Three-neutrino oscillations in matter and topological phases

V. A. Naumov

Scientific Research Institute for Applied Physics affiliated with Irkutsk State University
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Explicit expressions for the topological phases γ_N for a system of three mixed Dirac neutrinos, propagating in a medium with arbitrary density and composition distribution, are obtained within the framework of Berry's adiabatic approach. These expressions do not depend on the parametrization of the vacuum mixing matrix V. It is proved that the Berry phases γ_N^B which appear for a periodic variation of the parameters of the system vanish identically, whereas for a noncyclic evolution $\gamma_N \equiv 0$ (in a special gauge) if all matrix elements of the matrix V, the phase of CP-violation, and the differences of the squared neutrino masses $m_i^2 - m_j^2$ $(i \neq j)$ are nonzero. Exact expressions have been obtained for the mixing matrix of neutrinos in a medium and of the adiabatic evolution operator have been obtained. A recursion scheme for the computation of corrections to the adiabatic approximation is described.

INTRODUCTION

A common trait of Schrödinger evolutions of dynamical systems with Hamiltonians depending on time via a set of adiabatic parameters $\{\lambda_\alpha(t)\}$ is the occurrence of topological phases,[1,2] which enter into the quantum transition amplitudes together with dynamical phases. Neutrino oscillations in a medium with variable density $\rho(x)$ (depending on the coordinate $x \approx ct$),[3–5] are a typical example of such an evolution. The role of the parameters λ_α is played by the refractive indices of the neutrinos $(\nu_e,\nu_\mu,\nu_\tau,\ldots)$ in the medium. It is legitimate to ask the question of whether topological phases arise in a neutrino system and what their influence is on the probabilities of neutrino oscillations.

If one neglects the absorption of neutrinos and the possible contribution of off-diagonal neutral currents, then topological phases can manifest themselves only within a system of three or more mixing neutrinos, since the vacuum mixing matrix, and thus the Hamiltonian of the 2ν-system, is real. In the present paper we study the evolution of a three-flavor system of Dirac neutrinos (ν_e,ν_μ,ν_τ) in a medium with an arbitrary (smooth) distribution of density and composition, for arbitrary parameters of the vacuum mixing (including the parameter of CP-violation). The properties of the spectrum and of the eigenvectors of the Hamiltonian, the necessary and sufficient conditions for the appearance of Abelian topological phases, and their connection with the phase of CP-violation are investigated in detail. An exact mixing matrix for neutrinos in a medium, the adiabatic evolution operator, and a recursion scheme for computing corrections to the adiabatic approximation are constructed.

Three-neutrino oscillations in matter have been discussed over the last few years by many authors (see Refs. 6–18, and the review articles Refs. 3–5)[1)] mainly in relation to applications of the Mikheev–Smirnov–Wolfenstein effect.[20,21] A number of important results relating to the effects of T-violation on neutrino oscillations in a medium has been obtained in Refs. 22–24 by means of numerical and analytic methods. The question of the possible role of topological phases in the solution of the solar neutrino problem, if the neutrinos have a magnetic moment, was discussed in the recent papers.[25]

The analysis proposed below may be of interest for various applications in neutrino astrophysics (neutrinos originating in the Sun, collapsing stars, cosmic rays, etc.), cosmology (neutrino oscillations in the early Universe), as well as for future "geophysical" experiments (imaging of the Earth by means of neutrino beams from an accelerator or reactor).

2. THE EVOLUTION EQUATION

The evolution operator[2)] $S(t) = \|S_{\alpha\beta}(t)\|$ of a system of three mixing Dirac neutrinos with definite flavors (ν_e,ν_μ,ν_τ), propagating in a medium with arbitrary density and composition distribution satisfies the Wolfenstein equation[20]

$$i\dot{S}(t) = [VH^0V^+ + W(t)]S(t) \tag{2.1}$$

with the initial condition $S(0) = I$. Here $V = \|V_{\alpha i}\|$ is the vacuum mixing matrix which, in general, depends on three mixing angles θ_i and on the parameter δ of CP-violation (the "Dirac phase"),[3)] $V^+V = I$; I denotes the unit matrix;

$$H^0 = \|H_{ij}^0\| \approx \|(p_\nu + m_i^2/2p_\nu)\delta_{ij}\|,$$

where m_i are the eigenvalues of the mass operator of the 3ν-system, p_ν is the momentum of the neutrino (is assumed that $p_\nu^2 \gg \max(m_i^2)$, $c = 1$); the W-matrix describing the interactions of neutrinos with matter. Assuming the absence of off-diagonal neutral currents[20] we have

$$W = \|W_{\alpha\beta}(t)\| = -p_\nu\|(n_\alpha(t) - 1)\delta_{\alpha\beta}\|,$$

where $n_\alpha(t)$ is the refractive index for neutrinos of flavor α. In the case of a usual "cold" medium (Earth, Sun)

$$n_\alpha(t) = 1 + \frac{2\pi}{p_\nu^2}\sum_f \mathcal{M}_{\alpha f}(p_\nu, 0)\mathcal{N}_f(x),$$

where $\mathcal{M}_{\alpha f}$ is the amplitude for coherent scattering of a ν_α on a particle of type f ($f = e, p, n,\ldots$), $\mathcal{N}_f(x)$ is the concentration of scatterers f at the point $x = x(t) \approx x_0 + t$, and x_0 is the coordinate of the source. The amplitude for forward $\bar{\nu}_\alpha f$-scattering differs from $\mathcal{M}_{\alpha f}(p_\nu,0)$ only in sign, so that the equation of evolution for a system of three antineutrinos is obtained from Eq. (2.1) by the substitutions $V \mapsto V^*$ and $W \mapsto -W$. In the present paper we shall neglect neutrino

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absorption, by setting $\operatorname{Im} n_\alpha(t)=0$. Then the evolution operator is unitary $S^+S=I$. In addition, we assume the existence of continuous derivatives $\dot{\mathcal{N}}_f(x)$ at each point $x=x(t)$ of the medium, though this restriction is not obligatory.

The probabilities of the transitions $\nu_\alpha \mapsto \nu_\beta$ during the time t,

$$\mathcal{P}_{\alpha\beta}(t)=|S_{\beta\alpha}(t)|^2, \tag{2.2}$$

do not depend on the common phase of the evolution operator. This can be used for the simplification of Eq. (2.1). We carry out the transformation

$$S(t)\to\exp\left[-\frac{i}{3}\int_0^t \operatorname{Sp}(H^0+W(\tau))\,d\tau\right]S(t).$$

Then Eq. (2.1) gets replaced by

$$i\dot{S}(t)=H[\mathbf{q}(t)]S(t) \tag{2.3}$$

with the same initial condition $S(0)=I$. Here we have introduced the effective Hamiltonian

$$H[\mathbf{q}]=\begin{bmatrix} \mathscr{W}_e-q_e & \mathscr{H}_\tau & \mathscr{H}_\mu^* \\ \mathscr{H}_\tau^* & \mathscr{W}_\mu-q_\mu & \mathscr{H}_e \\ \mathscr{H}_\mu & \mathscr{H}_e^* & \mathscr{W}_\tau-q_\tau \end{bmatrix}, \tag{2.4}$$

which depends on time only through the "adiabatic parameters" q_α:

$$q_\alpha(t)=p_\nu[n_\alpha(t)-\langle n(t)\rangle],\quad \langle n(t)\rangle=\frac{1}{3}\sum_\alpha n_\alpha(t). \tag{2.5}$$

The constant quantities W_α and H_α in Eq. (2.4) are defined in terms of the vacuum mixing matrix and the neutrino masses:

$$\mathscr{W}_\alpha=\sum_i |V_{\alpha i}|^2\Delta_i,\quad \mathscr{H}_\alpha=\eta_\alpha^{\beta\gamma}\sum_i V_{\beta i}V_{\gamma i}^*\Delta_i, \tag{2.6}$$

$$\Delta_i=\frac{m_i^2-\langle m^2\rangle}{2p_\nu},\quad \langle m^2\rangle=\frac{1}{3}\sum_i m_i^2, \tag{2.7}$$

where $\eta_\alpha^{\beta\gamma}=1$ if $\alpha\beta\gamma$ is a cyclic permutation of $e\mu\tau$ and $\eta_\alpha^{\beta\gamma}=0$ otherwise.[4)] Obviously

$$\sum_i\Delta_i=\sum_\alpha \mathscr{W}_\alpha=\sum_\alpha q_\alpha(t)=0.$$

It follows from the last equality that the evolution of a three-neutrino system in an arbitrary medium is defined by not more than two functions of t: the independent components of the vector $\mathbf{q}=(q_e,q_\mu,q_\tau)$.

If one restricts oneself to the approximation of an effective four-fermion interaction for the amplitude $\mathcal{M}_{af}(P_\nu,0)$, then (in the case of a cold electrically neutral unpolarized medium), only one adiabatic parameter survives:[20,27]

$$q_e=-2q_\mu=-2q_\tau=-(2^{3/2}/3)G_F\mathcal{N}_e$$

(here G_F is the Fermi constant). Account of one-loop corrections to the amplitude to first order in $\alpha=1/137$ yields[8]

$$q_e-q_\mu=-2^{1/2}G_F\mathcal{N}_e[1+O(\alpha r_\mu)],$$

$$q_\tau-q_\mu=\frac{3\alpha G_F r_\tau}{2^{3/2}\pi\sin^2\theta_W}\left[(\mathcal{N}_p+\mathcal{N}_n)\ln r_\tau +\left(\mathcal{N}_p+\frac{2}{3}\mathcal{N}_n\right)+O(r_\tau\ln r_\tau)\right],$$

where θ_W is the Weinberg angle, $r_\alpha=m_\alpha^2/m_W^2$, m_α and m_W are the masses of the charged lepton and of the W-boson. In particular, for $\mathcal{N}_e=\mathcal{N}_p=\mathcal{N}_n$ (a neutral isoscalar medium) $q_\tau-q_\mu\approx 5.1\cdot 10^{-5}(q_e-q_\mu)$. A small difference between q_τ and q_μ is potentially important for astrophysical and cosmological applications.[8,28] In the case of a polarized medium or a hot plasma the expressions of the adiabatic parameters are modified substantially,[27,28] but their explicit form will not be needed in the present paper.

3. THE ENERGY SPECTRUM

We consider the eigenvalue problem for the Hamiltonian (2.4) for fixed $\mathbf{q}$:

$$H[\mathbf{q}]|\mathcal{U}_N[\mathbf{q}]\rangle=\mathscr{E}_N[\mathbf{q}]|\mathcal{U}_N[\mathbf{q}]\rangle. \tag{3.1}$$

The characteristic polynomial $\mathscr{D}_H(\mathscr{E})$ of the matrix H has the form

$$\mathscr{D}_H(\mathscr{E})=\det(\mathscr{E}I-H)=\mathscr{E}^3-3w^2\mathscr{E}-2v^3, \tag{3.2}$$

where

$$3w^2={}^1/_2\boldsymbol{\omega}^2+\mathbf{h}^2,\quad 2v^3=\prod_\alpha\omega_\alpha+2\cos\varphi\prod_\alpha h_\alpha-\sum_\alpha\omega_\alpha h_\alpha^2 \tag{3.3}$$

and

$$\boldsymbol{\omega}^2=\sum_\alpha\omega_\alpha^2,\quad \mathbf{h}^2=\sum_\alpha h_\alpha^2,\quad \varphi=\sum_\alpha\varphi_\alpha,$$

$$\omega_\alpha=\mathscr{W}_\alpha-q_\alpha,\quad h_\alpha=|\mathscr{H}_\alpha|,\quad \varphi_\alpha=\arg\mathscr{H}_\alpha. \tag{3.4}$$

Thus, the characteristic equation $\mathscr{D}_H(\mathscr{E})=0$ is a Cardano equation, and since $w\leqslant|v|$ (a consequence of the Hermitean nature of H), its roots $\mathscr{E}_n$ can be written in trigonometric form (cf. Ref. 19):

$$\mathscr{E}_N=2w\cos[{}^1/_3\arccos(v^3/w^3)-{}^2/_3\pi(2N+1)],\quad N=-1,0,+1. \tag{3.5}$$

It is easy to see from Eqs. (3.5) and (3.2) that

$$-2w\leqslant\mathscr{E}_-\leqslant-w,\quad -w\leqslant\mathscr{E}_0\leqslant w,\quad w\leqslant\mathscr{E}_+\leqslant 2w,$$
$$\sum_N\mathscr{E}_N=0,\quad \sum_N\mathscr{E}_N^2=6w^2,\quad \sum_N\mathscr{E}_N^3=6v^3. \tag{3.6}$$

The two levels coincide for $w^2=v^2$: $\mathscr{E}_0=\mathscr{E}_-=-v$, $\mathscr{E}_+=2v$, if $v>0$ and for $\mathscr{E}_0=\mathscr{E}_+=-v$, $\mathscr{E}_-=2v$ if $v<0$. It will be shown below that that with the exception of a single nontrivial case, degeneracy is possible in the 3ν system only in the absence of mixing.

For the sequel we shall need some properties of the adjoint matrix

$$B(\mathscr{E})=\begin{bmatrix} \xi_e(\mathscr{E}) & \mathscr{B}_\tau(\mathscr{E}) & \mathscr{B}_\mu^*(\mathscr{E}) \\ \mathscr{B}_\tau^*(\mathscr{E}) & \xi_\mu(\mathscr{E}) & \mathscr{B}_e(\mathscr{E}) \\ \mathscr{B}_\mu(\mathscr{E}) & \mathscr{B}_e^*(\mathscr{E}) & \xi_\tau(\mathscr{E}) \end{bmatrix}. \tag{3.7}$$

The functions $\xi_\alpha(\mathscr{E})$ and $\mathscr{D}_\alpha(\mathscr{E})$ which occur here (the algebraic complements of the transposed characteristic matrix) have the form

$$\begin{aligned} \xi_\alpha(\mathscr{E}) &= B_{\alpha\alpha}=\eta_\alpha^{\beta\gamma}(\mathscr{E}-\omega_\beta)(\mathscr{E}-\omega_\gamma)-h_\alpha^2, \\ \mathscr{B}_\alpha(\mathscr{E}) &= \eta_\alpha^{\beta\gamma}B_{\beta\gamma}=\mathscr{H}_\alpha(\mathscr{E}-\omega_\alpha)+\eta_\alpha^{\beta\gamma}\mathscr{H}_\beta^*\mathscr{H}_\gamma^*. \end{aligned} \tag{3.8}$$

It is easy to see that they obey the relations

$$\begin{aligned} &\xi_\alpha(\mathscr{E})\mathscr{B}_\alpha(\mathscr{E})-\eta_\alpha^{\beta\gamma}\mathscr{B}_\beta^*(\mathscr{E})\mathscr{B}_\gamma^*(\mathscr{E})=\mathscr{H}_\alpha\mathscr{D}_H(\mathscr{E}), \\ &\eta_\alpha^{\beta\gamma}\xi_\gamma(\mathscr{E})\xi_\gamma(\mathscr{E})-|\mathscr{B}_\alpha(\mathscr{E})|^2=(\mathscr{E}-\omega_\alpha)\mathscr{D}_H(\mathscr{E}). \end{aligned} \tag{3.9}$$

Introducing the notation

$$\begin{aligned} &b_{N\alpha}=|\mathscr{B}_\alpha(\mathscr{E}_N)|=h_\alpha[(\mathscr{E}_N-\omega_\alpha)^2+2(\mathscr{E}_N-\omega_\alpha)\mathscr{X}_\alpha\cos\varphi+\mathscr{X}_\alpha^2]^{1/2}, \\ &\psi_{N\alpha}=\arg[\mathscr{B}_\alpha(\mathscr{E}_N)], \qquad \psi_N=\sum_\alpha\psi_{N\alpha}, \\ &\xi_{N\alpha}=\xi_\alpha(\mathscr{E}_N), \qquad \mathscr{X}_\alpha=\eta_\alpha^{\beta\gamma}h_\beta h_\gamma/h_\alpha=h_\alpha^{-2}\prod_\beta h_\beta \end{aligned} \tag{3.10}$$

and making use of Eqs. (3.9), we obtain the identities:

$$\xi_{N\alpha}b_{N\alpha}=\exp(\pm\psi_N)\eta_\alpha^{\beta\gamma}b_{N\beta}b_{N\gamma}, \qquad b_{N\alpha}^2=\eta_\alpha^{\beta\gamma}\xi_{N\beta}\xi_{N\gamma}. \tag{3.11}$$

It can be seen from Eq. (3.11) that, in particular, for fixed N all functions $\xi_{N\alpha}=\xi_{N\alpha}[\mathbf{q}]$ and $b_{N\alpha}=b_{N\alpha}[\mathbf{q}]$ can vanish only simultaneously. Since, as follows from Eqs. (3.6), (3.10), and (3.11),

$$\xi_N\equiv\sum_\alpha\xi_{N\alpha}=3(\mathscr{E}_N^2-w^2). \tag{3.12}$$

the set of zeros $\mathbb{Z}_N$ of the functions $\xi_{N\alpha}[\mathbf{q}]$ and $b_{N\alpha}[\mathbf{q}]$ is determined by the equation $\mathscr{E}_N^2=w^2$, which is equivalent to condition of degeneracy of the levels $w^2=v^2$. Obviously, $\mathbb{Z}_0=\mathbb{Z}_\pm$ and $\mathbb{Z}_\mp=\varnothing$ for $w=\pm v\neq 0$. For $q\notin\mathbb{Z}_0$ the phase ψ_N can take on only values which are multiples of π, $\psi_N=K_N\pi$, and according to Eqs. (3.6) and (3.12)

$$\operatorname{sign}\xi_{N\alpha}[\mathbf{q}]=\operatorname{sign}\xi_N[\mathbf{q}]=\cos\psi_N=(-1)^{1+N}.$$

Since the geometric indices of the eigenvalues of the Hermitean matrix are equal to one, for $w^2=v^2$ the minimal polynomial of the Hamiltonian is $\Psi_H(\mathscr{E})=(\mathscr{E}-2v)(\mathscr{E}+v)$, and consequently

$$\mathscr{D}_H(\mathscr{E})/\Psi_H(\mathscr{E})=\mathscr{E}+v$$

is a common divisor of of the elements of the adjoint matrix $B(\mathscr{E})$ (see, e.g., Ref. 29), i.e.,

$$\mathscr{B}_\alpha(\mathscr{E})=\mathscr{H}_\alpha(\mathscr{E}+v), \quad \xi_\alpha(\mathscr{E})=(\mathscr{E}-\mathscr{R}_\alpha)(\mathscr{E}+v).$$

From here we derive the necessary conditions for degeneracy at the point $\mathbf{q}_0\in\mathbb{Z}_0$:

$$(v[\mathbf{q}_0]+\omega_\alpha[\mathbf{q}_0])h_\alpha=e^{\pm i\varphi}\eta_\alpha^{\beta\gamma}h_\beta h_\gamma, \tag{3.13}$$

$$\mathscr{R}_\alpha=v[\mathbf{q}_0]-\omega_\alpha[\mathbf{q}_0], \quad v[\mathbf{q}_0]\mathscr{R}_\alpha=h_\alpha^2-\eta_\alpha^{\beta\gamma}\omega_\beta[\mathbf{q}_0]\omega_\gamma[\mathbf{q}_0]. \tag{3.14}$$

The condition (3.13) can be satisfied only if the following dynamical invariant vanishes:

$$\mathscr{J}=\operatorname{Im}\prod_\alpha\mathscr{H}_\alpha=\sin\varphi\prod_\alpha h_\alpha. \tag{3.15}$$

Two qualitatively different cases are possible: 1) $\Pi_\alpha h_\alpha\neq 0$, $\sin\varphi=0$ and 2) $\Pi_\alpha h_\alpha=0$, $h^2\neq 0$, with arbitrary phase φ.

In the first case we obtain with the help of Eqs. (3.13) and (3.14), we obtain

$$\omega_\alpha[\mathbf{q}_0]=\mathscr{X}_\alpha\cos\varphi-v[\mathbf{q}_0], \quad 3v[\mathbf{q}_0]=\cos\varphi\sum_\alpha\mathscr{X}_\alpha, \tag{3.16}$$

where $\cos\varphi=\pm 1$. It follows from Eq. (3.16) that $v^2[\mathbf{q}_0]=\frac{1}{9}(\vec{\mathscr{X}}^2+2\mathbf{h}^2)$ and from Eqs. (3.16) and (3.13) that $w^2[\mathbf{q}_0]=\frac{1}{6}(\vec{\mathscr{X}}^2+2\mathbf{h}^2)$, which contradicts the condition $w^2[\mathbf{q}_0]=v^2[\mathbf{q}_0]\neq 0$. Thus $\mathbb{Z}_0=\varnothing$ for $\Pi_\alpha h_\alpha\neq 0$.

In the second case, as can be seen from Eq. (3.13), at least two components of the vector $\mathbf{h}$ must vanish. Let $h_\alpha=h_0>0$, $h_\beta=0$, $\beta\neq\alpha$. One can show that this case reduces to the two-neutrino problem with the effective mixing angle

$$\theta=\frac{1}{2}\operatorname{arctg}\left(\eta_\alpha^{\beta\gamma}\frac{2h_0}{\mathscr{W}_\gamma-\mathscr{W}_\beta}\right)\neq 0$$

and the degeneracy is lifted (see Appendix 2).

If nothing is stated to the contrary, we shall consider in the sequel that $\mathscr{J}\neq 1$. Then the degeneracy of the levels is excluded and consequently all the assumptions of the adiabatic theorem[30] are satisfied. This means, in particular, that in the system under consideration only the suppression of adiabatic topological phases is possible.[1,2]

4. THE EIGENVECTORS

In constructing the eigenvector system $\{|\mathscr{U}_N[\mathbf{q}]\rangle\}$ we note that the matrices $P_N=\xi_N^{-1}B(\mathscr{E}_N)$ are idempotent $P_N^2=P_N$ and satisfy the following relations

$$HP_N=\mathscr{E}_NP_N, \quad [H,P_N]=0. \tag{4.1}$$

Since $\mathscr{E}_N\neq\mathscr{E}_M$ for $N\neq M$ (and $\mathscr{J}\neq 0$), it follows from Eq. (4.1) that

$$P_NP_M=\delta_{NM}P_N, \qquad \sum_N P_N=I. \tag{4.2}$$

Thus P_N is an elementary projection operator onto the subspace spanned by the vector $|\mathscr{U}_N\rangle$,

$$P_N|\mathscr{U}_M\rangle=\delta_{NM}|\mathscr{U}_N\rangle, \quad P_N=|\mathscr{U}_N\rangle\langle\mathscr{U}_N|. \tag{4.3}$$

Denoting

$$u_{N\alpha}[\mathbf{q}]=(\xi_{N\alpha}[\mathbf{q}]/\xi_N[\mathbf{q}])^{1/2}, \quad z_N=(-1)^{1+N} \tag{4.4}$$

and making use of the identity (3.11), one can can represent the matrix P_N in the form

$$P_N=z_N\begin{vmatrix} z_Nu_{Ne}^2 & u_{N\mu}u_{Ne}\exp(+i\psi_{N\tau}) & u_{N\tau}u_{Ne}\exp(-i\psi_{N\mu}) \\ u_{Ne}u_{N\mu}\exp(-i\psi_{N\tau}) & z_Nu_{N\mu}^2 & u_{N\tau}u_{N\mu}\exp(+i\psi_{Ne}) \\ u_{Ne}u_{N\tau}\exp(+i\psi_{N\mu}) & u_{N\mu}u_{N\tau}\exp(-i\psi_{Ne}) & z_Nu_{N\tau}^2 \end{vmatrix}. \tag{4.5}$$

Let $\mathscr{U}_{N\alpha}[\mathbf{q}]$ denote the α-component of the vector $|\mathscr{U}_N[\mathbf{q}]\rangle$. The Eqs. (4.3) and (4.5) yield

$$\begin{aligned} &|\mathscr{U}_{N\alpha}|=u_{N\alpha}, \quad \arg\mathscr{U}_{N\alpha}\equiv\varkappa_{N\alpha}=-{}^1/_3\eta_\alpha^{\beta\gamma}(\tilde\psi_{N\beta}-\tilde\psi_{N\gamma})+{}^1/_3\varkappa_N, \\ &\tilde\psi_{N\alpha}=\psi_{N\alpha}+2\pi K_{N\alpha}, \end{aligned} \tag{4.6}$$

where $K_{N\alpha}$ are integers satisfying the condition $\Sigma_\alpha K_{N\alpha} = K_N$ and $\varkappa_N = \Sigma_\alpha \varkappa_{N\alpha}$ are arbitrary functions of $\mathbf{q}$. It is convenient to fix the gauge setting $\varkappa_N = \text{const.}$[5] According to Eqs. (4.2) and (4.3) the vectors

$$|\mathscr{U}_N\rangle = (u_{Ne}\exp\{i\varkappa_{Ne}\},\quad u_{N\mu}\exp\{i\varkappa_{N\mu}\},\quad u_{N\tau}\exp\{i\varkappa_{N\tau}\})^T$$

form a complete orthonormal set. Consequently,

$$\langle\mathscr{U}_N[\mathbf{q}]|\mathscr{U}_M[\mathbf{q}]\rangle = \delta_{NM},\qquad \sum_N |\mathscr{U}_N[\mathbf{q}]\rangle\langle\mathscr{U}_N[\mathbf{q}]| = I, \tag{4.7}$$

$$\sum_\alpha u_{N\alpha}[\mathbf{q}]u_{M\alpha}[\mathbf{q}]\exp\{\pm i(\varkappa_{N\alpha}[\mathbf{q}]-\varkappa_{M\alpha}[\mathbf{q}])\} = \delta_{NM}, \tag{4.7a}$$

$$\sum_N u_{N\alpha}[\mathbf{q}]u_{N\beta}[\mathbf{q}]\exp\{\pm i(\varkappa_{N\alpha}[\mathbf{q}]-\varkappa_{N\beta}[\mathbf{q}])\} = \delta_{\alpha\beta}. \tag{4.7b}$$

5. THE MIXING MATRIX OF THE NEUTRINOS IN A MEDIUM

The mixing matrix in the medium, $V^m[\mathbf{q}(t)]$, which relates the eigenstates of the neutrinos $|\nu^m(p_\nu;t)\rangle$ and the states with definite flavor $|\nu^f(p_\nu;t)\rangle$, satisfies the conditions[3,14]

$$\begin{aligned} &H[\mathbf{q}]V^m[\mathbf{q}] = V^m[\mathbf{q}]H^D[\mathbf{q}],\quad V^m[0] = V,\\ &H^D[\mathbf{q}] = \operatorname{diag}(\mathscr{E}_{N_1}[\mathbf{q}],\ \mathscr{E}_{N_2}[\mathbf{q}],\ \mathscr{E}_{N_3}[\mathbf{q}]). \end{aligned} \tag{5.1}$$

The labeling of the diagonal elements H^D is determined by the neutrino mass hierarchy,[6] which follows from Eq. (3.6) and the obvious equality $H^D_{ii}[0] = \Delta_i$. It can be seen from Eq. (5.1) that the columns of the matrix $V^m[\mathbf{q}]$ coincide, up to a phase factor, with the eigenvectors of the Hamiltonian $H[\mathbf{q}]$, and consequently

$$V^m[\mathbf{q}] = U[\mathbf{q}]D(\chi[\mathbf{q}]), \tag{5.2}$$

where

$$U[\mathbf{q}] = (|\mathscr{U}_{N_1}[\mathbf{q}]\rangle,\quad |\mathscr{U}_{N_2}[\mathbf{q}]\rangle,\quad |\mathscr{U}_{N_3}[\mathbf{q}]\rangle)^T, \tag{5.3}$$

$$D(\chi) = \operatorname{diag}[\exp(i\chi_1),\quad \exp(i\chi_2),\quad \exp(i\chi_3)], \tag{5.4}$$

and $\chi_i = \chi_i[\mathbf{q}]$ are arbitrary functions ("Majorana phases"). Imposing the conditions $\chi_i[0] = 0$, we obtain the equations

$$u_{N_i\alpha}[0] = |V_{\alpha i}|,\quad \varkappa_{N_i\alpha}[0] = \arg V_{\alpha i}, \tag{5.5}$$

which fix the constant phases $\varkappa_N$; according to Eqs. (4.5) and (5.5)

$$\varkappa_{N_i} = \sum_\alpha \arg V_{\alpha i}.$$

We point out here some important properties of the quantity $\mathscr{J}$ and of the "rephasing invariants" $J_{\alpha i}[\mathbf{q}]$ (see Refs. 31, and 23, 24), which are constructed from the elements of the matrix V^m according to the rule

$$J_{\alpha i}[\mathbf{q}] = \eta_\alpha^{\beta\gamma}\eta_i^{jk}V_{\beta j}^m V_{\gamma k}^m (V_{\beta k}^m V_{\gamma j}^m)^* = \eta_\alpha^{\beta\gamma}\eta_i^{jk}U_{\beta j}U_{\gamma k}(U_{\beta k}U_{\gamma j})^*.$$

It follows from the unitarity of the matrix U that all the invariants $J_{\alpha i}[\mathbf{q}]$ have identical imaginary parts, namely

$$J[\mathbf{q}] = \operatorname{Im} J_{\alpha i}[\mathbf{q}] = -\operatorname{Im}(\eta_\alpha^{\beta\gamma}\eta_i^{jk}U_{\alpha i}U_{\beta j}U_{\gamma k}\det U^+). \tag{5.6}$$

Making use of Eqs. (3.15) and (5.1)–(5.4), we obtain the identity

$$\mathscr{J} = \operatorname{Im}\prod_\alpha\sum_i \eta_\alpha^{\beta\gamma}U_{\beta i}H_{ii}^D U_{\gamma i}^*, \tag{5.7}$$

which explains the dynamical meaning of the quantity $\mathscr{J}$. With the help of Eqs. (5.6) and (5.7) one can show that $\mathscr{J}$ and $J[\mathbf{q}]$ are related by

$$\mathscr{J} = J[\mathbf{q}]\prod_L \eta_L^{MN}(\mathscr{E}_M[\mathbf{q}]-\mathscr{E}_N[\mathbf{q}]), \tag{5.8}$$

hence

$$\mathscr{J} = J[0]\prod_i \eta_i^{jk}(\Delta_j-\Delta_k) = J[0]\prod_i \eta_i^{jk}\frac{m_j^2-m_k^2}{2p_\nu}. \tag{5.9}$$

Thus the assertion made in Sec. 3, that the spectrum of H is nondegenerate for $\mathscr{J}\neq 0$ becomes obvious.

6. THE TOPOLOGICAL PHASES

Following Berry[1] we define the Abelian topological phases $\gamma_N = \gamma_N[\mathbf{q}(t)]$ by means of the differential equation

$$\dot\gamma_N[\mathbf{q}] = i\langle\mathscr{U}_N[\mathbf{q}]|\dot{\mathscr{U}}_N[\mathbf{q}]\rangle. \tag{6.1}$$

Making use of Eq. (4.6) we obtain

$$\langle\mathscr{U}_N|\dot{\mathscr{U}}_N\rangle = i\sum_\alpha u_{N\alpha}^2\dot\varkappa_{N\alpha} = -\frac{i}{3}\sum_\alpha u_{N\alpha}^2\eta_\alpha^{\beta\gamma}(\psi_{N\beta}-\psi_{N\gamma}). \tag{6.2}$$

On the other hand, Eqs. (3.10) and (3.11) imply the equality

$$\dot\psi_{N\alpha} = \mathscr{J}b_{N\alpha}^{-2}(\dot{\mathscr{E}}_N+\dot q_\alpha). \tag{6.3}$$

Introducing the gauge field $\mathscr{A}_N[\mathbf{q}]$ with the components (the Berry connections)

$$\mathscr{A}_N^\alpha = i\left\langle \mathscr{U}_N\left|\frac{\partial}{\partial q_\alpha}\right|\mathscr{U}_N\right\rangle = \frac{\mathscr{J}}{3\xi_N}\eta_\alpha^{\beta\gamma}\left(\frac{1}{\xi_{N\beta}}-\frac{1}{\xi_{N\gamma}}\right) \tag{6.4}$$

and taking account of Eqs. (6.2)–(6.4), one can rewrite the equation (6.1) in the form

$$\dot\gamma_N[\mathbf{q}] = \sum_\alpha \mathscr{A}_N^\alpha\dot q_\alpha \equiv \mathscr{A}_N\dot{\mathbf{q}}, \tag{6.5}$$

whence[7]

$$\gamma_N[\mathbf{q}(t)]-\gamma_N[\mathbf{q}(0)] = \int_C \mathscr{A}_N\,d\mathbf{q}. \tag{6.6}$$

The integration in Eq. (6.6) is between the points $\mathbf{q}(0)$ and $\mathbf{q}(t)$, along a contour C in the plane $\mathbb{Q} = \{\mathbf{q}:\Sigma_\alpha \mathbf{q}_\alpha = 0\}$. Since the field $\mathscr{A}_N$ is regular on the $\mathbb{Q}$-plane [guaranteed by the presence of the factor $\mathscr{J}$ in the right-hand side of (6.4)], the integral (6.6) always exists and is finite. Of course $\mathscr{A}_N^\alpha \propto \mathscr{J}$ (and, in addition $\Sigma_\alpha\mathscr{A}_N^\alpha = 0$) only in the chosen gauge $\Sigma_\alpha æ_{N\alpha} = \text{const}$ (we call it the $\mathscr{J}$ gauge). Going over to the unitarily equivalent basis $|\widetilde{\mathscr{U}}_N\rangle = \exp\{i\chi_N[\mathbf{q}]\}|\mathscr{U}_N\rangle$ the connections and the topological phases transform according to

$$\mathscr{A}_N^\alpha \to \mathscr{A}_N^\alpha - \partial\chi_N/\partial q_\alpha,\quad \gamma_N\to\gamma_N-\chi_N[\mathbf{q}(t)]+\chi_N[\mathbf{q}(0)].$$

An obvious gauge-invariant consequence of the nonsingular character of the field $\mathscr{A}_N[\mathbf{q}]$, is the vanishing of the Berry phases

$$\gamma_N{}^B \equiv \gamma_N[\mathbf{q}(T)] = \oint_C \mathscr{A}_N\, d\mathbf{q}, \qquad \mathbf{q}(T) \overset{\text{def}}{=} \mathbf{q}(0),$$

which appear if the parameters of the medium vary periodically, or equivalently, the identical vanishing of the curvature tensors

$$\mathscr{F}_N{}^{\alpha\beta} = \frac{\partial \mathscr{A}_N{}^\beta}{\partial q_\alpha} - \frac{\partial \mathscr{A}_N{}^\alpha}{\partial q_\beta}.$$

This means that $\gamma_N[\mathbf{q}]$ does not depend on the choice of integration path $C \in \mathbb{Q}$, i.e., the distribution of composition of matter between the points $x(0)$ and $x(t)$.

In the four-fermion approximation the expression for the topological phases simplifies considerably:

$$\gamma_N[\mathbf{q}(t)] = \frac{3}{2} \int_{q_e(0)}^{q_e(t)} \mathscr{A}_N{}^e\, dq_e$$

$$= -2^{1/2} G_F \mathscr{J} \int_{\mathscr{N}_e(x_0)}^{\mathscr{N}_e(x)} \frac{1}{3\xi_N} \left(\frac{1}{\xi_{N\mu}} - \frac{1}{\xi_{N\tau}} \right) d\mathscr{N}_e.$$

The equalities $\gamma_N^B = 0$ and $F_N^{\alpha\beta} = 0$ become obvious in this case.

These results do not depend on the parameterization of the vacuum mixing matrix, but in order to clarify the physical meaning of the $\mathscr{J}$-gauge it is useful to list the explicit form of the invariant $\mathscr{J}$ in the Kobayashi–Maskawa representation. With the help of Eq. (5.9) and the formulas from Appendix 1 we find

$$\mathscr{J} = \sin\delta \sin\theta_1 \prod_i \left[\eta_i{}^{jk} \frac{1}{4p_\nu} (m_j{}^2 - m_k{}^2) \sin 2\theta_i \right]. \tag{6.7}$$

As can be seen from Eq. (6.7), $\gamma_N = 0$ (in the $\mathscr{J}$-gauge) if some mixing angles θ_i vanishes, then either the Dirac phase $\delta = 0$ or $\delta = \pi$ (There is no *CP*-violation), or the neutrino mass spectrum is degenerate. In particular, topological phases are absent in the two-neutrino system.

7. THE EVOLUTION OPERATOR AND TRANSITION PROBABILITIES

It follows from the unitarity of the evolution operator that the differences

$$\eta_\alpha{}^{\beta\gamma} [\mathscr{P}_{\beta\gamma}(t) - \mathscr{P}_{\gamma\beta}(t)] = -\eta_\alpha{}^{\beta\gamma} (|S_{\beta\gamma}(t)|^2 - |S_{\gamma\beta}(t)|^2)$$

are equal to the same universal functions $\mathscr{P}(t)$ and $\mathscr{P}(t) \neq 0$ which is not identically zero, if $\mathscr{J} \neq 0$. Using this fact one can write the probabilities for off-diagonal transitions in the form:

$$\begin{aligned} \mathscr{P}_{\beta\gamma} &= {}^1/_2 (1 + \mathscr{P} + \mathscr{P}_{\alpha\alpha} - \mathscr{P}_{\beta\beta} - \mathscr{P}_{\gamma\gamma}), \\ \mathscr{P}_{\gamma\beta} &= {}^1/_2 (1 - \mathscr{P} + \mathscr{P}_{\alpha\alpha} - \mathscr{P}_{\beta\beta} - \mathscr{P}_{\gamma\gamma}), \end{aligned} \tag{7.1}$$

where α,β,γ is any cyclic permutation of e,μ,τ. Let $\zeta_\alpha(t)$ denote the fraction of ν_α in the summary neutrino flux at a distance $x(t)$ from the source. With the help of (7.1) we obtain

$$\zeta_\alpha(t) = \mathscr{P}_{\alpha\alpha}(t) + \frac{1}{2} \left[1 - \sum_\beta \mathscr{P}_{\beta\beta}(t) \right] [1 - \zeta_\alpha(0)]$$

$$- \frac{1}{2} \mathscr{P}(t) \eta_\alpha{}^{\beta\gamma} [\zeta_\beta(0) - \zeta_\gamma(0)]. \tag{7.2}$$

Thus, for the calculation of the experimentally measurable quantities $\zeta_\alpha(t)$ one must find the four functions: $\mathscr{P}_{ee}(t)$, $\mathscr{P}_{\mu\mu}(t)$, $\mathscr{P}_{\tau\tau}(t)$ and $\mathscr{P}(t)$.

We start out with the following representation of $S(t)$

$$S(t) = U[\mathbf{q}(t)] D(-\Omega(t)) X(t) U^+[\mathbf{q}(0)]. \tag{7.3}$$

Here $X(t)$ is an unknown unitary matrix, $D(\Omega)$ is a unitary diagonal matrix of the type (5.4): $D(\Omega) = \|\exp(i\Omega_j)\delta_{jk}\|$;

$$\Omega_j(t) = \Phi_{N_j}(t) - \gamma_{N_j}[\mathbf{q}(t)]$$

are the total phases, and $\Phi_N(t)$ are the dynamical phases, defined by the equation

$$\Phi_N(t) = \int_0^t \mathscr{E}_N[\mathbf{q}(\tau)]\, d\tau.$$

As is easily seen, the representation (7.3) is a matrix rewriting of the Born–Fock expansion for the amplitudes $S_{\beta\alpha}$ in terms of the complete set of vectors:

$$|\mathscr{P}_N\rangle = \exp(i\gamma_N) |\mathscr{U}_N\rangle.$$

Satisfying the Born–Fock–Simon condition[2] $\langle \mathscr{S}_N | \dot{\mathscr{S}}_N \rangle = 0$. We also note that the evolution operator $S^m(t)$ of the state $|\nu^m(p_\nu;t)\rangle$ equals $D[-(\chi + \Omega)]X$ and, consequently $|S_{ji}^m(t)|^2 = |X_{ji}(t)|^2$.

Substituting (7.3) into the original equation (2.3) and taking account of the definition (5.3) we obtain the evolution equation for the matrix X:

$$i\dot{X}(t) = F(t) X(t) \tag{7.4}$$

with the initial condition $X(0) = I$. Here we have introduced a new "Hamiltonian"

$$F(t) = D(\Omega(t)) Y(t) D(-\Omega(t)), \tag{7.5}$$

which represents a Hermitean matrix with vanishing elements along the diagonal. The matrix Y in (7.5) is defined by the eigenvectors of the Hamiltonian (2.4):

$$Y = \begin{vmatrix} 0 & y_3 & y_2^* \\ y_3^* & 0 & y_1 \\ y_2 & y_1^* & 0 \end{vmatrix}, \qquad y_i = -i\eta_i{}^{jk} \langle \mathscr{U}_{N_j} | \dot{\mathscr{U}}_{N_k} \rangle.$$

The explicit form of the inner products are easy to find $\langle \mathscr{U}_M | \dot{\mathscr{U}}_N \rangle$ with $M \neq N$ if one makes use of the equations (3.1) and (2.4), (4.6):

$$\begin{aligned} \langle \mathscr{U}_M | \dot{\mathscr{U}}_N \rangle &= -\frac{\langle \mathscr{U}_M | \dot{H} | \mathscr{U}_N \rangle}{\mathscr{E}_M - \mathscr{E}_N}, \quad M \neq N, \\ \langle \mathscr{U}_M | \dot{H} | \mathscr{U}_N \rangle &= -\sum_\alpha u_{M\alpha} u_{N\alpha} \exp\{-i(\varkappa_{M\alpha} - \varkappa_{N\alpha})\} \dot{q}_\alpha. \end{aligned} \tag{7.6}$$

Since $\mathscr{E}_M \neq \mathscr{E}_N$, for $M \neq N$ the functions $y_i = y_i(t)$ are regular.

It is clear that the equations (2.3) and (7.4) are fully equivalent, but the latter is more convenient for perturbative calculations for functions $q_\alpha(t)$ which do not vary too fast. The standard computation scheme for $X(t)$ in the nth approximation is given by the recursion relation

$$X_n(t) = I - i \int_0^t F(\tau) X_{n-1}(\tau)\, d\tau, \qquad X_0(t) = I. \tag{7.7}$$

On account of the continuity of $\dot{q}_\alpha(t)$ the sequence $\{X_n\}$ converges absolutely and uniformly to the solution of Eq. (7.4) on any finite interval $(0,t_1)$.[29] It is easy to show that the matrices X_n are unitary up to terms of the order $X_n - X_{n-1} \equiv \delta X_n$, i.e.,

$$X_n^+ X_n = X_n X_n^+ = I + i \int_0^t (X_{n-1}^+ F\, \delta X_n - \delta X_n^+ F X_{n-1})\, d\tau.$$

We consider in greater detail the solution in the adiabatic approximation:

$$X(t) = X_0 = I, \quad |S_{ji}^m|^2 = \delta_{ji},$$

which is important for many applications of the theory of neutrino oscillations in matter.[3] The criterion for applicability of this approximation is well known from quantum mechanics:

$$\max_{M \neq N} \left| \frac{\langle \mathcal{U}_M | \dot{\mathcal{U}}_N \rangle}{\mathcal{E}_M - \mathcal{E}_N} \right|^2 \ll 1. \tag{7.8}$$

According to Eq. (7.6) the condition (7.8) is satisfied if for all $M \neq N$

$$\Big(\sum_\alpha u_{M\alpha} u_{N\alpha} |\dot{q}_\alpha| \Big)^2 \ll (\mathcal{E}_M - \mathcal{E}_N)^4.$$

Near the Mikheev–Smirnov resonances,[21] i.e., near the points $\mathbf{q}_R$ of maximal proximity of the levels $\mathcal{E}_0[\mathbf{q}_R]$ and $\mathcal{E}_S[\mathbf{q}_R]$ (where $S = +$ or $-$) the condition (7.8) can be written in the form

$$(\dot{W}[\mathbf{q}_R])^2 \ll (\mathcal{E}_0[\mathbf{q}_R] - \mathcal{E}_S[\mathbf{q}_R])^4.$$

The adiabatic evolution operator of the 3ν system

$$S^A(t) = U[\mathbf{q}(t)] D(-\Omega(t)) U^+[\mathbf{q}(0)] \tag{7.9}$$

depends equally on the dynamical and topological phases, and since it is gauge invariant,[1,2] the topological effects cannot be removed by a redefinition of the basis $\{|\mathcal{U}_N\rangle\}$. It is sufficiently obvious that an analogous assertion is valid also in the general case of a nonadiabatic evolution.

From Eqs. (2.2) and (7.9) we determine the expressions for the probabilities of the adiabatic $\nu_\alpha \to \nu_\beta$ transitions:

$$\mathcal{P}_{\alpha\beta}(t) = \sum_i |U_{\alpha i}^0 U_{\beta i}^t|^2 + 2\mathrm{Re} \times \sum_i \eta_i^{jk} U_{\alpha j}^0 U_{\beta k}^t (U_{\alpha k}^0 U_{\beta j}^t)^* \exp(i\Omega_{jk}^t), \tag{7.10}$$

where, for the sake of brevity, we have introduced the notations

$$U_{\alpha i}^t = U_{\alpha i}[\mathbf{q}(t)], \quad \Omega_{jk}^t = \Omega_j(t) - \Omega_k(t).$$

As we can see, the expression for $\mathcal{P}_{\alpha\beta}(t)$ contains three kinds of phases: local phases ($\varkappa_{N\alpha}$), dynamical phases (Φ_N), and topological phases γ_N.

Of particular interest is the case of periodically varying parameters of the medium, $\mathbf{q}(t+T) = \mathbf{q}(t)$. This case is approximately realized in the special case when a neutrino beam intersects the globe of the Earth, and is important for the analysis of underground neutrino experiments. On account of the equality $\gamma_N^\beta = 0$ the topological effects do not affect the transition amplitudes for times which are integer multiples (K) of the periods T, and the expression of the evolution operator becomes very simple:

$$S^A(KT) = U[\mathbf{q}(0)] D(-K\Phi(T)) U^+[\mathbf{q}(0)]. \tag{7.11}$$

From Eq. (7.11) follow formulas for the transition probabilities generalizing the results of Refs. 22–24, which were derived for media with constant density. Taking account of Eqs. (5.3) and (5.6) and the completeness of the set of vectors $|\mathcal{U}_N[\mathbf{q}]\rangle$, Eq. (4.7b), we obtain

$$\mathcal{P}_{\alpha\alpha}(KT) = 1 - 4 \sum_L \eta_L^{MN} \left[u_{M\alpha}^0 u_{N\alpha}^0 \sin\left(\frac{1}{2} K \Phi_{MN}^T \right) \right]^2.$$

$$\mathcal{P}(KT) = 4J(\mathbf{q}(0)) \sum_L \eta_L^{MN} \sin(K\Phi_{MN}^T), \tag{7.12}$$

where

$$\Phi_{MN}^T = \Phi_M(T) - \Phi_N(T), \quad u_{M\alpha}^0 = u_{M\alpha}[\mathbf{q}(0)].$$

It can be seen from (7.12), (7.1), and (5.6) that the difference in the probabilities for the transitions $\nu_\beta \to \nu_\gamma$ and $\nu_\gamma \to \nu_\beta$ ($\beta \neq \gamma$) is due to *CP*-violation in the leptonic sector in the presence of three-neutrino mixing. As can be seen from Eq. (7.2), the four quantities $\mathcal{P}_{ee}$, $\mathcal{P}_{\mu\mu}$, $\mathcal{P}_{\tau\tau}$ and $\mathcal{P}$ cannot be simultaneously determined in a neutrino experiment with a beam of fixed flavor composition. Nevertheless, one could attempt to measure the effects of CP- (and *T*-) violation in underground detectors of the next generation (Super Kamiokande, MACRO, LVD, etc.) making use of neutrino beams from accelerators and atmospheric neutrinos. This question requires separate consideration.

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APPENDIX 1

The vacuum mixing matrix V in the Kobayashi–Maskawa representation[26] has the form,

$$V = \left\| \begin{matrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{matrix} \right\|,$$

$$\det V = -e^{i\delta},$$

$$s_i = \sin\theta_i, \quad c_i = \cos\theta_i, \quad 0 \leqslant \theta_i < \pi/2, \quad -\pi < \delta \leqslant \pi.$$

From here, making use of the definition (2.6), we obtain the expressions for the parameters $\mathcal{W}_\alpha$ and $\mathcal{H}_\alpha$, occurring in the Hamiltonian (2.4)

$$\mathcal{W}_e = c_1^2 \Delta_1 + s_1^2 \Delta_{23},$$

$$\mathcal{W}_\mu = c_2^2 (s_1^2 \Delta_1 + c_1^2 \Delta_{23}) + s_2^2 \Delta_{32} + 2 c_1 s_2 c_2 s_3 c_3 c_\delta (\Delta_3 - \Delta_2),$$

$$\mathcal{W}_\tau = s_2^2 (s_1^2 \Delta_1 + c_1^2 \Delta_{23}) + c_2^2 \Delta_{32} - 2 c_1 s_2 c_2 s_3 c_3 c_\delta (\Delta_3 - \Delta_2),$$

$$\mathcal{H}_e = s_2 c_2 (s_1^2 \Delta_1 + c_1^2 \Delta_{23} - \Delta_{32}) + c_1 s_3 c_3 (s_2^2 e^{i\delta} - c_2^2 e^{-i\delta}) (\Delta_3 - \Delta_2),$$

$$\mathcal{H}_\mu = s_1 [c_1 s_2 (\Delta_{23} - \Delta_1) - c_2 s_3 c_3 e^{+i\delta} (\Delta_3 - \Delta_2)],$$

$$\mathcal{H}_\tau = s_1 [c_1 c_2 (\Delta_{23} - \Delta_1) + s_2 s_3 c_3 e^{-i\delta} (\Delta_3 - \Delta_2)].$$

Here $\Delta_{ij} = c_3^2 \Delta_i + s_3^2 \Delta_j$, $c_\delta = \cos\delta$, and the other notations are those introduced in the main text. The imaginary parts J of the rephasing invariants [see Eq. (5.6)] in the vacuum case are equal to

$$J[0]={}^1/_8 \sin\delta \sin\theta_1 \prod_i \sin 2\theta_i.$$

We list a few formulas which are independent of the parametrization of the matrix V and are useful for the analysis of the vacuum oscillations. Making use of the definitions (2.6), (3.4), and (3.8), and applying several times the identities

$$(\det V)\, V_{\alpha i}^{*}=\eta_\alpha^{\beta\gamma}\eta_i^{jk}(V_{\beta j}V_{\gamma k}-V_{\beta k}V_{\gamma j}).$$

which follow from the unitarity of V, it is easy to derive the relations

$$h_\alpha^2=\mathscr{W}_\alpha^2+\mathrm{h}^2-\sum_i |V_{\alpha i}|^2\Delta_i^2,\quad \mathrm{h}^2={}^1/_2(\Delta^2-\vec{\mathscr{W}}^2),$$

$$\xi_{N,\alpha}[0]=\xi_i|V_{\alpha i}|^2,\quad \mathscr{B}_{N,\alpha}[0]=\xi_i\eta_\alpha^{\beta\gamma}V_{\beta i}V_{\gamma i}^{*},$$

where

$$\xi_i=\sum_\alpha \xi_{N,\alpha}[0]=3(\Delta_i^2-\Delta^2/6),\quad \Delta^2=\sum_i\Delta_i^2,$$

$$\Delta^2/6=(1/8p_\nu^2)(\langle m^4\rangle-\langle m^2\rangle^2)={}^1/_3(\mathrm{h}^2+{}^1/_2\mathscr{W}^2)=\vec{w}^2[0].$$

From here follow, in particular, the identities (5.5).

APPENDIX 2

We consider the case $h_\beta=h_0\delta_{\alpha\beta}$, $h_0\neq 0$, when a degeneracy of the spectrum of the Hamiltonian (2.4) is possible, and consequently the condition of orthogonality and completeness (4.7) of the system of eigenvectors of H can be violated. The transitions $\nu_\alpha\leftrightarrow\nu_\beta$ $(\alpha\neq\beta)$ are forbidden and the equation (2.3) becomes equivalent to the evolution equation of the two-neutrino system (ν_β,ν_γ) of the form

$$i\dot{S}_\alpha(t)=H_\alpha(t)S_\alpha(t),\quad S_\alpha(0)=I,$$

where

$$S_\alpha=\eta_\alpha^{\beta\gamma}\begin{vmatrix} S_{\beta\beta} & S_{\beta\gamma}\\ S_{\gamma\beta} & S_{\gamma\gamma}\end{vmatrix},\qquad H_\alpha=\eta_\alpha^{\beta\gamma}\begin{vmatrix}\omega_\beta & \mathscr{H}_\alpha^{*}\\ \mathscr{H}_\alpha & \omega_\gamma\end{vmatrix}.$$

The transformation

$$S_\alpha(t)\to\tilde{S}(t)=\exp\left(-\frac{i}{2}\int_0^t\omega_\alpha(\tau)\,d\tau\right)D_\alpha S_\alpha(t),$$

$$H_\alpha(t)\to\tilde{H}(t)=D_\alpha(H_\alpha(t)+{}^1/_2\omega_\alpha(t)I)D_\alpha^{*}$$

with the diagonal unitary matrix

$$D_\alpha=\mathrm{diag}\,(\exp(i\varphi_\alpha/2),\quad \exp(-i\varphi_\alpha/2))$$

leads to a real effective Hamiltonian

$$\tilde{H}(t)=\begin{vmatrix}-D\cos 2\theta+Q(t) & D\sin 2\theta\\ D\sin 2\theta & D\cos 2\theta-Q(t)\end{vmatrix}.$$

Here we have introduced the following notations:

$$D=(Q_0^2+h_0^2)^{1/2},\quad \mathrm{tg}\,2\theta=h_0/Q_0,\quad Q_0={}^1/_2(W_\gamma-W_\beta),$$

$$Q(t)={}^1/_2p_\nu[n_\gamma(t)-n_\beta(t)],\quad \eta_\alpha^{\beta\gamma}=1.$$

The eigenvalues of $\tilde{H}(t)$ are equal to $\pm\varepsilon(t)$, where

$$\varepsilon=(D^2-2DQ\cos 2\theta+Q^2)^{1/2},$$

and consequently $(\theta\neq 0)$ degeneracy of the levels is impossible. The corresponding eigenvectors $|\tilde{\mathscr{U}}_\pm(t)\rangle$ can be chosen to be real

$$|\tilde{\mathscr{U}}_\pm\rangle=2^{-1/2}\begin{vmatrix}(1\pm\omega/\varepsilon)^{1/2}\\ \pm(1\mp\omega/\varepsilon)^{1/2}\end{vmatrix},\quad \omega=Q-D\cos 2\theta.$$

One can check directly that

$$\langle\tilde{\mathscr{U}}_N|\tilde{\mathscr{U}}_M\rangle=\delta_{NM},\ \langle\tilde{\mathscr{U}}_N|\dot{\tilde{\mathscr{U}}}_N\rangle=0,$$

$$\langle\tilde{\mathscr{U}}_+|\dot{\tilde{\mathscr{U}}}_-\rangle=-\langle\dot{\tilde{\mathscr{U}}}_+|\tilde{\mathscr{U}}_-\rangle=\frac{D\sin 2\theta}{2\varepsilon^2}\dot{Q}=\frac{\sin 2\theta^m}{2\varepsilon}\dot{Q},$$

where θ^m is the mixing angle in the medium:

$$\theta^m=\mathrm{arctg}\left[\left(\frac{\varepsilon-\omega}{\varepsilon+\omega}\right)^{1/2}\right].$$

In terms of the θ^m the eigenvectors and the mixing matrix in the medium $\tilde{V}^m$ which diagonalizes the Hamiltonian $\tilde{H}$:

$$\tilde{V}^{m+}\tilde{H}\tilde{V}^m=\mathrm{diag}\,(-\varepsilon,\varepsilon),$$

Eigenvectors could be written in standard form (see for example Ref. 3–5):

$$|\tilde{\mathscr{U}}_-\rangle=\begin{vmatrix}\cos\theta^m\\ -\sin\theta^m\end{vmatrix},$$

$$|\tilde{\mathscr{U}}_+\rangle=\begin{vmatrix}\sin\theta^m\\ \cos\theta^m\end{vmatrix},$$

$$\tilde{V}^m=\begin{vmatrix}\cos\theta^m & \sin\theta^m\\ -\sin\theta^m & \cos\theta^m\end{vmatrix}.$$

The equations for the evolution operators $\tilde{S}(t)$ and the transition probabilities $\mathscr{P}_{\beta\gamma}=|S_{\gamma\beta}|^2=|\tilde{S}_{\gamma\beta}|^2$ are obtained (with the obvious modifications) in the same manner as in the general case of 3ν-oscillations (Sec. 7), and we shall not list them here.

[1)] The general expressions for the probabilities of three-neutrino oscillations in the case $\rho=$ const were first obtained in Ref. 19.

[2)] Everywhere in the sequel the Greek indices $\alpha,\beta,\ldots$ are used for labeling flavors (e,μ,τ), lower-case Latin indices $i,j,\ldots$ take on the values 1,2,3, and upper-case Latin letters $N,M,\ldots$ take on the values $+,-,0$ (see the following section).

[3)] The explicit form of $V(\theta_i,\delta)$ in the Kobayashi–Maskawa representation[26] is given in Appendix 1. The mixing matrix for three antineutrinos is $V^*(\theta_i,0)=V(\theta_i,-0)$.

[4)] The symbols η_i^{jk} and η_L^{MN}, to be used later, are defined analogously.

[5)] The constants $\varkappa_N$ are uniquely fixed by a transition to the vacuum limit. The arbitrariness in the choice of the numbers $K_{N\alpha}$ can be used to control concrete computations.

[6)] For example, in the case of the "direct hierarchy" $(m_1^2\leqslant m_2^2\leqslant m_3^2)$ $N_2=0$, $N_3=+1$.

[7)] Without loss of generality one can set in Eq. (6.6) $\gamma_N[\mathbf{q}(0)]=0$.

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