a^2)}\times\left\{c_1\left[\frac{x^2\psi}{2}+O(x^3)\right]+c_2\left[-\frac{1}{\pi\psi}+O(x)\right]+c_3[x+O(x^2)]\right\}. \tag{A14}$$

The energy density of the matter thus has a logarithmic singularity as $x\to 0$, but

$$\lim_{x\to 0}\left(V_{inv}\Big/\frac{\delta\varepsilon_{inv}}{\bar{\varepsilon}}\right)=0.$$

[1)] It will be shown below that the LTGS equations reduce to a singularly perturbed third-order equation which is nonlinear in several parameters. As a result, the solutions of this third-order equation have bifurcation properties.

[2)] In other words, the situation is analogous to that which prevails in the theory of a free quantum field in a curved space–time.[23,24]

[3)] We are ignoring terms which are relativistically small, of order $\overline{T}/m$ in comparison with the terms we have written out, in the dissipative part of the energy-momentum tensor and the heat-flux vector.

[4)] The ratio of the mass of the baryon subsystem to the mass of the total system, $\Omega_b = \mathfrak{M}_b/\mathfrak{M}$, is usually estimated to be $0.01 \leqslant \Omega_b \leqslant 0.1$ (Refs. 4 and 6).

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