Amplification and absorption of drift waves at a resonant magnetic surface

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(Submitted 9 July 1991)
Zh. Eksp. Teor. Fiz. **100**, 1797–1810 (December 1991)

Propagation of collision drift waves transverse to the plane plasma layer in a sheared magnetic field is discussed. It is shown that, depending on the plasma parameters, amplification or absorption of waves occurs in the vicinity of the magnetic surface on which the longitudinal component of the wave vector vanishes. The process is associated with the formation of magnetic islands on this surface caused by a wave passing through it and it is accompanied by changes in the profiles of the density, temperature and magnetic field shear near this surface. The wave energies are changed owing to the variation of the thermal energy of the plasma. For instance, wave amplification is caused by plasma expansion along the direction of the gradient.

One of the most significant factors in understanding the dynamics of drift turbulence is the concept of the "source" and the "sink" of the energy of the drift oscillations.

The energy exchange between the oscillations and the plasma for drift turbulence is typically assumed to be a bulk process, that is, conditions prevail in a certain volume such that at each point of it the wave starts to grow (or to diminish) at a certain rate. Processes of a different type, however, can occur in inhomogeneous media in which a wave propagates in the direction of inhomogeneity and loses or absorbs energy in the vicinity of a certain point (or a surface in the three-dimensional case). This local energy exchange between a wave and the medium can be illustrated by the following effects. They include the absorption of oscillations near the resonance points in plane-parallel flows of an ideal fluid, which is stable if the velocity profile has no inflection points (the Rayleigh theorem),[1,2] the absorption and amplification of electromagnetic waves near points where the refraction index tends to infinity,[3] and similar phenomena occurring when waves of other types propagate in plasma.[2]

The present study demonstrates that a similar process of local energy exchange between the wave and the medium occurs also for the drift waves propagating in a plasma in a sheared magnetic field. We can give the following qualitative description of this process. A long-wavelength electrostatic drift wave ($k_\perp \rho_s \ll 1$, $\rho_s = (T/M_i)^{1/2}/\omega_{Bi}$) propagates in a plasma slab where $T_0 = T_0(x)$, $n_0 = n_0(x)$, and $B = (0,\ B_y(x),\ B_z)$. The shear gives rise to a variation of the angle between the external magnetic field and the wave vector along the axis x, that is, the longitudinal component of the wave vector, $k_\parallel$, is a function of x. A perturbation of the magnetic field cannot be ignored for small $k_\parallel$ and therefore the propagation of waves in the region where $k_\parallel$ is small must be described with the equations for the drift Alfven waves. At the resonant surface $x = x_r$ on which $k_\parallel$ vanishes the solutions of these equations exhibit a singularity in which the current along the magnetic field grows without bound. This means that additional effects must be taken into consideration near this surface which limit this singularity. If the wave frequency is lower than the electron collision frequency, $\omega < \nu_e$, then we must take into account the fact that the electric and heat conductivities along the magnetic field are finite. For $\omega > \nu_e$ a more significant contribution is made by the kinetic effects, and the kinetic description of the longitudinal electron motion must be employed in the range $\omega/k_\parallel > V_{Te}$. When the dissipation processes are taken into consideration we obtain energy exchange between the wave and the plasma, that is, amplification or absorption of the wave.

Note that the above process has much in common with the development of the drift tearing mode[4–7] and is described by the same equations. By analogy with spontaneous and stimulated light emission we may refer to it as the drift tearing mode stimulated by the drift wave. The ion sound effects giving rise to "shear" attenuation[8] are not taken into consideration.

An analysis of the local processes of amplification and absorption typically encounters some mathematical difficulties since it involves solving differential equations with variable coefficients. Since the dissipation processes are weak we can employ the technique of matched asymptotic expansions,[9] that is, we can ignore dissipation in the region outside a thin dissipation layer and then we can match the resulting solution with the solution describing the dissipation region. This technique simplifies the problem considerably but does not solve it entirely in the case under consideration. Therefore, additional assumptions must be made to solve the problem.

Most importantly, two restrictions are applied to the wave frequency. On the one hand, only the collisional case is treated, with $\omega < \nu_e$. On the other hand, the width of the "Alfven" layer in which $\omega/k_\parallel > c_A$ holds is assumed to be larger than the width of the dissipation layer. Under these conditions it is simpler to find the solution in the vicinity of the resonant surface. The regime under which these conditions are satisfied for the drift tearing mode is known as the semicollisional regime.[4]

The ion temperature is assumed to be zero. This assumption is explained by the fact that when a wave propagates toward the resonant surface in a plasma with parameters typical of thermonuclear fusion devices the transverse component of the wave vector varies from $k_\perp \rho_s \ll 1$ to $k_\perp \rho_s \gg 1$. In the case $T_e \approx T_i$ the effects caused by the Larmor motion of ions can be correctly taken into account only if the differential equations are replaced with a fairly complicated system of integrodifferential equations.[10]

Finally, another condition applied to the wave vector implies that at large distances from the resonant surface k_x must be much greater than k_y.

Note that the above assumptions are determined not so

 0038-5646/91/120994-08$03.00

much by the physical considerations as by the mathematical difficulties encountered in the solution of a more general problem. It may be expected that the qualitative results are not changed when we eliminate the last two assumptions. We shall discuss this question in more detail in the Conclusion.

The structure of this paper is as follows. The first section presents the derivation of the equations describing the process. In the second section we employ the technique of matched asymptotic solutions to derive a solution describing the propagation of the drift wave across the resonant surface. In the third section the equations for the quantities of second order in amplitude averaged over the wave period are analyzed. The analysis indicates the sources (sinks) of the energy determining amplification (absorption) of the waves. In the Conclusion the results and their applicability ranges are discussed.

1. MAIN EQUATIONS

Consider a long-wavelength electrostatic drift wave propagating in a plasma slab where all unperturbed parameters depend only on the coordinate x. Take the frame of reference to be that in which the direction of the x axis is opposite to the direction of the density gradient, while the z axis is perpendicular to the wave vector. The unperturbed magnetic field has the form $\mathbf{B} = \mathbf{B}_0(\mathbf{e}_z + \mathbf{e}_y x/L_s)$. We consider wave propagation in the vicinity of the resonance surface $x = 0$ on which the component $k_\parallel = k_y x/L_s$ of the wave vector along the magnetic field vanishes. For $|x/L_s| > v^*/c_s$, where $v^* = -(dn/dx)T/nM_i\omega_{Bi}$ and $c_s = (T/M_i)^{1/2}$, the motion of ions along the magnetic field makes a significant contribution and the drift wave is converted into an ion-sound wave. Below we shall ignore the longitudinal motion of ions, that is, we consider only the range of $|x/L_s| < v^*/c_s$. Since typically we have $v^*/c_s \ll 1$ we ignore the effects of the second and higher orders in x/L_s.

As noted in the introduction in this paper, we have considered only the case $\omega < \nu_e$ and therefore we can describe the electron motion with the hydrodynamic equations:[11]

$$\frac{dn}{dt} + n\nabla_\parallel v_\parallel = 0, \tag{1.1}$$

$$0 = -enE_\parallel - \nabla_\parallel p + R_\parallel, \tag{1.2}$$

$$\frac{3}{2} n \frac{dT}{dt} + nT\nabla_\parallel v_\parallel + \nabla_\parallel q_\parallel = Q_\parallel, \tag{1.3}$$

Here $v_\parallel$ and $E_\parallel$ are the components of the electron velocity and the electric field along the magnetic field,

$$R_\parallel = -0{,}51 mn\nu_e v_\parallel - 0{,}71 n\nabla_\parallel T,$$

$$q_\parallel = 0{,}71 nTv_\parallel - 3{,}16 \frac{nT}{m\nu_e}\nabla_\parallel T,$$

$$Q_\parallel = -R_\parallel v_\parallel,$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_\parallel \nabla_\parallel + \mathbf{v}_E \nabla_\perp,$$

$$\nabla_\parallel = \frac{\mathbf{B}}{B}\nabla, \qquad \mathbf{v}_E = \frac{c}{B}[\mathbf{E}, \mathbf{e}_z].$$

We have omitted the inertial term in (1.2) because it is small (of order ω/ν_e) in comparison with the first term in $R_\parallel$. As is typically done for the drift Alfven waves, in a plasma with a small parameter β (here $\beta = 8\pi nT/B^2$) we may ignore the longitudinal component of the perturbed magnetic field and describe the transverse components in terms of the longitudinal component of the vector potential:

$$\tilde{\mathbf{B}} = [\nabla A_\parallel, \mathbf{e}_\parallel] \approx [\nabla A_\parallel, \mathbf{e}_z].$$

We express $\tilde{v}_\parallel$ and $E_\parallel$ in terms of $A_\parallel$ and the potential φ of the electric field:

$$\tilde{v}_\parallel = \frac{c}{4\pi en}\Delta A_\parallel,$$

$$E_\parallel = -\nabla_\parallel \varphi - c^{-1}\partial A_\parallel/\partial t.$$

We introduce the dimensionless quantities

$$\mathbf{x} = \mathbf{x}/\rho_s,\ t' = t\omega_{Bi},\ n' = n/n_0(0),\ T' = T/T_0(0),$$

$$\Phi = e\varphi/T_0(0),\ A_\parallel' = A_\parallel e c_A/cT_0(0),$$

where

$$\rho_s = c_s/\omega_{Bi},\ \omega_{Bi} = eB_0/cM,\ c_A = B_0/(4\pi M n_0(0))^{1/2}.$$

Then we omit the prime sign from the dimensionless parameters and rewrite (1.1)–(1.3) in the form

$$\frac{dn}{dt} + n\frac{d}{dz}\left(\frac{\Delta A}{n}\right) = 0, \tag{1.4}$$

$$\frac{\partial A}{\partial t} + \frac{d\Phi}{dz} - \frac{T}{n}\frac{dn}{dz} - \alpha_1\frac{dT}{dz} - \eta\Delta A = 0, \tag{1.5}$$

$$\frac{3}{2}n\frac{dT}{dt} + nT\frac{d}{dz}\left(\frac{\Delta A}{n}\right) + \frac{d}{dz}\left(\alpha_3 T\Delta A - \frac{3}{2}\alpha_2\frac{T}{\eta}\frac{dT}{dz}\right) = \eta(\Delta A)^2 + \alpha_3\Delta A\frac{dT}{dz}, \tag{1.6}$$

where

$$A = A_0 + A_\parallel,\ A_0 = -(c_A/\omega_{Bi}L_s)x^2/2,$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\Delta A}{n}\frac{d}{dz} + \{\Phi, \ldots\},$$

$$\frac{d}{dz} = (c_A/c_s)\nabla_\parallel = -\{A, \ldots\},$$

$$\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}, \qquad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

$$\eta = 0{,}51(c_A^2/c_s^2)(m/M)(\nu_e/\omega_{Bi}n),$$

$$\alpha_1 = 1{,}71,\ \alpha_2 = 1{,}07,\ \alpha_3 = 0{,}71.$$

The operator d/dz is the gradient in the direction of the magnetic field, expressed in units of ω_{Bi}/c_A. Note that in this frame of reference we have $\partial/\partial z = 0$.

The nonlinear equations (1.4)–(1.6) will be employed in the third section to derive the equations of second order in the amplitude. Another equation is derived immediately in the linearized form. This equation is the condition of charge conservation, $\operatorname{div} j = 0$. The ion temperature is always assumed to be zero and we have

$$\operatorname{div}\mathbf{j}_\perp = \operatorname{div}(en\mathbf{v}_I), \text{ where } \mathbf{v}_I = -\frac{c}{B\omega_{Bi}}\frac{\partial}{\partial t}\nabla_\perp\varphi$$

is the additional inertial term in the drift velocity of ions. As a result we obtain the following dimensionless expression:

$$\frac{\partial}{\partial t}\Delta\Phi - \{A_0, \Delta A\} = 0. \tag{1.7}$$

We linearize of (1.4)–(1.6) and write the perturbed parameters as $\Phi = \Phi(x)\exp(-i\omega t + ik_y y)$. We assume that in the dissipation range the unperturbed current velocity $\Delta A_0/n_0$ is much smaller than the phase velocity $\omega/k_\parallel$ of the wave along the magnetic field. When we eliminate $n(x)$ and $T(x)$ from the equations we obtain

$$\Delta\Phi = \frac{k_\parallel}{\omega}\Delta A, \tag{1.8}$$

$$\left(k_\parallel^2/\omega - i\eta + \frac{\alpha_4 k_\parallel^2/\omega}{1+i\alpha_2 k_\parallel^2/\omega\eta}\right)\Delta A + \left(\omega - \omega_n^* - \frac{\alpha_1 \omega_T^*}{1+i\alpha_2 k_\parallel^2/\omega\eta}\right) \times\left(A - \frac{k_\parallel}{\omega}\Phi\right) = 0, \tag{1.9}$$

where

$$k_\parallel = k_\parallel' x,\quad k_\parallel' = \frac{c_A}{\omega_{Bi}L_s}k_y,\quad \omega_n^* = -k_y\frac{\partial n_0}{\partial x},$$

$$\omega_T^* = -k_y\frac{\partial T_0}{\partial x},\quad \alpha_4 = 1{,}95,\quad \Delta = \partial^2/\partial x^2 - k_y^2.$$

Let us analyze (1.9) in more detail. If we ignore the dissipation effects and let η tend to zero, (1.9) will have the following form, which is typical for drift Alfven waves:

$$\frac{k_\parallel^2}{\omega}\Delta A + (\omega - \omega_n^*)\left(A - \frac{k_\parallel}{\omega}\Phi\right) = 0. \tag{1.10}$$

For the drift waves with $|k_\parallel/\omega| \gg 1$ (written in terms of dimensional units this condition is equivalent to $|k_\parallel/\omega| \gg c_A^{-1}$) we may ignore A, which is much smaller than $\Phi k_\parallel/\omega$. Then (1.8) and (1.10) yield the following equation describing electrostatic drift waves:

$$\Delta\Phi + (\omega_n^*/\omega - 1)\Phi = 0. \tag{1.11}$$

The coefficients in (1.11) no longer depend on $k_\parallel$ and for $\omega_n^* = \text{const}$ a solution of this equation is a set of plane waves.

The dissipation terms in (1.9) depend on the finite electrical conductivity of the plasma (the term $i\eta\Delta A$) and the finite heat conductivity in the direction of the magnetic field. It can be easily seen that both effects become significant for $|x| < (\omega\eta)^{1/2}/k_\parallel'$. We assume below that the width of the dissipation layer, $\Delta_D = (\omega\eta)^{1/2}/k_\parallel'$, is small in comparison with the width of the "Alfven region," $\Delta_A = \omega/k_\parallel'$, that is,

$$\omega \gg \eta. \tag{1.12}$$

This condition is compatible with the condition $\omega \ll \nu_e$ (which has the form $\omega \ll \nu_e/\omega_{Bi}$ in dimensionless variables) only if the plasma is sufficiently hot, that is, $\beta \gg m/M$.

Note that (1.8) and (1.9) differ from the equations of Ref. 7, which describe the semicollisional regime of the drift tearing mode, only in small terms related to the time-dependent thermal force, which can be ignored in the case under consideration.

2. SOLUTION OF THE LINEARIZED EQUATIONS DESCRIBING SCATTERING OF THE DRIFT WAVE

External solution. We ignore dissipation and find the solution outside the dissipation layer. Eliminating A from (1.8) and (1.10) we obtain

$$\frac{\partial}{\partial x}\hat{D}\frac{\partial\Phi}{\partial x} - k_y^2\hat{D}\Phi = 0, \tag{2.1}$$

$$\hat{D} = x^2(\partial^2/\partial x^2 + k_x^2) - (\omega/k_\parallel')^2(\omega_n^*/\omega - 1),$$
$$k_x^2 = \omega_n^*/\omega - 1 - k_y^2.$$

If we assume $\partial/\partial x \gg k_y$ and ignore the second term in (2.1) we can readily derive a solution of this equation:

$$\Phi = \int_0^\xi \{(\pi\xi/2)^{1/2}[C_1 H^{(1)}_{\nu+1/2}(\xi) + C_2 H^{(2)}_{\nu+1/2}(\xi)] + C_3 f(\xi)\}d\xi + C_4, \tag{2.2}$$

$$A = \frac{k_\parallel}{\omega}\left\{\frac{\partial}{\partial\xi}[(\pi\xi/2)^{1/2}[C_1 H^{(1)}_{\nu+1/2}(\xi) + C_2 H^{(2)}_{\nu+1/2}(\xi)] + C_3 f(\xi)] + \Phi\right\}, \tag{2.3}$$

where

$$\nu = -\frac{1}{2} + \left[\frac{1}{4} + \left(\frac{\omega}{k_\parallel'}\right)^2(\omega_n^*/\omega - 1)\right]^{1/2},$$

$$f(\xi) = \xi^{1/2}S_{-3/2,\nu+1/2}(\xi) = \frac{\pi\xi^{1/2}}{2}\left[J_{\nu+1/2}(\xi)\int_\xi^\infty Y_{\nu+1/2}(\xi)\xi^{-3/2}d\xi - Y_{\nu+1/2}(\xi)\int_\xi^\infty J_{\nu+1/2}(\xi)\xi^{-3/2}d\xi\right], \tag{2.4}$$

Here $J_{\nu+1/2}(\xi)$, $Y_{\nu+1/2}(\xi)$ and $H^{(1,2)}_{\nu+1/2}(\xi)$ are the Bessel functions of the first, second, and third kinds, $S_{-3/2,\nu+1/2}(\xi)$ is the Lommel function,[12] and $\xi = k_x|x|$. For the sake of definiteness the point $\xi = 0$ is taken as the lower limit of integration in (2.2), which is permissible for $\nu < 1$.

For large $|x|$ the asymptotic solution of (2.2) has the form

$$\begin{aligned}\Phi \sim &-C_1\exp[i(k_x|x| - \pi\nu/2)] \\ &-C_2\exp[-i(k_x|x| - \pi\nu/2)] - C_3/(k_x|x|) \\ &+\pi^{-1/2}\Gamma(1/2 - \nu/2)\Gamma(\nu/2+1)[C_1\exp(-i\pi\nu/2) + C_2\exp(i\pi\nu/2) \\ &-2^{-3}\pi^{-1/2}C_3\sin(\pi\nu)\Gamma(\nu/2)\Gamma(-\nu/2 - 1/2)] + C_4,\end{aligned} \tag{2.5}$$

where $\Gamma(z)$ is the gamma function. For small $|x|$ it is more convenient to use the following asymptotic expression for the function A:

$$\begin{aligned}A \sim \frac{x}{|x|}\{&[C_1\exp(-i\pi\nu) + C_2\exp(i\pi\nu)]\delta_1|x|^{\nu+1} \\ &+ i(C_1 - C_2)\delta_2|x|^{-\nu} \\ &+2^{-2}\pi^{-1/2}C_3\Gamma(-\nu/2 - 1/2)\Gamma(\nu/2) \\ &[-\sin(\pi\nu/2)\delta_1|x|^{\nu+1} + \cos(\pi\nu/2)\delta_2|x|^{-\nu}]\} \\ &+C_3 x|x|k_\parallel' k_x/[\omega(1-\nu)(2+\nu)] + C_4 k_\parallel' x/\omega, \\ \delta_1 = &\frac{k_\parallel'}{\omega}\frac{\pi^{1/2}(\nu+1)}{2\cos(\pi\nu)\Gamma(\nu+3/2)}\left(\frac{k_x}{2}\right)^\nu, \\ \delta_2 = &\frac{k_\parallel'}{\omega}\frac{\pi^{1/2}\nu}{2\cos(\pi\nu)\Gamma(1/2-\nu)}\left(\frac{k_x}{2}\right)^{-\nu-1}.\end{aligned} \tag{2.6}$$

The last two terms in (2.6) correspond to regular solutions near the resonant surface, where the dissipation effects can be ignored. Hence, the constants C_3^+ and C_4^+ appearing in the solution of (2.2) and (2.3) for $x > 0$ and the constants C_3^- and C_4^- for $x < 0$ are related by the following expressions:

$C_3^+=-C_3^-, \; C_4^+=C_4^-.$

Let us now consider (2.5). For $\Phi\sim$const we have $A\sim x$ and therefore we must take the constant in (2.5) to be equal to zero for $x>0$ and $x<0$ in order to obtain a solution bounded at infinity. Using these conditions we can express C_3 in terms of the drift-wave amplitudes:

$$C_3^+=-C_3^-=\frac{4\pi^{1/2}}{\sin(\pi\nu)\Gamma(\nu/2)\Gamma(-\nu/2-1/2)}$$
$$[(C_1^+-C_1^-)\exp(-i\pi\nu/2)+(C_2^+-C_2^-)\exp(i\pi\nu/2)]. \qquad (2.7)$$

We see that the asymptotic approximation for the solution under consideration at large distances from the resonant surface consists of electrostatic drift waves with the amplitudes C_1 and C_2 and the magnetohydrodynamic asymptotic solution described by $\Phi\sim C_3/(k_x|x|)$ and $A\sim(k_\parallel/\omega)\Phi$. When we take into consideration the second term in (2.1) the latter becomes exponential, rather than algebraic: $A\sim(k_\parallel/\omega)\Phi\sim\exp(-k_y|x|)$. The time dependence of the perturbed quantities has been taken in the form of $\exp(-i\omega t)$ and therefore the phase velocity of the C_1 wave is directed away from the resonant surface and that of the C_2 wave is directed towards the resonant surface. But since the x component of the group velocity for the potential drift waves has a sign opposite the sign of the x component of the phase velocity,

$$\frac{\partial\omega}{\partial k_x}=-\frac{2k_x^2}{1+k^2}\frac{\omega}{k_x},$$

in the first case the wave packet propagates towards the resonant surface and in the second case away from it. The C_1 wave will accordingly be referred to as the wave incident on the resonant surface and the C_2 wave as the wave propagating away from it.

Internal solution. Let us find a solution in the vicinity of the resonance surface to which the asymptotic solution (2.6) with non-integer exponents can be matched. In the range $|x|\ll\Delta_A=\omega/k'_\parallel$ we have for this solution $\Phi\sim Ak_\parallel/\omega\ll A$ and $k_y\ll\partial/\partial x$. Then we obtain from (1.9)

$$\left(k_\parallel^2/\omega-i\eta+\frac{\alpha_4k_\parallel^2/\omega}{1+i\alpha_2k_\parallel^2/\omega\eta}\right)\frac{\partial^2A}{\partial x^2}$$
$$+\left(\omega-\omega_n^*-\frac{\alpha_1\omega_T^*}{1+i\alpha_2k_\parallel^2/\omega\eta}\right)A=0. \qquad (2.8)$$

Let us introduce a new variable $\zeta=x/\Delta_D$ where $\Delta_D=(\omega\eta)^{1/2}/k'_\parallel$. Then we can rewrite (2.8) as

$$(\zeta^2-ia_1)(\zeta^2-ia_2)\frac{\partial^2A}{\partial\zeta^2}-\nu(\nu+1)(\zeta^2-ig)A=0, \qquad (2.9)$$

where

$$a_{1,2}=\{1+\alpha_2+\alpha_4\pm[(1+\alpha_2+\alpha_4)^2-4\alpha_2]^{1/2}\}/2\alpha_2,$$
$$a_1=0{,}27,\quad a_2=3{,}49,\quad g=\frac{1}{\alpha_2}\left(1+\alpha_1\frac{\omega_T^*}{\omega_n^*-\omega}\right),$$

while ν is given by (2.4). This equation has the solution

$$A=D_1(\zeta^2-ia_1)^{-\nu/2}\psi(a,q,\nu/2,\nu/2+1,1/2,\nu+3/2,\zeta^2/(\zeta^2-ia_1))$$
$$+D_2\zeta(\zeta^2-ia_1)^{-(\nu+1)/2}\psi(a,q_1,\nu/2+1/2,$$
$$\nu/2+3/2,3/2,\nu+3/2,\zeta^2/(\zeta^2-ia_1)), \qquad (2.10)$$

where

$$a=a_2/(a_2-a_1)=1{,}08,\quad q=\nu[a_2+(\nu+1)g]/4(a_2-a_1),$$
$$q_1=q+(\nu/2+3/4)a,$$

and the function $\psi(a,q,\alpha,\beta,\gamma,\delta,z)$ is a solution of the Heun equation,[13]

$$z(z-1)(z-a)\frac{\partial^2\psi}{\partial z^2}+[(\alpha+\beta+1)z^2$$
$$-(\alpha+\beta+1+a(\gamma+\delta)-\delta)z+a\gamma]\frac{\partial\psi}{\partial z}+(\alpha\beta z-q)\psi=0,$$

which for $|a|>1$ can be written as a power series converging for $|z|<1$:

$$\psi(a,q,\alpha,\beta,\gamma,\delta,z)=\sum_{n=0}^{\infty}c_nz^n,$$
$$c_0=1,\quad c_1=q/a\gamma,$$
$$a(n+1)(n+\gamma)c_{n+1}=\{(a+1)n^2$$
$$+[\alpha+\beta-\delta+a(\gamma+\delta-1)]n+q\}c_n-(n-1+\alpha)(n-1+\beta)c_{n-1}. \qquad (2.11)$$

The first term in (2.10) is an even function of ζ and the second term is an odd function. For any real values of the variable ζ the absolute value of the argument of the functions in (2.10) is smaller than unity, that is, the entire real axis is in the circle of convergence for the series (2.11) after the transformation $z=\zeta^2/(\zeta^2-ia_1)$ has been performed. To match the solution (2.10) to the external solution we must consider the asymptotic representation of (2.10) for $\zeta\to\pm\infty$. It can be found with the use of the invariance properties of the Heun equation.[13] Let us write down two linearly independent solutions of the Heun equation in the vicinity of the point $z=1$. In the intersection of the circles $|z|<1$ and $|z-1|<a-1$ the function (2.11) can be written as a linear combination of these solutions:

$$\psi(a,q,\alpha,\beta,\gamma,\delta,z)=E[(1-z)^{1-\delta}\psi(a_3,q_2,\alpha-\delta+1,$$
$$\beta-\delta+1,2-\delta,\alpha+\beta-\gamma-\delta+1,(1-z)a_3)+B\psi(a_3,q_3,\alpha,\beta,\delta,$$
$$\alpha+\beta-\gamma-\delta+1,(1-z)a_3)), \qquad (2.12)$$

where

$$a_3=1/(1-a),$$
$$q_2=[-q+(\alpha-\delta+1)(\beta-\delta+1)+\gamma(\delta-1)a]a_3,$$
$$q_3=(-q+\alpha\beta)a_3,$$

and E and B are constants which may depend on all the parameters of the function ψ. In order to determine these constants we must find values of the functions (2.12) at two different points. Unfortunately this can not be done analytically and thus we had to perform numerical summation of the series (2.11).

Equation (2.12) suggests that for $z\to1$ we have

$$\psi(a,q,\alpha,\beta,\gamma,\delta,z)\sim E[(1-z)^{1-\delta}+B].$$

Denote by B_1 and E_1 the values of the constants B and E for the sets of parameters in the first term of (2.10) and by B_2 and E_2 these constants for the set of parameters in the second term. Then for $|\zeta|=|x|/\Delta_D\gg1$ we obtain

$$A\sim\tilde{D}_1(|x|^{1+\nu}+\beta_1|x|^{-\nu})+\tilde{D}_2\frac{x}{|x|}(|x|^{1+\nu}+\beta_2|x|^{-\nu}),\tag{2.13}$$

where

$$\tilde{D}_i=D_iE_i\exp\,[i\pi(\nu/2+{}^1/_4)]\,a_i^{-\nu-{}^1/_2}\Delta_D^{-\nu-1},$$
$$\beta_i=B_i\exp\,[-i\pi(\nu/2+{}^1/_4)]\,(a_i^{{}^1/_2}\Delta_D)^{2\nu+1}.$$

A significant disadvantage of the solution (2.10) is the need to perform numerical summation of the series in order to find the asymptotic solution for large $|\zeta|$. It is useful to derive a simpler and clearer solution by analyzing a special case of the equation (2.9). Let us assume that the following condition is satisfied for the wave frequency:

$$g=a_2.\tag{2.14}$$

Then we obtain from (2.9)

$$(\zeta^2-ia_1)\frac{\partial^2A}{\partial\zeta^2}-\nu(\nu+1)A=0.\tag{2.15}$$

Note that a similar equation can be derived from (2.8), where the terms dependent on the finite heat conductivity in the direction of the magnetic field must be eliminated. It is therefore useful to analyze the solutions of the equation (2.15) in order to understand the contributions made by each of the dissipation effects to the process under consideration.

The solution of (2.15) may be expressed in terms of the associated Legendre functions $P_\nu^1(\chi)$:

$$A=(\chi^2-1)^{{}^1/_2}[G_1P_\nu^1(\chi)+G_2P_\nu^1(-\chi)],$$
$$\chi=\zeta a_1^{-{}^1/_2}\exp(-i\pi/4).\tag{2.16}$$

For large ζ we obtain the following asymptotic representation for this solution:

$$A\sim\{G_1\exp\,[\mp i\pi(\nu+1)/2]+G_2\exp\,[\pm i\pi(\nu+1)/2]\}\gamma_1|x|^{1+\nu}$$
$$+\{G_1\exp(\pm i\pi\nu/2)+G_2\exp(\mp i\pi\nu/2)\}\gamma_2|x|^{-\nu},\tag{2.17}$$

where

$$\gamma_1=[\Gamma(\nu+{}^1/_2)/2\Gamma(\nu)]\pi^{-{}^1/_2}[2\Delta_D^{-1}a_1^{-{}^1/_2}\exp(i\pi/4)]^{1+\nu},$$
$$\gamma_2=[\Gamma(-\nu-{}^1/_2)/2\Gamma(-\nu-1)]\pi^{-{}^1/_2}[2\Delta_D^{-1}a_1^{-{}^1/_2}\exp(i\pi/4)]^{-\nu},$$

and the upper sign in the exponential factors corresponds to the positive x while the lower sign corresponds to the negative x. If we represent (2.17) as a sum of even and odd functions we can readily show that in the special case described by (2.14) we have for the constants in (2.13):

$$B_1=-\operatorname{cotan}(\pi\nu/2)\frac{\Gamma(-\nu-{}^1/_2)\Gamma(\nu)}{\Gamma(-\nu-1)\Gamma(\nu+{}^1/_2)}2^{-1-2\nu},$$
$$B_2=\tan^2(\pi\nu/2)B_1.\tag{2.18}$$

Matching of the external and internal solutions. Let us compare the asymptotic behavior of the external and internal solutions in the range $\Delta_D\ll|x|\ll\Delta_A$ in order to find the relationship between their arbitrary constants. Assume that we have $C_1^-=0$, that is, the wave is incident on the plane $x=0$ from the direction of positive x. When we equate the coefficients with the same powers in (2.6) and (2.13) for $x>0$ and $x<0$, take (2.7) into consideration and eliminate the constants $\tilde{D}_1$ and $\tilde{D}_2$, we can determine the relative energies of the reflected and transmitted waves:

$$|C_2^+/C_1^+|^2=((\vartheta_2-\vartheta_1)\sin2\alpha)^2/Z,\tag{2.19}$$
$$|C_2^-/C_1^+|^2=\{[(1+\vartheta_1\vartheta_2)\cos\alpha-(\vartheta_1+\vartheta_2)\cos2\alpha]^2$$
$$+(1-\vartheta_1\vartheta_2)^2\sin^2\alpha\}/Z,\tag{2.20}$$

Here we have

$$Z=(1+\vartheta_1^2-2\vartheta_1\cos\alpha)(1+\vartheta_2^2-2\vartheta_2\cos\alpha),$$
$$\alpha=\pi(\nu/2+{}^1/_4),\quad \vartheta_1=\vartheta_2\,\mathrm{tg}^2(\pi\nu/2)B_1/B_2,$$
$$\vartheta_2=B_2(a_1^{{}^1/_2}\Delta_Dk_z/2)^{2\nu+1}(1+1/\nu)\Gamma({}^1/_2-\nu)/\Gamma(\nu+{}^3/_2).$$

Figures 1–3 show the relative energy variation $\Delta E=|C_2^+/C_1^+|^2+|C_2^-/C_1^+|^2-1$ as a function of the squared wave vector $k^2=\omega_n^*/\omega-1$ for different values of the parameters ω_T^*/ω_n^*, η/ω and $(\omega/k_\parallel')^2\approx2\beta(L_s/L_n)^2$. If the magnetic field shear is not too small ($\nu<1/2$) wave amplification takes place for small temperature gradients. As the parameter ω_T^*/ω_n^* grows (Fig. 1) wave amplification is replaced with wave absorption. A change in the parameter η/ω (see Fig. 2) produces only a slight effect on the growth of ΔE, while an increase in η/ω increases the absolute magnitude of the absorption or amplification.

Figure 4 shows the relative energies of the reflected and transmitted waves for two values of the parameter ω_T^*/ω_n^*. It can be seen from the plots that for $\Delta E>0$ almost the entire energy is concentrated in the transmitted wave. When amplification gives way to absorption the amplitude of the reflected wave grows and can become greater than the amplitude of the transmitted wave. In the special case described by (2.14) we have $\vartheta_1=\vartheta_2$ and the amplitude of the reflected wave is identically equal to zero. These results suggest that partial reflection of the wave is caused by a perturbation of the longitudinal temperature gradient owing due to finite longitudinal heat conductivity.

3. CHANGE IN THE PLASMA PARAMETERS IN THE DISSIPATION LAYER

Let us determine the changes in the magnetic field and the density and pressure of the plasma caused by the wave. To do this we must analyze Eqs. (1.4)–(1.6) to second order in the amplitude. When we average these equations over the wave period we obtain the required expressions describing the rate of growth of the additional terms for the initial plasma parameters. For simplicity we assume $\nu<1/2$. Under these circumstances the energy exchange between the plas-

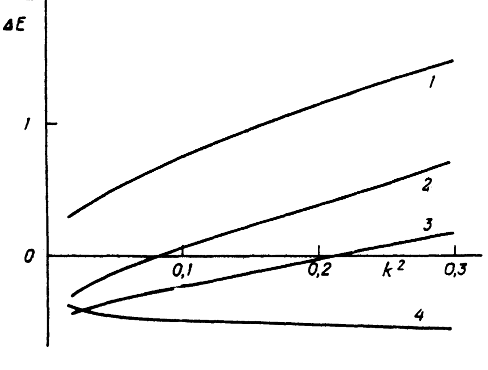

FIG. 1. Relative energy variation ΔE as a function of $k^2=\omega_n^*/\omega-1$ for $(\omega/k_\parallel')^2=1$, $\eta/\omega=0.1$, and $\omega_T^*/\omega_n^*=0$ (curve *1*); 0.2 (curve *2*); 0.4 (curve *3*); 1 (curve *4*).

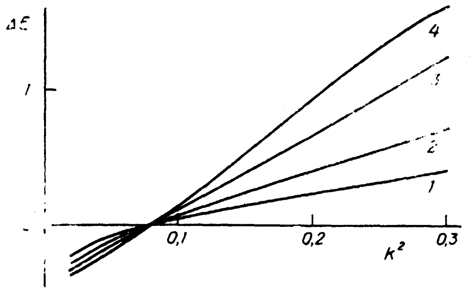

FIG. 2. Relative energy variation ΔE as a function of $k^2 = \omega_n^*/\omega - 1$ for $(\omega/k_\parallel^*)^2 = 1$, $\omega_T^*/\omega_n^* = 0.2$ and $\eta/\omega = 0.05$ (curve *1*); 0.1 (curve *2*); 0.2 (curve *3*); 0.3 (curve *4*).

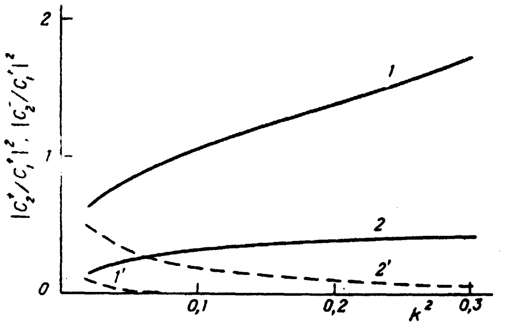

FIG. 4. Relative energies of the transmitted (solid lines 1 and 2) and the reflected (dashed lines 1' and 2') waves for $(\omega/k_\parallel^*)^2 = 1$, $\eta/\omega = 0.1$, and $\omega_T^*/\omega_n^* = 0.2$ (curves *1, 1'*); 1 (curve *2, 2'*).

ma and the wave is concentrated near the resonant surface and therefore we may limit the analysis to the region of $|x| \ll \Delta_A$. Since in this region we have $\Phi \ll A$, $k_y \ll \partial/\partial x$, and $\Delta A_0/n_0 \ll \omega/k_\parallel$ we obtain after averaging (here and below angle brackets denote averages)

$$\frac{\partial \bar{n}_2}{\partial t} = \langle \{A_1, \Delta A_1\} \rangle, \tag{3.1}$$

$$\frac{\partial \bar{A}_2}{\partial t} - \eta \overline{\Delta A_2} = \left\langle \frac{T_1}{n_0}\left[\frac{\partial A_1}{\partial y}\frac{\partial n_0}{\partial x} - \frac{\partial A_0}{\partial x}\frac{\partial n_1}{\partial y}\right]\right\rangle - \left\langle \frac{n_1}{n_0}\frac{\partial A_1}{\partial y}\frac{\partial T_0}{\partial x}\right\rangle - \left\langle\left\{A_1, \frac{T_0}{n_0}n_1 + \alpha_1 T_1\right\}\right\rangle, \tag{3.2}$$

$$\frac{3}{2}\frac{\partial}{\partial t}(\bar{n}_2 T_0 + n_0 \bar{T}_2) - 2\eta \Delta A_0 \overline{\Delta A_2} = \left\langle \frac{\partial A_1}{\partial y}\frac{\Delta A_1}{n_0}\frac{\partial (n_0 T_0)}{\partial x}\right\rangle + \left\langle\left(\frac{T_0}{n_0}n_1 + T_1\right)\frac{\partial A_0}{\partial x}\frac{\partial \Delta A_1}{\partial y}\right\rangle + \eta\langle(\Delta A_1)^2\rangle + \alpha_3\left\langle \Delta A_1\left(\frac{\partial A_1}{\partial y}\frac{\partial T_0}{\partial x} - \frac{\partial A_0}{\partial x}\frac{\partial T_1}{\partial y}\right)\right\rangle + \langle\{A_1, \dots\}\rangle. \tag{3.3}$$

The subscripts 0, 1, and 2 refer to quantities of the zeroth, first and second orders, respectively, in the wave amplitude and the expression $\langle\{A_1,\dots\}\rangle$ in (3.3) denotes all terms of this type. These terms are not significant for further analysis since after averaging they are converted to the form $(\partial/\partial x)(\dots)$ and do not contribute to the integral over x.

A comparison of the appropriate terms in (3.2) and (3.3) indicates that the change in the magnetic energy $\partial W_M/\partial t = -\int(\Delta A_0 \partial A_2/\partial t)dx$ is smaller by a factor $\Delta A_0 k_\parallel/n_0\omega \ll 1$ than the change in the thermal energy. This means that amplification and absorption of the waves are

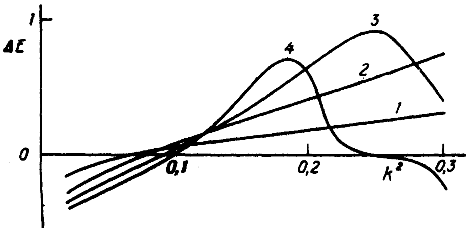

FIG. 3. Relative energy variation ΔE as a function of $k^2 = \omega_n^*/\omega - 1$ for $(\omega/k_\parallel^*)^2 = 0.2$, $\eta/\omega = 0.1$, and $\omega_T^*/\omega_n^* = 0.2$ (curve *1*); 1 (curve *2*); 2 (curve *3*); 3 (curve *4*).

determined by the change in the thermal energy of the plasma in the dissipation layer.

The first two terms on the right-hand side of (3.3) give the work of the pressure forces (to within terms of the form $\{A,\dots\}$ and signs) while the third and fourth terms give the work of the friction force and the thermal force, respectively. Their sum gives the work of the wave field $\langle j_\parallel E_\parallel\rangle = \langle \Delta A_1 \partial A_1/\partial t\rangle$. To analyze their respective contributions we should consider not only their sum but also each of them separately.

Take the solution of the linear problem in the region of $|x| \ll \Delta_A$ in the form

$$A_1 = A(\zeta)\cos[k_y y - \omega t + \varphi(\zeta)], \quad \zeta = x/\Delta_D. \tag{3.4}$$

Then (2.9) yields

$$\Delta A_1 = \Delta_D^{-2}\frac{\partial^2 A_1}{\partial \zeta^2} = \rho(\zeta)A(\zeta)\cos[k_y y - \omega t + \varphi_e(\zeta)],$$

$$\rho(\zeta) = \frac{\omega_n^* - \omega}{\eta}\left[\frac{\zeta^4 + g^2}{(\zeta^4 + a_1^2)(\zeta^4 + a_2^2)}\right]^{1/2}, \tag{3.5}$$

$$\varphi_e(\zeta) - \varphi(\zeta) = \arctan(a_1/\zeta^2) + \arctan(a_2/\zeta^2) - \arctan(g/\zeta^2).$$

We can derive from the linearized equations (1.4) and (1.5)

$$n_1 = (k_\parallel/\omega)\Delta A_1, \tag{3.6}$$

$$T_1 = (1/\alpha_1 k_\parallel)\left((\omega_n^* - \omega + \alpha_1\omega_T^*)A_1 - (k_\parallel^2/\omega)\Delta A_1 - \eta k_y \int \Delta A_1\, dy\right). \tag{3.7}$$

Using (3.4)–(3.7) we obtain

$$\left\langle \frac{\partial A_1}{\partial y}\frac{\Delta A_1}{n_0}\frac{\partial(n_0 T_0)}{\partial x}\right\rangle = -(\omega_T^* + \omega_n^*)\frac{\rho A^2}{2}\sin(\varphi_e - \varphi), \tag{3.8}$$

$$\left\langle\left(\frac{T_0}{n_0}n_1 + T_1\right)\frac{\partial A_0}{\partial y}\frac{\partial \Delta A_1}{\partial x}\right\rangle = [\omega_T^* + (\omega_n^* - \omega)/\alpha_1]\frac{\rho A^2}{2}\cdot \sin(\varphi_e - \varphi) - \frac{\eta}{\alpha_1}\frac{\rho^2 A^2}{2}, \tag{3.9}$$

$$\eta\langle(\Delta A_1)^2\rangle = \eta\frac{\rho^2 A^2}{2}, \tag{3.10}$$

$$\alpha_3\left\langle \Delta A_1\left(\frac{\partial A_1}{\partial y}\frac{\partial T_0}{\partial x} - \frac{\partial A_0}{\partial x}\frac{\partial T_1}{\partial y}\right)\right\rangle = \frac{\alpha_3}{\alpha_1}\left[(\omega_n^* - \omega)\frac{\rho A^2}{2}\sin(\varphi_e - \varphi) - \eta\frac{\rho^2 A^2}{2}\right], \tag{3.11}$$

$$\Delta A_1\frac{\partial A_1}{\partial t}=-\omega\frac{\rho A^2}{2}\sin(\varphi_c-\varphi),$$

where

$$\sin(\varphi_c-\varphi)=\frac{(a_1+a_2-g)\zeta^4+a_1a_2g}{[(\zeta^4+a_1^2)(\zeta^4+a_2^2)(\zeta^4+g^2)]^{1/2}}.$$

Substituting (3.4)–(3.7) into the equation (3.1) which describes the change in the plasma density in the dissipation layer we obtain

$$\frac{\partial n_2}{\partial t}=-\frac{\partial}{\partial x}\left[k_y\frac{\rho A^2}{2}\sin(\varphi_c-\varphi)\right]. \qquad (3.13)$$

When the temperature gradient is small, $|\omega_T^*|\ll\omega_n^*-\omega$, and wave amplification occurs, the expressions (3.9)–(3.11) are of order $(\omega_n^*/\omega-1)\ll 1$ relative to (3.8). The change in the thermal energy of the plasma in this case is then caused by the work done by the unperturbed plasma pressure. As the parameter ω_T^*/ω_n^* grows and wave amplification is replaced with wave absorption the quantities (3.8)–(3.11) have the same order of magnitude.

Let us consider the the phase shift $\varphi_c-\varphi$ between ΔA_1 and A_1 as a function of ζ. For small ω_T^* when $0<g<a_1+a_2$ the phase shift is always positive, it equals $\pi/2$ for $\zeta=0$ and tends to zero as $|\zeta|$ grows. Accordingly, the flux $\Gamma_x=(1/2)k_y\rho A^2\sin(\varphi_c-\varphi)$ everywhere has a direction opposite to that of the density gradient, the work of the unperturbed pressure is positive, and the work of the wave field given by (3.12) is negative. The change in the thermal energy of the plasma,

$$\frac{3}{2}\frac{\partial}{\partial t}\int(n_2T_0+n_0T_2)dx=-\frac{\omega}{2}\int\rho A^2\sin(\varphi_c-\varphi)dx \qquad (3.14)$$

is also negative under these conditions. As the temperature gradient grows g becomes greater than a_1+a_2 and for large $|\zeta|$ the phase shift becomes negative. If in (3.14) the integral over the region in which $\varphi_c-\varphi<0$ holds is greater than the integral over the region in which $\varphi_c-\varphi>0$ holds, then the total change in the thermal energy of the plasma is positive and the wave is amplified rather than absorbed.

We see that for small parameters ω_T^*/ω_n^* wave amplification is determined by the outward expansion of the plasma while the magnetic field structure in the dissipation layer is disrupted. For higher values of the parameter ω_T^*/ω_n^* we find regions at the external boundaries of the dissipation layer where the work of the wave field is positive. As a result the total change in the thermal energy of the plasma may become positive and, accordingly, wave absorption occurs.

CONCLUSION

The paper analyzes the local energy exchange between a plasma and a drift wave in the vicinity of a magnetic surface on which the longitudinal component of the wave vector vanishes. In the absence of dissipation the wave exhibits a singularity on this surface (the current along the magnetic field diverges). Therefore, the presence of a small dissipation produces a significant change in the wave energy. The magnetic field in the vicinity of the resonance magnetic surface through when a wave passes through it exhibits asymmetric magnetic islands [related to the odd function in (2.10)]. The fluxes of heat and particles along the perturbed magnetic field give rise to nonvanishing mean fluxes along the direction of the inhomogeneity. If the magnetic field shear is not too small ($\nu<1/2$) amplification occurs for small temperature gradients owing to plasma expansion in the dissipation layer. As the temperature gradient grows, wave amplification is replaced with wave absorption while the thermal energy of plasma in the dissipation layer increases.

We have made some additional assumptions to simplify the analysis. Let us see how the results are altered if we remove some of the assumptions.

The process of wave scattering for $k_y^2\sim k_x^2$ must not differ significantly from that analyzed above for $k_y^2\ll k_x^2$ since in the Alfven layer $|x|<\Delta_A$ inside which the wave and the plasma exchange energy we observe a decrease in the characteristic scale of variation of the magnetic field along the x axis, so that the terms proportional to k_y^2 become insignificant.

The results also must not change significantly if we take into consideration the finite ion temperature, since the processes inside the Alfven layer depend primarily on the electron dynamics and the ion motion has only a weak effect on them. The effect of the ion Larmor motion on the growth rate of the drift tearing mode has been analyzed elsewhere.[10]

The characteristic transverse dimension of the density perturbations occurring in tokamaks is typically comparable to $L_s v^*/c_s$. Therefore analysis of the processes involving linear drift waves in the vicinity of the resonance surface must take into account the ion sound effects, which have been shown[14–16] to make a significant contribution for $|k_\parallel|\geqslant\omega_n^*/c_s$. In most tokamaks, however, the drift perturbations are highly nonlinear and the velocity of transverse motion in them is comparable to the diamagnetic drift velocity.[17] For such perturbations (which may be referred to as eddies) the processes near the resonance surface are spatially separated from those at $|k_\parallel|\geqslant\omega_n^*/c_s$ and thus it is reasonable to treat them separately.

In conclusion, note that an important feature of the effect analyzed in this paper is that the processes of amplification and absorption of the drift waves in the vicinity of the resonant magnetic surface depend on the characteristic dimensions over which the density, plasma temperature, and magnetic field shear vary and, in turn, affect these parameters. This feature may be of interest in connection with the self-organization of plasma.[18]

I am grateful to V. I. Petviashvili, O. P. Pogutse, and V. V. Yankov for discussions.

[1] C. C. Lin, *The Theory of Hydrodynamic Stability*, Cambridge (1955).
[2] A. V. Timofeev, in *Problems of Plasma Theory*, Vol. 17, B. B. Kadomtsev (Ed.), Consultants Bureau, New York (1991).
[3] V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasma*, Pergamon, Oxford (1970).
[4] J. F. Drake and Y. C. Lee, Phys. Fluids **20**, 1343 (1977).
[5] N. T. Gladd, J. F. Drake, C. L. Chang, and C. S. Liu, Phys. Fluids **23**, 1182 (1980).
[6] M. Rosenberg, R. R. Dominguez, W. Pfeiffer, and R. E. Waltz, Phys. Fluids **23**, 2022 (1980).
[7] J. F. Drake, T. M. Antonsen, A. B. Hassam, and N. T. Gladd, Phys. Fluids **26**, 2509 (1983).
[8] L. D. Pearlstein and H. L. Berk, Phys. Rev. Lett. **23**, 220 (1969).
[9] A. Nayfeh, *Perturbation Methods*, Wiley, New York (1973).
[10] R. Farengo, Y. C. Lee, and P. N. Guzdar, Phys. Fluids **26**, 3515 (1983).
[11] S. I. Braginsky, in *Reviews of Plasma Physics*, Vol. 1, M. A. Leontovich

(Ed.), Consultants Bureau, New York (1963).
[12] A. Erdelyi, *Higher Transcendental Functions,* Vols. 1 and 2, McGraw-Hill, New York (1953).
[13] E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen,* Leipzig (1954).
[14] Y. C. Lee and Liu Chen, Phys. Rev. Lett. **42**, 708 (1979).
[15] Liu Chen, P. N. Guzdar, J. Y. Hsu, P. K. Kaw, C. Oberman, and R. White, Nuclear Fusion **19**, 373 (1979).
[16] C. L. Chang, J. F. Drake, N. T. Gladd, and C. S., Phys. Fluids **23**, 1998 (1980).
[17] P. C. Liewer, Nuclear Fusion **25**, 543 (1985).
[18] B. B. Kadomtsev, Fiz. plasmy **13**, 771 (1987).

Translated by Oleg Glebov