"Matsubara" approach in classical statistical physics: the method and a simple example

A. A. Abrikosov, Jr.

Institute of Theoretical and Experimental Physics
(Received 10 July 1991)
Zh. Eksp. Teor. Fiz. **100**, 2014–2025 (December 1991)

The path-integration technique of classical mechanics is generalized to classical statistical physics. Gibbs averages are represented as transition amplitudes in a phase space supplemented with time and ghosts. The partition function is BRST-invariant. The harmonic oscillator is treated as an example. The classical solution is derived, and zero modes are distinguished. The quantum fluctuations have an equidistant spectrum which is not the same as the Matsubara spectrum.

1. INTRODUCTION

In research in quantum field theory in Euclidean space-time it has become customary to seek analogies in classical statistical physics. The latter field has developed into a sort of proving ground for testing new methods. The methods of quantum field theory are used to advantage in quantum statistics. An important role here is played by the Matsubara approach, in which the temperature is replaced by an imaginary time. The wave functions of (Fermi-) Bose fields are (anti-) periodic along this coordinate, with a period equal to the reciprocal of the temperature, $\beta = 1/T$ (Ref. 1).

Let us continue this chain of analogies and introduce a "time" in classical statistical physics. Here is our motivation for doing so. In the first place, it may prove useful to raise the dimensionality of the space, $d \to d+1$, in view of the effectiveness of the ε expansion in the theory of phase transitions. Second, in perturbation theory it is convenient to introduce a temporal separation of interactions at spatially coincident points. This approach has been the reason for the success of field methods in quantum statistics. Finally, it is interesting to note that in quantum field theory (in particular, QCD), with $\beta = 1/g^2$, the charge becomes a spatial variable. It characterizes the properties of the system at long range and can play the role of an ir cutoff.

We work from the representation of the evolution operator for classical mechanics constructed in Refs. 2, as a path integral. We introduce a Gibbs average in it. A specific feature of our method is the use of Hamilton's equations of motion. This approach makes it possible to also find kinetic characteristics of systems in thermodynamic equilibrium. The use of a physical time distinguishes this approach from the method of stochastic quantization.[3] In that method, the statistical problem is solved by relaxing the system to thermal equilibrium as the system interacts with a reservoir. A Langevin equation is used in which the effect of the reservoir is taken into account as a Gaussian random force. This approach is very convenient for numerical calculations. The Langevin equation includes a characteristic relaxation time of the system, Γ; the thermalization occurs over a time $t \gg \Gamma$. The time scale Γ itself is arbitrary.

We intend here to look at the opposite limiting case, $t \ll \Gamma$, in which the dynamics is determined by Hamilton's equations. There is no difficulty in generalizing the path integration to a kinetic equation or in solving nonequilibrium problems. It would be interesting to attempt to combine the two approaches and to entrust to Langevin forces the Gibbs averaging.

In addition to a statistical quantization, a procedure of introducing a "fifth time," which leads to convergent formulas, has been described.[4] Instead of a relaxation equation one studies a Schrödinger equation which differs from the former by the Wick rotation $t \to it$. It has been asserted that in this manner one can determine a partition function for systems having an action with a lower bound. This may be a sort of analytic-continuation method, but the physical meaning of the results is not obvious.

Let us outline the present paper. We begin with a description of the path-integration formalism in classical mechanics and for a kinetic equation. We then show how to generalize this formalism to problems of statistical physics. Finally, we calculate the partition function of a harmonic oscillator as an example. Unfortunately, our method is more complicated than the usual methods in this simple case, and it requires field-theoretical approaches. Indeed, we would hardly expect that replacing an ordinary integral by a path integral would simplify a problem. However, when we are considering a complex multidimensional system we may find it beneficial to raise the dimensionality of the space.

2. THE PATH INTEGRAL IN CLASSICAL MECHANICS

Following Refs. 2, we will show how classical dynamics can be described with the help of a path integral. In the Hamilton formalism, a mechanical motion with $2n$ degrees of freedom (these results can easily be generalized to field systems) is described by the $2n$ equations[5]

$$\dot{\varphi}^a = \omega^{ab} \frac{\partial H}{\partial \varphi^b}, \tag{2.1}$$

where $H(\varphi^1 \ldots \varphi^{2n})$ is a Hamiltonian, φ^a are the coordinates in the $2n$-dimensional phase space of the system, and $\omega^{ab} = -\omega^{ba}$ is a symplectic 2-form. We can find the ordinary Hamilton's equations by separating the coordinates and the momenta, $\{\varphi^1 \ldots \varphi^{2n}\} = \{p_1 \ldots p_n, q^1 \ldots q^n\}$, and by setting $\omega^{q^i p_j} = -\omega^{p_j q^i} = \delta^i_j$. We assume that ω^{ab} is independent of the coordinates.

The solutions of Eqs. (2.1) completely determine the changes in the coordinates $\varphi^a_{\rm cl}(t)$ over time as a function of the initial conditions:

$$\psi_{cl}{}^a(t) = \varphi_{cl}{}^a(t, \{\varphi_i{}^a\}). \tag{2.2}$$

 0038-5646/91/121115-07$03.00

It is convenient to use "quantum-mechanical notation." In classical statistical physics and in classical kinetics, systems are described by a distribution function f, i.e., by the probability density for the system to have given values of the coordinates.[6,7] We call $f(\varphi)$ a "state vector" $|f\rangle$, and we introduce the scalar product

$$\langle f|g\rangle=\int f(\varphi)\,g(\varphi)\,d\varphi. \tag{2.3}$$

The state vectors corresponding to classical mechanics are $|\psi\rangle=\delta^{2n}(\varphi-\psi)$, where $\psi=\{\psi^a\}$ are phase coordinates.

We will show how to construct the evolution operator $S(t,t')$ for a mechanical system such that

$$\langle\varphi(t)|S(t,t_i|\varphi(t_i)\rangle=\delta^{2n}(\varphi^a(t)-\varphi_{cl}{}^a(t,\varphi_i)). \tag{2.4}$$

The time evolution of the distribution function is expressed by

$$|f(t)\rangle=S(t,t_i)\,|f(t_i)\rangle, \tag{2.5}$$

The function f must satisfy the kinetic equation[7]

$$\frac{df}{dt}=\omega^{ab}\partial_a f\partial_b H=\{H,f\}. \tag{2.6}$$

where $\{\ ,\ \}$ are the Poisson brackets.[1)] Actually, S is the Green's function of the kinetic equation.

The evolution operator (2.4) can be rewritten as a path integral of a δ^c-function. This integral is defined by the limit

$$\delta(\varphi_f-\varphi_{cl}(t_f,\varphi_i))=\left\langle\varphi_f\left|\int[d\varphi]\delta^c(\varphi-\varphi_{cl}(t))\right|\varphi_i\right\rangle$$
$$=\left\langle\varphi_f\left|\lim_{N\to\infty}\int\prod_{n=0}^{N-1}d\varphi_n\,\delta(\varphi_n-\varphi_{cl}(t_n,\varphi_i))\right|\varphi_i\right\rangle. \tag{2.7}$$

The points t partition the interval t_i, t_f into equal parts: $t_0=t_i, t_N=t_f, \varphi_{\rm cl}(t_0)=\varphi_i$. By construction, the carrier of the δ^c-function is the classical trajectory.

However, it is more convenient to replace $\varphi_{\rm cl}(t)$ as the argument of the δ^c-function in (2.7) by Hamilton's equations (2.1), which this function satisfies. The resulting expression found differs from (2.7) by a functional determinant:

$$\delta^c(\varphi-\varphi_{cl}(t))=\left|\det\frac{\delta(\dot\varphi^a-\omega^{ab}\partial_b H(\varphi))}{\delta\varphi^c}\right|\delta^c(\dot\varphi^a-\omega^{ab}\partial_b H(\varphi)). \tag{2.8}$$

Introducing Lagrange multipliers λ_a, we can represent the δ^c-function on the right side of (2.8) by a path integral:

$$\delta^c(\dot\varphi^a-\omega^{ab}\partial_b H)=\int\left[\frac{d\varphi^a\,d\lambda_a}{2\pi}\right]\exp i\int_{t_i}^{t_f}\lambda_a(\dot\varphi^a-\omega^{ab}\partial_b H(\varphi))\,dt, \tag{2.9}$$

Trajectories which connect points φ_i and φ_f in phase space are considered in this integral. For brevity, we will omit the factors $(2\pi)^{-1}$ in the measure $[d\varphi d\lambda]$.

To evaluate the functional determinant (2.8) we need to introduce ghosts, i.e., anticommuting Grassmann fields $\bar c_a$ and c^a:

$$\det\frac{\delta(\dot\varphi^a-\omega^{ab}\partial_b H)}{\delta\varphi^c}=\int[d\bar c_a\,dc^a]\exp i\int_{t_i}^{t_f}i\bar c_a(\dot c^a-\omega^{ab}\partial_b\partial_c H(\varphi)c^c)\,dt. \tag{2.10}$$

Combining (2.8) and (2.10), we find

$$\langle\varphi_f|S(t_f,t_i)|\varphi_i\rangle=\left\langle\varphi_f\left|\int[d\bar c\,dc\,d\varphi\,d\lambda]\exp i\int_{t_i}^{t_f}\mathscr L\,dt\right|\varphi_i\right\rangle, \tag{2.11}$$

where the action A is given by

$$A=\int\mathscr L\,dt=\int dt[\lambda_a(\dot\varphi^a-\omega^{ab}\partial_b H)+i\bar c_a(\dot c^a-\omega^{ab}\partial_b\partial_c H c^c)]. \tag{2.12}$$

[Do not confuse the Lagrangian $\mathscr L(\varphi,\lambda,\bar c,c)$ with the true Lagrangian.]

Equations of motion for the ghosts $\bar c$ and c are found by varying the action in (2.12)

$$\partial_t c^a-\omega^{ab}\partial_b\partial_c H(\varphi)c^c=0, \tag{2.13a}$$
$$\partial_t\omega^{ab}\bar c_b-\omega^{ab}\partial_b\partial_c H(\varphi)\omega^{cd}\bar c_d=0 \tag{2.13b}$$

These are Jacobi equations, which are satisfied by the distances $\delta\varphi^a=\varphi_1^a-\varphi_2^a$ between two closely spaced classical trajectories.

In choosing the ghost action as in (2.10), we are fixing φ_i, φ_f, and the ghosts c_i and c_f as boundary conditions. Imposing boundary conditions on the fields and the conjugate ghosts ($\varphi^a\leftrightarrow c^a$; $\lambda_a\leftrightarrow\bar c_a$) is a natural and convenient approach in a study of the properties of S. However, this approach is not obligatory, and in the following section of this paper we will fix c_f and $\bar c_i$.

The fields φ^a, λ^a and c^a, $\bar c_a$ satisfy canonical commutation relations. From the standard definition of the one-time commutator,

$$\langle[A_1(t),A_2(t)]_\pm\rangle=\lim_{\varepsilon\to0}\langle A_1(t+\varepsilon)A_2(t)\pm A_2(t)A_1(t-\varepsilon)\rangle \tag{2.14}$$

we find

$$\langle[\varphi^a,\lambda_b]_-\rangle=i\delta_b{}^a,\quad\langle[\bar c_b,c^a]_+\rangle=\delta_b{}^a,\quad\langle[\varphi^a,\varphi^b]_-\rangle=0. \tag{2.15}$$

The latter equation shows that we are again dealing with classical mechanics and that the physical variables commute.

A noteworthy property of action (2.12) is its BRST-invariance.[2,8] We introduce the auxiliary Grassmann variables $\bar\theta$, θ, and we define them with the help of commuting supercoordinates $\tilde q^a$ and supermomenta $\tilde p_a$,

$$\tilde q^a=\varphi^a+\bar\theta c^a,\quad\tilde p_a=\lambda_a\bar\theta\theta-i\bar c_a\theta, \tag{2.16}$$
$$\langle[\tilde q^a,\tilde p_b]_-\rangle=0. \tag{2.17}$$

We can then write the action in terms of superfields:

$$A=\int dt\,d\theta\,d\bar\theta[\tilde p_a\dot{\tilde q}^a-H(\tilde q^a-\omega^{ab}\tilde p_b)]. \tag{2.18}$$

The invariance of (2.18) under the shifts $\theta\to\theta+\varepsilon$ and $\theta\to\theta+\varepsilon$ means that action (2.12) is invariant under supersymmetry transformations (BRST and anti-BRST transformations)

$$\varphi^a\to\varphi^a+i\omega^{ab}\bar c_b\varepsilon,\quad c^a\to c^a-\varepsilon\omega^{ab}\lambda_b,\quad\lambda_a,\bar c_a\to\lambda_a,\bar c_a, \tag{2.19a}$$
$$\varphi^a\to\varphi^a+\bar\varepsilon c^a,\quad\bar c_a\to\bar c_a+i\lambda_a\bar\varepsilon,\quad\lambda_a,c^a\to\lambda_a,c^a \tag{2.19b}$$

where ε and $\bar\varepsilon$ are Grassmann parameters. The boundary conditions on the fields c_i and c_f do not break the supersymmetry. Below we will meet another version: Homogeneous

boundary conditions on λ_i and c_f impart supersymmetry to the partitioned function.

3. CLASSICAL STATISTICAL PHYSICS IN REAL TIME

A fundamental problem in classical statistical physics and in Euclidean quantum field theory is to calculate the partition function

$$Z(J_a)=\int[d\varphi^a]\exp\{-\beta H(\varphi)-iJ_a\varphi^a\}. \tag{3.1}$$

As usual, β is the reciprocal of the temperature, and the φ's are coordinates in phase space. For the time being, we will incorporate the sources J_a in the Hamiltonian.

We can show that the Gibbs distribution function[6]

$$\rho(\varphi)=\exp\left[-\beta H(\varphi)\right] \tag{3.2}$$

can be thought of as the result of the evolution of the system in real time. Admittedly, in place of Planck's constant, which does not figure in classical mechanics, we are obliged to introduce another quantum of action—the constant $\hat{h}$—so that we can express β in time units: $\beta\to\beta\hat{h}$. The final results will not depend on this quantity.

The variables φ in (3.1) and (3.1) can be thought of as coordinates on a hyperplane $t=\beta\hat{h}$ in the phase space of the system supplemented with the time. The evolution determined by Eqs. (2.1) determines a family of curves in the (φ, t) space which have a unique intersection with planes of $t=$ const. Energy is conserved along the trajectories:

$$H(\varphi_{cl}(t))=\text{const} \tag{3.3}$$

We can thus construct an operator for the statistical evolution of the system. Using (2.11) and (2.12), we write

$$\tilde{S}(t_f,t_i)=\int[d\bar{c}\,dc\,d\varphi\,d\lambda]\exp i\int_{t_i}^{t_f}\left(\mathscr{L}+\frac{iH}{\hat{h}}\right)dt. \tag{3.4}$$

The Gibbs distribution function ρ in (3.2) can easily be expressed in terms of this operator:

$$\rho(\varphi^a)=\int\langle\varphi^a|\tilde{S}(\beta\hat{h},0)|\varphi_i\rangle d\varphi_i. \tag{3.5}$$

We will call the expression in the argument of the exponential function $\tilde{S}(\beta\hat{h},0)$ the action $\tilde{A}=\int_0^{\beta\hat{h}}dt(L+iH/\hat{h})$.

We see that a Gibbs distribution is established over a time $\beta\hat{h}$ in the course of the statistical evolution, with equiprobable initial conditions (Fig. 1). In contrast with the quantum case, where an imaginary Euclidean time is used in the Matsubara technique, the motion in (3.4) occurs along genuine classical trajectories in real time.

A second distinction is the appearance of an arbitrary scale $\hat{h}$. Strictly speaking, we can introduce an arbitrary function of the time f such that

$$\int_{-\infty}^{\infty}f(t)\,dt=\beta\hat{h}. \tag{3.6}$$

In this case we can write

$$\rho(\varphi)=\int\left\langle\varphi\left|\int[d\bar{c}\,dc\,d\varphi\,d\lambda]\exp i\int_{-\infty}^{\infty}\left(\mathscr{L}+\frac{iHf}{\hat{h}}\right)dt\right|\varphi_i\right\rangle d\varphi_i. \tag{3.7}$$

This flexibility can be realized with the help of the new variable f in the path integral and the Lagrange multiplier η:[2)]

$$\rho(\varphi)=N^{-1}\int\left\langle\varphi\left|\int[d\bar{c}\,dc\,d\varphi\,d\lambda\,df]d\eta\right.\right.$$
$$\left.\times\exp i\int_{-\infty}^{\infty}\left(\mathscr{L}+\frac{iHf^2}{\hat{h}}+\eta f^2\right)dt-i\eta\beta\hat{h}|\varphi_i\right\rangle d\varphi_i. \tag{3.8}$$

In this paper we restrict the discussion to the choice [see (3.5)]

$$f^2=\theta(t)\theta(\beta\hat{h}-t)$$

The third distinction can be clarified by discussing the boundary conditions. In the quantum case the path integral specifies a density matrix, and we require that the fields be periodic (or antiperiodic) in the imaginary time. This requirement cannot be made in classical mechanics in real time. It turns out that instead of carrying out an integration over the initial conditions φ_i in (3.5) we need to set the corresponding Lagrange multiplier equal to zero.

To explain this point, we break up the time interval $[0, \beta\hat{h})$ into N subintervals, and we treat the path integral in (3.5) as the limit of a multiple integral as $N\to\infty$. We start with the definition of the Gibbs distribution, (3.2), and we specify the fields φ at the point $t_N=\beta\hat{h}$. We then introduce an integration over φ at the points $t_k=\beta\hat{h}k/N$. The Lagrangion fields λ prune the extra degrees of freedom and fix $\varphi(t_k)=\varphi_{\rm cl}(t_k)$. The ghosts $\bar{c}_k$ and c_k correct the measure of integration in accordance with $[d\varphi d\lambda]$. This procedure is shown graphically in Fig. 2. The points are integration variables. Since the variables φ, λ and $\bar{c}$, c do not commute, we have put $\bar{c}$ and λ at interior points of the intervals (t_k,t_{k+1}), and we have specified φ and c at points t_k. Note that the

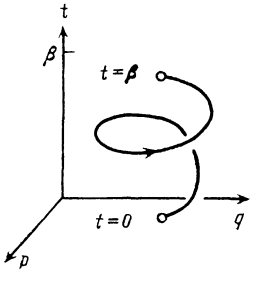

FIG. 1. Instead of points in the phase space (the pq plane) we consider the part of the trajectory over a time proportional to the reciprocal of the temperature, β.

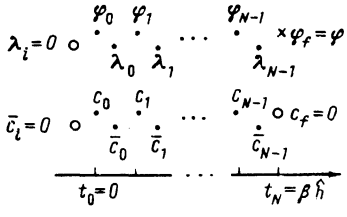

FIG. 2. Addition integrable (●) and fixed (○) variables in the distribution function. ×—Physical phase coordinates.

multiplicities of the integrals in the definition of $\rho(\varphi)$ are the same in terms of all the variables.

Figure 2 also explains how to choose the boundary conditions. The variables φ^a, λ_a and $c^a, \bar{c}_a$ are conjugates of each other, like ordinary coordinates and momenta.

The transition from one set to the other is made by means of Fourier transforms. Integration over the boundary value gives rise to the zeroth Fourier harmonic. As the boundary conditions in (3.5) we should thus adopt $\lambda_0 = 0$, $\bar{c} = 0$, and $c_N = 0$ (the fixed variables are shown by the circles in Fig. 2). When this choice is made, the numbers of physical variables and auxiliary variables (including the fixed variables) are the same, so we can go over to the continuum limit. The conditions $\bar{c}_i = 0$, $c_f = 0$ eliminate the zero modes from the spectrum of ghosts, preventing ρ from vanishing. We thus have

$$\rho(\varphi) = \langle \varphi,\ c=0 | \hat{S}(\beta\hat{\hbar},\ 0) | \lambda=0,\ \bar{c}=0 \rangle. \tag{3.9}$$

Expression (3.9) gives the Gibbs distribution in terms of the evolution in real time. The partition function $Z(\beta,J)$ is [see (3.1)]

$$Z(\beta, J) = \int \rho(\varphi) \exp(-iJ_a\varphi^a)\, d\varphi = \langle \lambda_a = J_a,\ c=0 | \hat{S}(\beta\hat{\hbar}, 0) | \lambda=0,\ \bar{c}=0 \rangle. \tag{3.10}$$

Admittedly, when we change the boundary conditions we should correct the action, replacing a term: $i\int \lambda_a \dot{\varphi}^a dt \to -i\int \varphi^a \dot{\lambda}_a dt$. Accordingly, Eq. (3.10) contains

$$\bar{A} = i\bar{c}_i c_i + \int_0^{\beta\hat{\hbar}} dt \left[-\dot{\lambda}_a \varphi^a - \lambda_a \omega^{ab} \partial_b H + i\bar{c}_a (\dot{c}^a - \omega^{ab}\partial_b \partial_c H c^c) + \frac{iH}{\hat{\hbar}} \right]. \tag{3.11}$$

Interestingly, the partition function in (3.10) is also invariant under supersymmetry transformations. First, the action (3.11), which can be put in the form

$$\bar{A} = \int_0^{\beta\hat{\hbar}} \frac{dt\, d\theta\, d\bar{\theta}}{\hat{\hbar}} [-\breve{q}^a \dot{\breve{p}}_a - H(\breve{q}^a - \omega^{ab}\breve{p}_b) \exp(-i\bar{\theta}\theta)] \tag{3.12}$$

with commuting variables $[\breve{p}, \breve{q}] = 0$,

$$\breve{q}^a = \varphi^a + i\hat{\hbar}^{1/2}\bar{c}_b\theta\omega^{ba}, \quad \breve{p}_a = \hat{\hbar}\lambda_a\bar{\theta}\theta + \hat{\hbar}^{1/2}\omega_{ab}\bar{\theta}c^b. \tag{3.13}$$

is supersymmetric. Second, one can verify that the boundary conditions $\lambda_i = 0$, $c_f = 0$ do not break the supersymmetry.

To conclude this section of the paper we note that the BRST invariance does not prevent us from introducing an arbitrary function of the time $f(t)$ in the definition of the partition function [see (3.6)]. In this case the action becomes

$$\bar{A} = \int_{-\infty}^{\infty} dt\, \frac{d\theta\, d\bar{\theta}}{\hat{\hbar}} [-\breve{q}^a \dot{\breve{p}}_a - H(\breve{q}^a - \omega^{ab}\breve{p}_b) \exp[-if(t)\bar{\theta}\theta]]. \tag{3.14}$$

Condition (6) obviously does not break the supersymmetry.

4. FREE FIELD; PARTITION FUNCTION OF A HARMONIC OSCILLATOR

We will demonstrate how this method works in the very simple problem calculating the partition function of a harmonic oscillator. This is a zeroth approximation, i.e., a starting point for constructing a perturbation theory for other problems. Admittedly, incorporating the time evolution complicates the problem here, since it becomes necessary to evaluate a path integral instead of a double integral. The arsenal of quantum field theory must be brought to bear on the problem: find classical solutions and average them over quantum fluctuations, incorporating zero modes. Let us see how this is done.

The Hamiltonian of the harmonic oscillator is

$$H(p, q) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}, \tag{4.1}$$

where p and q are the momentum and coordinate, m is the mass, and ω is the resonant frequency. The corresponding partition function is determined by a Gaussian integral. Using the sources J_p and J_q, we write this partition function as

$$Z(\beta, J) = \int \frac{dp\, dq}{2\pi} \exp[-\beta H - iJ_p p - iJ_q q] = \frac{1}{\beta\omega} \exp\left\{ -\frac{mJ_p^2}{2\beta} - \frac{J_q^2}{2m\omega^2\beta} \right\}. \tag{4.2}$$

In calculations in real time, the constant $\hat{\hbar}$, with the dimensionality of an action, arises in intermediate steps. Since physical properties do not depend on this constant, it is convenient to switch to dimensionless variables. We introduce the matrix notation

$$\varphi = \begin{pmatrix} \dfrac{1}{(m\omega\hat{\hbar})^{1/2}} & p \\ \left(\dfrac{m\omega}{\hat{\hbar}}\right)^{1/2} & q \end{pmatrix}, \quad J = \begin{pmatrix} (m\omega\hat{\hbar})^{1/2} & J_p \\ \left(\dfrac{\hat{\hbar}}{m\omega}\right)^{1/2} & J_q \end{pmatrix}. \tag{4.3}$$

In terms of these variables the Hamiltonian of the system is

$$H = {}^1/_2 \hat{\hbar}\omega\varphi^2, \quad \varphi^2 = \varphi^T\varphi. \tag{4.4}$$

When we introduce the dimensionless proper time $\tau = \omega t$, the equations of motion become

$$\partial_\tau \varphi = \Omega\varphi, \quad \Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.5}$$

We also make the replacement $\beta \to \omega\hat{\hbar}\beta$. The partition function in real time then takes the form of (3.10) with the action

$$\bar{A} = i\bar{c}_i c_i + \int_0^{\beta} d\tau \left[-\varphi(\partial_\tau - \Omega)\lambda + i\bar{c}(\partial_\tau - \Omega)c + \frac{i\varphi^2}{2} \right]. \tag{4.6}$$

To study the capabilities of the method it is interesting to try to change the order: first evaluate the path integral over $[d\varphi]$ and then evaluate that over $[d\lambda]$ {if we begin with $[d\lambda]$, we immediately go back to Eq. (4.2)}. We recall that in the preceding sections of this paper we omitted coefficients of $(2\pi)^{-1/2}$ of the differentials $d\varphi$ and $d\lambda$. We now put them back in. As a result, the Gaussian integrals are normalized (to unity):

$$\int dx (2\pi)^{-1/2} \exp(-x^2/2) = 1.$$

After integrating over $[d\varphi]$, we find

$$Z(J) = \left\langle \lambda_f = J \left| \int [d\lambda] \exp\left\{ -{}^1/_2 \int [(\partial_\tau - \Omega)\lambda]^2 d\tau \right\} \right| \lambda_i = 0 \right\rangle Z_c, \tag{4.7}$$

where Z_c is the ghost-dependent part of the partition function.

From the equations of motion of the fields λ which arise when the effective Lagrangian in (4.7) is varied,

$$(\partial_\tau-\Omega)^2\lambda=0, \tag{4.8}$$

we can find a classical solution which satisfies the boundary conditions[3)]

$$\lambda_{cl}=\frac{\tau}{\beta}[\exp\Omega(\tau-\beta)]J. \tag{4.9}$$

Introducing the shift of variables $\lambda\to\lambda_{\rm cl}+\lambda$ in (4.7), we find

$$Z(J)=\exp(-J^2/2\beta)Z(0). \tag{4.10}$$

As usual, incorporating the classical solution leads to the argument of the exponential function [which is the same as the argument of the exponential function on the right side of (4.2)], and the preexponential coefficient is determined by quantum fluctuations. In the case at hand, this coefficient is universal, independent of the sources.

In the calculations of $Z(0)=Z_\lambda Z_c$ it is convenient to free the variables at the points 0, β and to fix the boundary conditions by means of corresponding δ-functions. We then write

$$Z_\lambda(0)\propto\int d\eta_i\,d\eta_f\,d\lambda_i\,d\lambda_f\exp(i\eta_f\lambda_f-i\eta_i\lambda_i)\langle\lambda_f|\hat S_\lambda(\beta,0)|\lambda_i\rangle, \tag{4.11a}$$

$$Z_c(0)=\int d\bar\xi_f\,dc_f\,d\bar c_i\,d\xi_i\exp(-\bar\xi_f c_f-\bar c_i\xi_i)\langle c_f|\hat S_c(\beta,0)|\bar c_i\rangle. \tag{4.11b}$$

The only subtlety to be kept in mind below is that the spectra of λ, $\bar c$, and c contain zero modes. These modes satisfy Jacobi equations (2.13) [the equation for λ is the same as Eq. (2.13b) for $\bar c$]. For free fields, the latter equations are the same as the equations of motion

$$(\partial_\tau-\Omega)f=0,\quad f=\lambda,c,\bar c. \tag{4.12}$$

In the case of fixed boundary conditions, there would be no zero modes. When the boundaries are freed, classical trajectories close to the main trajectory, (4.9), come into play, and the distances between them satisfy Jacobi equations. The explicit expressions for the zero modes are

$$f^0=\exp[\Omega(\tau-\beta/2)]F,\quad F=\Lambda,\bar C,C. \tag{4.13}$$

In the calculation of $Z(0)$, we should integrate separately over the quantities Λ, $\bar C$, and C. The parts of expressions (4.11) which depend on the zero modes are

$$Z_\lambda(0)=\int d\Lambda\exp[(i\eta_f e^{\beta\Omega/2}-i\eta_i e^{-\beta\Omega/2})\Lambda]$$
$$=\delta(e^{\beta\Omega/2}\eta_i-e^{-\beta\Omega/2}\eta_f), \tag{4.14a}$$

$$Z_c(0)=\int d\bar C\,dC\exp-[\bar c_i^0c_i^0+\bar c_i'c_i^0+\bar c_f^0c_f'+\bar\xi_f c_f^0+\bar c_i^0\xi_i]$$
$$=\exp[\bar c_i'e^{-\beta\Omega}c_f'+\bar\xi_f e^{\beta\Omega}\xi_i]. \tag{4.14b}$$

To find (4.14b), we separated the zero and nonzero modes, $c=c^0+c^1$, $\bar c=\bar c^0+\bar c^1$, and we used

$$\int\bar c^0(\partial_\tau-\Omega)c'\,d\tau=\bar c_f^0c_f'-\bar c_i^0c_i'. \tag{4.15}$$

After (4.14) is substituted into partition function $Z(0)=Z^0\cdot Z^1$, we easily see that the boundary conditions are altered in the parts which do not contain zero modes. Integrating over the sources η, ξ_i, and $\bar\xi_f$, and also over the fields $\bar c_i'$, we find (det $e^{\beta\Omega}=1$)

$$Z_\lambda'=\int d\lambda\left\langle e^{\beta\Omega}\lambda\left|\int[d\lambda']\exp\left\{-\tfrac12\int[(\partial_\tau-\Omega)\lambda']^2d\tau\right\}\right|\lambda\right\rangle, \tag{4.16a}$$

$$Z_c'=\int dc\left\langle e^{\beta\Omega}c\left|\int[d\bar c'\,dc']\exp\left\{-\int\bar c'(\partial_\tau-\Omega)c'\,d\tau\right\}\right|c\right\rangle. \tag{4.16b}$$

Surprisingly, our classical problem has become analogous to a quantum-mechanical problem. In both cases, we are to evaluate the trace of an evolution operator over a finite time β. Admittedly, the boundary conditions in the classical case are not periodic, and they contain a relative rotation ("twist") in the p, q plane:

$$f_f'=\begin{pmatrix}f_p'\\f_q'\end{pmatrix}_f=e^{\beta\Omega}f_i',\qquad f=\lambda,c,\bar c. \tag{4.17}$$

However, we can take some approaches which are familiar in quantum statistics. In calculating the Feynman diagrams, we run into a summation over a discrete series of frequencies ω^+ for bosons and ω^- for ghosts. Because of the twist, these frequencies are shifted from the Matsubara frequencies by the resonant frequency of the oscillator:

$$\omega_n{}^+-1=\frac{2\pi}{\beta}n,\qquad \omega_n{}^--1=\frac{2\pi}{\beta}\left(n+\frac12\right). \tag{4.18}$$

Making use of the arbitrariness in the definition of the constant $\hat h$, we could of course bring these series of frequencies into coincidence with the Matsubara frequencies. To make the time interval equal to an integral number of periods, it would be sufficient to take $\hat h=2\pi k/\beta\omega$, where k is an integer. In complicated problems, unfortunately, this approach can be taken only for several multiple frequencies.[4)]

It is convenient to use column-matrix harmonics:

$$\begin{pmatrix}f\\g\end{pmatrix}_\pm=\sum_{-\infty}^{\infty}\exp(\Omega\omega_n{}^\pm\tau)\begin{pmatrix}f_n\\g_n\end{pmatrix}_\pm. \tag{4.19}$$

It would be interesting to pursue the analogy with the quantum case, noting that the matrix Ω, where $\Omega^2=-1$, has assumed the role played by the square root of negative unity (i). Classical statistical physics is thus converted into a field theory in an imaginary "matrix" time. However, it is not clear how to introduce wave functions which would have the meaning of probability amplitudes, and so forth.

We will not use a Fourier expansion; we will evaluate the partition function in a different way. We transform expression (4.11a) in the following way: we extend the integration over time to the interval $(-\pi N,\pi N)$ (we "sew tails" on front and back), and we impose periodic boundary conditions on the fields. We then take the limit $N\to\infty$. This method makes it possible, through the imposition of boundary conditions, to preserve the zero modes:

$$Z_\lambda(0)=\frac{\int d\lambda\, d\eta_i\, d\eta_f \exp[i\eta_f\lambda(\beta)-i\eta_i\lambda(0)]\langle\lambda|\exp\left\{-\frac{1}{2}\int_{-\pi N}^{\pi N}[(\partial_\tau-\Omega)\lambda]^2\,d\tau\right\}|\lambda\rangle}{\int d\lambda\, d\eta_0 \exp[i\eta_0\lambda(0)]\langle\lambda|\exp\left\{-\frac{1}{2}\int_{-\pi N}^{\pi N}[(\partial_\tau-\Omega)\lambda]^2\,d\tau\right\}|\lambda\rangle}+O\left(\frac{\beta}{N}\right). \tag{4.20}$$

After an integration over $d\lambda^0[d\lambda']$ we find

$$Z_\lambda=\int d\eta_i\, d\eta_f \exp\left\{-\frac{1}{2}[\eta_i G(0)\eta_i+\eta_f G(\beta)\eta_i +\eta_i G(-\beta)\eta_f+\eta_f G(0)\eta_f]\right\}\delta(e^{\beta\Omega/2}\eta_i-e^{-\beta\Omega/2}\eta_f). \tag{4.21}$$

The Green's function of the fields λ is

$$G(x-y)=(\partial_\tau-\Omega)^{-2}={}^1/_2|x-y|e^{\Omega(x-y)}. \tag{4.22a}$$

In any case, we introduce the ghost propagator

$$S(x-y)=(\partial_\tau-\Omega)^{-1}={}^1/_2\,\mathrm{sgn}\,(x-y)\,e^{\Omega(x-y)}. \tag{4.22b}$$

After substituting $G(x,y)$ into (4.21), and noting that the fields contain two components, we find

$$Z_\lambda(0)=1/\beta. \tag{4.23}$$

The factor $Z_c(0)$ [see (4.11b)] in the partition function is calculated in the standard way.[9] As a result we find

$$Z_c(0)=1. \tag{4.24}$$

Collecting (4.10), (4.23), and (4.24), we find

$$Z(J)=\frac{1}{\beta}\exp\left(-\frac{J^2}{2\beta}\right). \tag{4.25}$$

Aside from the switch to dimensionless variables, this expression is the same as (4.2). Q.E.D.

We see that the harmonic oscillator turns out to be a rather rich "toy field theory" in calculations in real time. We would point out the following circumstances.

First, if we wish to reproduce the exponential factor in the partition function, we need to consider the classical solution. This solution is determined by a system of Hamilton's equations for the fields φ and inhomogeneous Jacobi equations for the Lagrange multipliers λ.

Second, the zero modes, which are necessarily present in the spectrum, play an important role. They satisfy the homogeneous Jacobi equation. They arise because of the continuous dependence of the solution on the boundary conditions.

Third, incorporating quantum fluctuations leads to a preexponential factor which is independent of the sources for a free field. The nonzero modes satisfy the "twisted" boundary conditions and have an equidistant spectrum. This spectrum is shifted with respect to the Matsubara spectrum by the resonant frequency of the oscillator. The Lagrangians of both the boson and fermion degrees of freedom are expressed in terms of the Jacobi operator $\delta_b^a\partial_t-\omega^{ac}\partial_c\partial_b H$. The preexponential factor arises because of the difference between the boson and fermion frequencies.

Our last comment is of a technical nature. We have written the equations in terms of two components, and we have represented operators by 2×2 matrices. However, we could have gone over to holomorphic and antiholomorphic components:

$$f_\pm={}^1/_2(1\pm i\Omega)f. \tag{4.26}$$

That approach would have made it possible to completely separate variables in the case of a free field; it would apparently also be useful in solving other problems.

5. CONCLUSION

Let us list the basic results of this paper. We have shown that the path-integration formalism of classical mechanics can be generalized to classical statistical physics. The thermodynamic functions are calculated as path integrals in a phase space supplemented with the time. The partition function is BRST-invariant. Admittedly, we do not have a literal analogy with quantum statistics, since (first) the calculations are carried out in real time, and (second) instead of using periodic boundary conditions we have to fix the Lagrange multipliers at the ends of the interval.

In place of Planck's constant we are obliged to introduce a "quantum of action" $\hat{h}$, which has an arbitrary value in the classical case. The physical properties do not depend on this quantum of action.

The thermodynamic functions can be represented as the results of an evolution over an infinite time; alternatively, we could limit the time interval. We chose the second method. It turns out that in terms of the discrimination of classical solutions and zero modes the spectrum of quantum fluctuations may be combined with the Matsubara choice of scale.

Path integration can be used to study the dynamic characteristics. In that case, the scale would be dictated by the particular problem.

The capabilities of this approach appear to us to depend on the following considerations. We have seen in the example of a free field that the strongest exponential dependence of the partition function is determined by the classical solution. The value of this solution at the end of the interval, $t=\beta\hat{h}$, coincides with the saddle point in the ordinary partition function, whose calculation is simplified in the present case. It may turn out that this is true of all perturbation-theory problems in classical statistics. However, we know that the partition function in asymptotically free problems in Euclidean field theories (e.g., QCD or a two-dimensional n field) is not saturated by the classical solutions. Here is a place to try to find new approaches for analytic calculations. Perhaps the additional coordinate will prove to be a pawn which can be sacrificed in a trade for a queen.

I would like to thank A. A. Roslyĭ for friendly discussions and Ya. I. Kogan for moral support.

[1)] The operator $L=-\partial_a H\omega^{ab}\partial_b$ is called the "Liouville operator."

[2)] We have introduced a nonnegative function f^2 in place of f in the argument of the exponential function to avoid spoiling the convergence of

the integral. In addition, it is necessary to introduce a normalization factor $N = \int [df] \delta(f^2 - \beta\hat{h}) \cdot 2\pi$.

[3)] If we go back to action (4.6), we see that this formula specifies an extreme trajectory on which the equation $\varphi = -(i/\beta)\,[\exp\Omega(\tau - \beta)]J$ holds. The value of $\varphi(\beta)$ coincides with the saddle point of the ordinary partition function, (4.2).

[4)] Another way to get rid of the frequency shift is to introduce Heisenberg state vectors $\Psi_n = e^{\Omega\tau}\psi_n$ and to use the redefinition $\partial \to \partial + \Omega$.

[1] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, *Quantum Field-Theoretical Methods in Statistical Physics*, Fizmatgiz, Moscow, 1961 (Pergamon, New York, 1965); E. M. Lifshits and L. P. Pitaevskiĭ, *Statistical Physics*, Part 2, Fizmatgiz, Moscow, 1978 (Pergamon, Oxford, 1980).

[2] E. Gozzi, M. Reuter, and W. D. Thacker, "Hidden BRS invariance in classical mechanics," Preprint CERN-TH.5271/89; E. Gozzi and M. Reuter, "Algebraic characterization of ergodicity," Preprint CERN-TH.5241/89.

[3] A. A. Migdal, "Stochastic quantization of field theory," Usp. Fiz. Nauk **149**, 3 (1986) [Sov. Phys. Usp. **29**, 389 (1986)].

[4] J. Greensite and M. Halpern, Nucl. Phys. **B242**, 167 (1984).

[5] L. D. Landau and E. M. Lifshitz, *Mechanics*, Nauka, Moscow, 1965 (Addison-Wesley, Reading, Mass, 1969).

[6] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Part 1, Pergamon, Oxford, 1980.

[7] E. M. Lifshits and L. P. Pitaevskiĭ, *Physical Kinetics*, Nauka, Moscow, 1979 (Pergamon, Oxford, 1981).

[8] C. Becchi, A. Rouet, and R. Stora, Ann. Phys. (NY) **98**, 287 (1976); I. V. Tyutin, Preprint No. 39, P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow, 1975.

[9] L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory*, Nauka, Moscow, 1988 (a previous edition of this book was published in English translation by Addison-Wesley, Reading, Mass., 1981).

Translated by D. Parsons