Modified adiabatic approximation. Multiphoton-ionization model

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A new integral representation is obtained for a nonstationary wave function in the adiabatic approximation. This representation is asymptotically exact and imposes no significant constraints on the form of the Hamiltonian. A multiphoton ionization model is considered. The ionization spectrum reveals a characteristic interference effect connected with the system ionization by different parts of the laser pulse.

The use of perturbation-theory methods or of calculations involving expansions in certain complete bases is frequently quite difficult in investigations of processes in atomic-molecular systems in strong laser fields. This holds for strongly interacting states, when the calculation requires either a large number of basis functions or high orders of perturbation theory. In such cases, however, a strong electromagnetic field can be regarded with good accuracy as classical, and we arrive at a nonstationary Schrödinger equation with a time dependent Hamiltonian, which can be approximately solved by a diverse asymptotic methods.

If the field frequency is much higher than the characteristic atomic frequencies, $\omega \gg \omega_{at}$, an approximate solution can be obtained by averaging over fast oscillations of the external field. At a frequency $\omega \ll \omega_{at}$, however, averaging over "fast" intrinsic variables is possible. This approach is well known in physics as the adiabatic approximation. The solution of the nonstationary Schrödinger equation is sought then in the form of an asymptote in terms of the small parameter ω/ω_{at}.

Two physically important processes can be considered, excitation and ionization. In the first we deal with transitions between discrete atomic levels and the problem reduces to calculating the probabilities P_{pq} of the corresponding transitions (p and q are the quantum numbers of the initial and final states). In the investigation of ionization, the task of the theory is to calculate the distribution in the energy $P(E)$ of the emitted electrons.

The adiabatic approximation for the Hamiltonian $H(t)$ with a purely discrete spectrum was considered back in 1928 by Born and Fock.[1] Further development of the calculation methods, as well as their application to both model and real atomic-molecular systems, is well covered in a monograph by Nikitin and Umanskiĭ.[2]

Finding the distribution in energy $P(E)$ is a much more difficult task and the first results were obtained using adiabatic perturbation theory for "below-barrier" transitions to a continuous spectrum[3] and in the framework of exactly solvable models,[4,5] the general of which is the Demkov–Osherov model.[6–8]

These investigations have made it possible to grope for principles of a more general asymptotic approach containing no substantial restrictions on the form of the Hamiltonian $H(t)$. A consistent asymptotic setup was developed by Solov'ev.[9]

The present paper I derive for the wave function, in the adiabatic approximation, a new integral representation constituting an exact answer in the model, proposed in Ref. 5, with one discrete level against the background of a continuous spectrum. In principal order in the saddle-point method, the integral is equivalent to the "simple" exponential asymptotes obtained in Ref. 9. The last section of the paper deals with a model describing ionization of an atomic system with one discrete level by a low-frequency laser pulse. The ionization spectrum exhibits a characteristic interference effect connected with the ionization of the system by a different part of the laser pulse.

2. FORMULATION OF PROBLEM

Consider a nonstationary Schrödinger equation ($\hbar = m = e = 1$):

$$H(\omega t)\Psi(\mathbf{r},t) = i\frac{\partial}{\partial t}\Psi(\mathbf{r},t), \tag{1}$$

where ω is a small parameter having the meaning of field frequency in an atomic system of units. Of fundamental importance to us is the analytic dependence of the Hamiltonian $H(\omega t)$ on the time t. It is known that in this case the solution $\psi(\mathbf{r},t)$ is also an analytic function and only in this case will the entire exposition that follows be valid.

We seek the solution of (1) in the form

$$\Psi(r,t) = \int_C f(E)\exp(-iEt)\varphi(\mathbf{r},E)\,dE, \tag{2}$$

where C is a contour in the complex E plane, and $f(E)$ and $\varphi(\mathbf{r},E)$ are regarded as analytic continuations over the entire E plane. The exact meaning of the function $\varphi(\mathbf{r},E)$ and the choice of the contour C will be explained below. It is assumed in (2) that $\varphi(\mathbf{r},E)$ is given, and that $f(E)$ is an unknown function obtained by substituting the representation (2) of $\psi(\mathbf{r},t)$ in Eq. (1).

The normalization is specified for negative E in a form suitable for analytic continuation on the entire E plane:

$$\int \varphi(\mathbf{r},E)\varphi^*(\mathbf{r},E^*)\,d\mathbf{r} = 1. \tag{3}$$

In the adiabatic approach (АП) we seek $f(E)$ in the form of an asymptote as $\omega \to 0$. The crucial role is played in this case by the analytic properties of the corresponding adiabatic basis. Before starting the solution, we shall recall these properties briefly; a more complete exposition can be found in Solov'ev's review.[10]

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3. ANALYTIC PROPERTIES OF THE ADIABATIC TERMS AND OF THE ADIABATIC BASIS

The adiabatic basis and adiabatic terms of our problem are taken to be respectively the eigenfunctions and eigenvalues of the "instantaneous" Hamiltonian of the nonstationary problem:

$$H(\tau)\varphi_p(\mathbf{r},\tau)=E_p(\tau)\varphi_p(\mathbf{r},\tau), \tag{4}$$

where τ is a fixed parameter.

Terms having one and the same symmetry $E_p(\tau)$ are connected by branching points in the complex τ plane in such a way that they are branches of a single analytic function $E^s(\tau)$, where $s=1,2,\ldots$ number the irreducible representations of the symmetry group of the Hamiltonian $H(\tau)$. Terms of different symmetry turn out to be interconnected. Joining all the sheets τ of terms of given symmetry s we obtain a Riemann surface $E^s(\tau)$ on which it is by definition single-valued and analytic. The function $E^s(\tau)$ specifies the mapping $\{E^s(\tau):\tau\to E\}$ and correspondingly $\tau^s(E)-\{\tau^s(E):E\to\tau\}$.

In addition to the pair of complex-conjugate points connecting two terms, there exist branching points as the term emerges to the continuum, where it acquires a width.

The function $\tau(E)$ has a branching point on the boundary of the continuum at $E=0$, and also branching points connected with the extrema of $E(\tau)$. The latter are in general complex if the term emerges to a continuum.

The singularities of $E_p(\tau)$ in the complex τ plane are connected with singularities of the corresponding adiabatic wave functions. It is shown in Ref. 11 that the adiabatic wave functions can be represented in the form

$$\varphi_p(\mathbf{r},\tau)=C_p(\tau)\chi_p(\mathbf{r},\tau), \tag{5}$$

where

$$\chi_p(\mathbf{r},\tau)=\frac{A(\hat{r})\exp(-\varkappa r)}{(2\pi\varkappa)^{1/2}}[1+O(r^{-1})], \tag{6}$$

$$\int|A(\hat{r})|^2\,d\hat{r}=1,\quad \hat{r}=\mathbf{r}/r,\quad \varkappa=[-2E_p(\tau)]^{1/2}. \tag{7}$$

The functions $\chi_p(\mathbf{r},\tau)$ are bounded for all τ, and at branching points connecting two terms the factors $C_p(\tau)$ have the singularity

$$E_p(\tau)\underset{\tau\to\tau_c}{\approx}E_c+\mathrm{const}(\tau-\tau_c)^{1/2},\quad C_p(\tau)\underset{\tau\to\tau_c}{\approx}\mathrm{const}(\tau-\tau_c)^{-1/4}, \tag{8}$$

or in terms of E

$$C_p(\tau)\underset{E\to E_c}{\approx}\mathrm{const}(E-E_c)^{-1/2}. \tag{9}$$

It is important to note that these singularities appear only if τ_c and E_c are complex.

4. NONSTATIONARY WAVE FUNCTION IN THE ADIABATIC APPROXIMATION

An asymptotic expression for $f(E)$ as $\omega\to 0$, satisfying the initial condition

$$\Psi(\mathbf{r},t)\underset{t\to-\infty}{\longrightarrow}\varphi_n(\mathbf{r},E_n)\exp(-iE_nt), \tag{10}$$

was obtained in Ref. 9:

$$f(E)=(2\pi\omega)^{-1/2}\left[\frac{d\tau(E)}{dE}\right]^{1/2}\exp\left[\frac{i}{\omega}\int_{E_n}^{E}\tau(E')\,dE'\right][1+O(\omega)]. \tag{11}$$

In expression (11) E is regarded as a variable of a Riemann surface $\tau(E)$.

The function $\tau(E)$ is defined (in accordance with Sec. 3) as the inverse of $E(\tau)$, where $\tau=\omega\tau$ is the "slow" time.

We choose next the particular Riemann surface on one of the sheets τ of which is located the term $E_n(\tau)$ corresponding to the initial atomic state: $E_n(\tau)\to E_n$, $\tau\to-\infty$. In the corresponding adiabatic basis the mapping $\{E^s(\tau):\tau\to E\}$ specifies a function $\varphi(\mathbf{r},E)$ on the Riemann surface E. On each sheet E there will be mixed mappings from different sheets τ, so that $\varphi(\mathbf{r},E)$ can no longer be assigned a definite adiabatic state. Thus, for example, in the Demkov–Osherov state $E(\tau)$ has many sheets, while $\tau(E)$ has a single sheet, and a function $\varphi(\mathbf{r},E)$ is specified on one sheet E and represents all the adiabatic states $\varphi_p(\mathbf{r},E_p(\tau))$.

To satisfy the physical boundary conditions (1) we choose a contour C such that as $t\to\infty$ there remains a contribution from only the saddle point near E_n. This contour is directed from the saddle point E_n along the steepest-descent lines to the upper E half-plane (HP). As $t\to+\infty$ the exponential $\exp(-iEt)$ decreases to the lower HP, and as a result the contour C is deformed to the lower HP and hooks all the singularities with respect to E of the integrand. Different physical processes will correspond to contributions from these singularities.

The cols (branching points) on the negative E axis give the probabilities of populating the limiting atomic states. The branching point $\tau(E)$ ($\varphi(\mathbf{r},E)$) on the boundary of the continuum $E=0$ generates a contribution to the system ionization. Let us examine this in greater detail.

We draw a cut along the positive E axis and define the argument on its upper bank as follows:

$$-(-2E)^{1/2}=-\delta+ik,\ k>0,\ \delta>0, \tag{12}$$

which corresponds to the radiation condition.

As $t\to+\infty$ the contour C is deformed into the lower HP and the contribution to the integral (2) from the cut is a wave packet of ionized particles;

$$\Psi_{ion}(\mathbf{r},t)=(2\pi\omega)^{-1/2}\int_0^{\infty}\left[\frac{d\tau(E)}{dE}\right]^{1/2}\times\exp\left[\frac{i}{\omega}\int_{E_n}^{E}\tau(E')\,dE'-iEt\right]\varphi(\mathbf{r},E)\,dE. \tag{13}$$

The adiabatic wave function $\varphi(\mathbf{r},E)$ is normalized at $E<0$ to unity in accordance with (3). At $E>0$ it describes a particle going off to infinity with momentum $k=(2E)^{1/2}$:

$$\varphi\propto\exp(ikr)$$

and can be represented in the form [see Eqs. (5)–(7)]

$$\varphi(\mathbf{r},E)=C(E)\chi(\mathbf{r},E), \tag{14}$$

where $C(E)$ is the analytic continuation of the normalization constant of the bound state to positive E, and $\chi(\mathbf{r},E)$ is

the adiabatic wave function normalized to unity flux. Substituting (14) in (23) we obtain $\Psi_{\text{ion}}(\mathbf{r},t)$ in the form of an expansion in functions of the continuum $\chi(r,\mathrm{E})$ with a normalized flux, from which we obtain for the probability distribution of the particles in energy

$$P(E)=\frac{1}{2\pi\omega}\left|\left[\frac{d\tau(E)}{dE}\right]^{1/2}\exp\left(\frac{i}{\omega}\int_{E_n}^{E}\tau(E')dE'\right)C(E)\right|^2. \tag{15}$$

This result solves in principle the problem of calculating the probabilities in the adiabatic approximation, but is easy to see that the asymptote does not hold at the extremum points of $E(\tau)$:

$$\frac{dE}{d\tau}\to 0,\quad \frac{d\tau(E)}{dE}\to\infty. \tag{16}$$

On the complex E plane this is manifested by the appearance of additional cuts near which, generally speaking, it is necessary to consider a more complicated standard equation. This becomes particularly perceptible when $E(\tau)$ is a function with a large number of extrema, as in the problem of interaction between an atom and a laser emission field.

This raises the question of obtaining a modified asymptote free of this shortcoming. This is dealt with in the next section of this paper.

5. MODIFIED ADIABATIC ASYMPTOTE

We turn again to the representation of $\Psi(\mathbf{r},t)$ in the form (2); inverting the integral with respect to E we obtain

$$f(E)=(2\pi)^{-1}\int_D \exp(iEt)\langle\varphi(\mathbf{r},E)\,|\,\Psi(\mathbf{r},t)\rangle dt, \tag{17}$$

Expression (17) suggests a new representation of $f(E)$ in the form

$$f(E)=(2\pi)^{-1}\int_D \exp(iEt)F(t,E)dt, \tag{18}$$

$$F(t,E)=\langle\varphi(\mathbf{r},E)\,|\,\Psi(\mathbf{r},t)\rangle. \tag{19}$$

The function $F(t,E)$ is initially specified only on the real t axis, but we shall use later also its analytic continuation into the complex t plane.

We reduce thus the problem to finding a new unknown function $F(t,E)$. To derive an equation for $F(t,E)$ we recognize that $\Psi(\mathbf{r},t)$ satisfies the nonstationary Schrödinger equation (1), and then:

$$i\frac{\partial}{\partial t}F(t,E)=\langle\varphi(\mathbf{r},E)\,|\,i\frac{\partial}{\partial t}|\,\Psi(\mathbf{r},t)\rangle$$

$$=\langle\varphi(\mathbf{r},E)\,|\,H(\tau)\,|\,\Psi(\mathbf{r},t)\rangle, \tag{20}$$

where $\tau=\omega t$ is the "slow" time.

To abbreviate the notation we use hereafter the abbreviated designations $\varphi(\mathbf{r},E)\equiv\varphi(\mathrm{E})$ and $\varphi(\mathbf{r},\tau)\equiv\varphi(\tau)$.

We substitute $\Psi(\mathbf{r},t)$ in (2c) in the form (2), next $f(E)$ in the form (18), and write the result as a set of two equations:

$$\iint dE'dt'\exp[iE'(t'-t)]F(t',E')E'\langle\varphi(E)\,|\,\varphi(E')\rangle$$

$$=\iint dE'dt'\exp[iE'(t'-t)]F(t',E')E'\langle\varphi(E)\,|\,H(\tau)\,|\,\varphi(E')\rangle,$$

$$i\frac{\partial}{\partial t}F(t,E)=(2\pi)^{-1}\iint dE'dt'$$

$$\times\exp[iE'(t'-t)]F(t',E')E'\langle\varphi(E)\,|\,\varphi(E')\rangle. \tag{21}$$

The double integration can be represented (just as in the real case) either as a successive use of two single integrals over the complex planes E and t, or as an integral in $\mathbb{C}^2$ over a two-dimensional complex manifold.

Equations (21) for $F(t,E)$ must be supplemented by a boundary condition that follows directly from the boundary conditions for $\Psi(\mathbf{r},t)$ as $t\to-\infty$:

$$F(t,E)\underset{t\to-\infty}{\longrightarrow}\langle\varphi(\mathbf{r},E)\,|\,\varphi_n(\mathbf{r},E_n)\rangle\exp\left[-i\int^{t}E_n(\omega t')dt'\right]. \tag{22}$$

An exact solution of the system (21) for $F(t,E)$ is in general a more difficult task than the solution of the initial nonstationary Schrödinger equation, but in the asymptotic approach we need only guess correctly the form of the "strong" dependence of $F(t,E)$ on the small parameter ω (the "exponential function" and the "preexponential" coefficient preceding it) and obtain an easy-to-analyze recurrence procedure.

We change in (21) to the new variables $\tau'=\omega t'$ and $\tau=\omega t$:

$$\iint dE'\,d\tau'\exp\left[\frac{i}{\omega}E'(\tau'-\tau)\right]F(\tau',E')E'\langle\varphi(E)\,|\,\varphi(E')\rangle$$

$$=\iint dE'd\tau'\exp\left[\frac{i}{\omega}E'(\tau'-\tau)\right]$$

$$\times F(\tau',E')E'\langle\varphi(E)\,|\,H(\tau)\,|\,\varphi(E')\rangle,$$

$$i\omega\frac{\partial}{\partial\tau}F(\tau,E)=(2\pi\omega)^{-1}\iint dE'd\tau'\exp\left[\frac{i}{\omega}E'(\tau'-\tau)\right]$$

$$\times F(\tau',E')E'\langle\varphi(E)\,|\,\varphi(E')\rangle. \tag{23}$$

We shall seek $F(\tau,E)$ in the form

$$F(\tau,E)=X(E,\tau)\exp\left[-\frac{i}{\omega}\int^{\tau}Q(\tau')d\tau'\right], \tag{24}$$

$$X(E,\tau)=X_0(E,\tau)+\omega X_1(E,\tau)+\ldots, \tag{25}$$

where the functions $Q, X_0, X_1, \ldots$ are assumed to be smooth. Substituting $F(\tau,E)$ in the form (24) into the system (23) we note that the double integrals have a simple two-dimensional saddle point. The general result for integrals of this type is[12]

$$F(k)=\int_\gamma f(z)\exp[kS(z)]dz\approx k^{-n/2}\exp[kS(z^0)]\sum_{m=0}^{\infty}a_m k^{-m}, \tag{26}$$

where k is a large parameter, $z=(z^1,z^2,\ldots,z^n)\in\mathbb{C}^n$,

$dz = dz_1 ... dz_n$, and γ is an n-dimensional complex manifold. The point z^0 is obtained from the condition $S'_z(z^0) = 0$, $\det S''_{zz}(z^0) \neq 0$.

In the principal order in k we have

$$F(k) = \left(\frac{2\pi}{k}\right)^{-n/2} \exp[kS(z^0)] \times \{\det[-S_{zz}''(z^0)]\}^{-1/2} [f(z^0) + O(k^{-1})]. \quad (27)$$

In our case the conditions on a simple saddle points are

$$\frac{\partial}{\partial \tau'} S(\tau', E') = 0, \qquad \frac{\partial}{\partial E'} S(\tau', E') = 0, \quad (28)$$

$$S(\tau', E') = i\left[E'(\tau' - \tau) - \int^{\tau'} Q(\tau'')\, d\tau''\right]. \quad (29)$$

Substituting $S(\tau,E)$ in (28) we get

$$\tau' = \tau, \qquad E' = Q(\tau'). \quad (30)$$

The solution of this system for finding τ_{sad} and E_{sad} is obviously single-valued. Let us calculate the determinant in (27):

$$\det \left| - \begin{pmatrix} \partial^2 S/\partial \tau'^2 & \partial^2 S/\partial \tau' \partial E' \\ \partial^2 S/\partial \tau' \partial E' & \partial^2 S/\partial E'^2 \end{pmatrix} \right|_{\substack{\tau' = \tau \\ E' = Q(\tau')}} = -1. \quad (31)$$

We have thus proved the existence and uniqueness of a simple saddle point. Therefore a solution asymptotic as $\omega \to 0$ in the form (24) always exists for $F(\tau,E)$.

It remains to determine $Q(\tau)$ and $X_0(E,\tau)$ and verify whether the solution satisfies the boundary condition (22). Writing down on the right and on the left the terms to the zeroth power of ω in the system (23) we obtain

$$\langle \varphi(E) | \varphi(Q(\tau)) \rangle Q(\tau) = \langle \varphi(E) | H(\tau) | \varphi(Q(\tau)) \rangle, \qquad X_0(E, \tau) = X_0(Q(\tau), \tau) \langle \varphi(E) | \varphi(Q(\tau)) \rangle. \quad (32)$$

A solution of this system of functional equations is

$$Q(\tau) = E(\tau), \qquad X_0(E, \tau) = \langle \varphi(E) | \varphi(E(\tau)) \rangle \equiv \langle \varphi(E) | \varphi(\tau) \rangle \quad (33)$$

so that in principal order in ω the expression for $F(\tau,E)$ takes, apart from an inessential phase factor, the form

$$F(\tau, E) = \exp\left[-\frac{i}{\omega} \int^{\tau} E(\tau')\, d\tau'\right] \langle \varphi(E) | \varphi(\tau) \rangle [1 + O(\omega)]. \quad (34)$$

Substituting (34) in (17) and (2) we get

$$f(E) = (2\pi\omega)^{-1} \int_D \exp\left[\frac{i}{\omega}\left(E\tau - \int^{\tau} E(\tau')\, d\tau'\right)\right] \times \langle \varphi(E) | \varphi(\tau) \rangle d\tau [1 + O(\omega)], \quad (35)$$

$$\Psi(r,t) = (2\pi\omega)^{-1} \iint_{C\ D} \exp\left\{\frac{i}{\omega}\left[E\tau' - \int^{\tau'} E(\tau'')\, d\tau''\right] - iEt\right\} \times \langle \varphi(E) | \varphi(\tau') \rangle \varphi(E)\, dE\, d\tau' [1 + O(\omega)]. \quad (36)$$

Let us check whether the boundary conditions are satisfied. As $t \to \infty$ the integral in (36) has a simple saddle point

$$\tau' = \omega t, \qquad E = E(\tau'). \quad (37)$$

If the start of the contour ($\tau' \to -\infty$) along τ' is specified on the sheet corresponding to the limiting atomic state

$$E_n(\tau) \underset{\tau \to -\infty}{\longrightarrow} E_n(-\infty) \equiv E_n,$$

we obtain in the leading order in ω the correct boundary condition:

$$\Psi(\mathbf{r}, t) \underset{t \to -\infty}{\sim} \varphi_n(\mathbf{r}, E_n) \exp\left[-i \int^{t} E_n(\omega t')\, dt'\right] [1 + O(\omega)]. \quad (38)$$

In addition to the indicated condition on the τ contour D, the E and τ contours must be drawn such that when they are deformed in the saddle point one can neglect, in principal order in ω, the contributions from the remaining singularities.

6. TRANSITIONS AND IONIZATION IN THE MODIFIED ADIABATIC APPROXIMATION

For finite t we can regard the double integral (36) as two single ones. We can get rid of the integral over τ by calculating $f(E)$ by the saddle-point method as $\omega \to 0$. The integral over E for finite t, however, cannot be explicitly calculated. Only as $t \to \mp\infty$ is a simple saddle-point pattern produced, and it is precisely to this limit that the physical formulation of the measurements pattern corresponds (the calculation is similar in this sense to the S-matrix formalism).

Let us consider the calculation scheme in the limit as $t \to \mp\infty$. On the negative E axis there are simple two-dimensional cols that are determined from (37) and correspond to population of the limiting atomic states. As $t \to -\infty$ the initial state is formed according to (38). As $t \to +\infty$ we obtain contributions from the finite atomic states E_m in accordance with the contour D and the structure of the Riemann surface of the given $E^s(\tau)$. The coefficients preceding them yield an expression for the leading term of the asymptote, as $\omega \to 0$, of the probability amplitude A_{nm} of the inelastic transitions.

In full analogy with the exposition in Sec. 4, the contour C is deformed as $t \to +\infty$ in the lower HP and is linked to the cut, defined according to (12), along the positive E axis. The integral along the cut must be considered separately, for a role is played here by the asymptote, with respect to r, of the adiabatic wave function at positive energy [See Eq. (14)]:

$$\varphi \propto \exp(ikr), \qquad k = (2E)^{1/2}.$$

Therefore for $t \to +\infty$, $\omega \to 0$, and large r where the wave packet of the ionized particles is located, the condition (37) is modified into

$$\frac{\tau'}{\omega}-t+\frac{r}{k}=0, \qquad E=E(\tau'). \tag{39}$$

The treatment of this system is quite clear. Solving it for r, we obtain

$$r=k(t-t'), \quad t'=\tau(E)/\omega.$$

This equation shows that the wave packet is made up of particles produced at the instant t' at a finite distance $r'\sim 1$, when the adiabatic state was characterized by a momentum k and turned to be at a distance r at the instant t.

Each point E of the interval along the cut corresponds thus to a saddle point in the vicinity of large r, due to ionization of particles with a given energy E, and the entire integral over the positive energies, just like Eq. (10), has the meaning of a packet of ionized particles:

$$\Psi_{ion}(\mathbf{r},t)=\int_0^\infty f(E)\exp(-iEt)\varphi(\mathbf{r},E)\,dE. \tag{40}$$

For the probability $P(E)$ of the particle distribution in energy we obtain by analogy with (15)

$$P(E)=|f(E)C(E)|^2, \tag{41}$$

where $f(E)$ has the form (35). The only regions of importance in the integral (35) are those in whose vicinity the derivative of the exponent with respect to τ vanishes: $E=E(\tau)$. In these regions the matrix element is $\langle\varphi(E)|\varphi(\tau)\rangle=1$, so that the integral (35) is equivalent, with the same ω error, to the expression

$$f(E)=(2\pi\omega)^{-1}\int_D \exp\left\{\frac{i}{\omega}\left[E\tau-\int^\tau E(\tau')d\tau'\right]\right\}d\tau[1+O(\omega)]. \tag{42}$$

If all the roots of the equation $E=E(\tau)$ are simple, then $f(E)$ is equivalent, with a specified ω error, to the simple exponential asymptotes:

$$f(E)=(2\pi\omega)^{-1/2}\sum_{\substack{k,\\ \tau_k(E)=E}}\left[i\frac{d\tau_k(E)}{dE}\right]^{1/2}\exp\left\{\frac{i}{\omega}\left[E\tau_k(E)-\int^{\tau_k(E)}E(\tau')d\tau'\right]\right\}[1+O(\omega)], \tag{43}$$

which, after integration by parts, take the form of exponential asymptotes of the type (11):

$$f(E)=(2\pi\omega)^{-1/2}\sum_{\substack{k,\\ \tau_k(E)=E}}\left[i\frac{d\tau_k(E)}{dE}\right]^{1/2}\times\exp\left[\frac{i}{\omega}\int_{E_n}^{E}\tau_k(E)\,dE\right][1+O(\omega)]. \tag{44}$$

The lower limit of integration with respect to τ in (43) is chosen on the sheet $E_n(\tau)$ as $\tau\to\pm\infty$.

In the opposite case the integral over τ remains (it coincides sometimes with the integral representation for known special functions). If two simple saddle points are present and merge into a double one at a certain E_0, then $f(E)$ will be asymptotically equivalent to the Airy function (see Ref. 12).

We demonstrate next the application of the equations of the modified adiabatic approximation to one model of multiphoton ionization.

7. MULTIPHOTON-IONIZATION MODEL

The question of the ionization spectrum in the context of the adiabatic approximation is presently extensively discussed in the literature (see, e.g., Refs. 13 and 14).

We consider the calculation of the ionization spectrum $P(E)$ for a model with one discrete level $E_1(\tau)$ $(\tau=\omega t)$, when it constitutes a multiperiodic function in τ and does not go off to the continuum. This model describes multiphoton ionization of an atomic system by a low-frequency laser pulse.

We approximate the central part of the pulse by the expression

$$E_c(\tau)=E_1^\infty+A\cos(\tau), \quad E_1^\infty<0, \quad A>0, \quad |E_1^\infty/A|<1, \tag{45}$$

where E_1^∞ is the energy of the limiting atomic level, and A is indicative of the amplitude of the laser pulse. The decrease of the pulse can be assumed, for example, to be exponential. For the left-and right-hand wings we have respectively

$$E_l(\tau)=E_1^\infty+A\exp[\gamma(\tau+2\pi N)], \qquad E_r(\tau)=E_1^\infty+A\exp[-\gamma(\tau-2\pi N)], \tag{46}$$

where $\gamma=\gamma_d/\omega$, $\gamma>0$ and γ_d characterizes the pulse damping rate. For simplicity we have chosen in (46) a symmetric approximation.

The ionization spectrum is given by the general equation (41), where $f(E)$ takes the form (42). In our case, at $E>0$, all the roots of the equation $E=E(\tau)$ are simple and we cannot use the representation (43) for $f(E)$. The saddle points for (43) are of the form

$$\tau_k^c(E)=\pm i\ln\left\{\frac{E-E_1^\infty}{A}+\left[\left(\frac{E-E_1^\infty}{A}\right)^2-1\right]^{1/2}\right\}+2\pi k, \qquad k=-N,\ldots,N$$

for the central part,

$$\tau_k^{\,l}(E)=\frac{1}{\gamma}\ln\left(\frac{E-E_1^\infty}{A}\right)-2\pi N+\frac{1}{\gamma}2\pi k, \quad k=0,\pm1,\ldots \tag{47}$$

for the left-hand wing, and

$$\tau_k^{\,r}(E)=-\frac{1}{\gamma}\ln\left(\frac{E-E_1^\infty}{A}\right)+2\pi N+\frac{1}{\gamma}2\pi k, \quad k=0,\pm1,\ldots$$

for the right-hand wing. The saddle points on the real axis have no special physical meaning and are due simply to our rough approximation of the character of the decrease of the pulse wings (non-analytic joining at the points $\tau=\pm 2\pi N$).

We deform the contour D over the saddle points in the upper τ half-plane, where the integrand decreases, and sepa-

rate the sums from the left wing, the central part, and the right wind:

$$f_l(F) = (2\pi\omega)^{-1/2} \sum_{k=1}^{\infty} \left[i \frac{d\tau_k{}^l(E)}{dE} \right]^{1/2} \times \exp\left\{ \frac{i}{\omega} \left[E\tau_k{}^l(E) - \int_{-\infty}^{\tau_k{}^l(E)} E(\tau')\, d\tau' \right] \right\},$$

$$f_c(E) = (2\pi\omega)^{-1/2} \sum_{k=-N}^{N} \left[i \frac{d\tau_k{}^c(E)}{dE} \right]^{1/2} \times \exp\left\{ \frac{i}{\omega} \left[E\tau_k^c(E) - \int_{-\infty}^{\tau_k{}^c(E)} E(\tau')\, d\tau' \right] \right\}, \tag{48}$$

$$f_r(E) = (2\pi\omega)^{-1/2} \sum_{k=1}^{\infty} \left(i \frac{d\tau_k{}^r(E)}{dE} \right)^{1/2} \times \exp\left\{ \frac{i}{\omega} \left[E\tau_k{}^r(E) - \int_{-\infty}^{\tau_k{}^r(E)} E(\tau')\, d\tau' \right] \right\}.$$

Using next the expressions for the saddle points (47) and carrying out all the necessary calculations, we obtain

$$f_l(E) = (2\pi\omega)^{-1/2} B_1(E) S_1(E) \exp[i\Phi_1(E)],$$

$$f_r(E) = (2\pi\omega)^{-1/2} B_1(E) S_1(E) \exp\left[-i\Phi_1(E) - \frac{i}{\omega} \int_{-\infty}^{+\infty} E(\tau')\, d\tau' \right],$$

$$f_c(E) = (2\pi\omega)^{-1/2} B_2(E) S_2(E) \exp[i\Phi_2(E)], \tag{49}$$

$$\Phi_1(E) = \frac{i}{\omega}(E - E_1{}^{\infty}) \left\{ \frac{1}{\gamma} \left[\ln\left(\frac{E - E_1{}^{\infty}}{A} \right) - 1 \right] - 2\pi N \right\} + i\pi/4,$$

$$S_1(E) = \exp\left[-\frac{2\pi(E - E_1{}^{\infty})}{\gamma\omega} \right] \Big/ \left\{ 1 - \exp\left[-\frac{2\pi(E - E_1{}^{\infty})}{\gamma\omega} \right] \right\},$$

$$B_1(E) = [\gamma(E - E_1)]^{-1/2},$$

$$S_2(E) = \left\{ 2\sin\left[\frac{\pi(E - E_1{}^{\infty})}{\omega}(N+1) \right] \Big/ \sin\left[\frac{\pi(E - E_1{}^{\infty})}{\omega} \right] \right\} \times \cos\left[\frac{\pi(E - E_1{}^{\infty})}{\omega} N \right] - 1,$$

$$B_2(E) = \left\{ A \left[\left(\frac{E - E_1{}^{\infty}}{A} \right)^2 - 1 \right]^{1/2} \right\}^{-1/2} \exp[-T(E)],$$

$$T(E) = \frac{(E - E_1{}^{\infty})}{\omega} \ln\left\{ \frac{E - E_1{}^{\infty}}{A} + \left[\left(\frac{E - E_1{}^{\infty}}{A} \right)^2 - 1 \right]^{1/2} \right\} - \frac{A}{\omega} \left[\left(\frac{E - E_1{}^{\infty}}{A} \right)^2 - 1 \right]^{1/2},$$

$$\Phi_2(E) = -\frac{i}{\omega\gamma} A - i\pi/2.$$

We use the representation (41) for $P(E)$:

$$P(E) = |f(E)|^2 |C(E)|^2.$$

The factor $|C(E)|^2$ influences only the general form of the $P(E)$ dependence and we shall not consider it in detail, noting only that it is possible to choose for it an appropriate parametrization both in the case of short-range potentials and in the case with long-range Coulomb action.[15]

Of greater interest to us is $|f(E)|^2$. Using the earlier calculations, we have

$$f(E) = f_l(E) + f_c(E) + f_r(E),$$

$$|f(E)|^2 = |f_l(E)|^2 + |f_c(E)|^2 + |f_r(E)|^2 + 2\,\mathrm{Re}[f_l(E) f_r(E)] + 2\,\mathrm{Re}[[f_l(E) + f_r(E)] f_c(E)],$$

$$|f_l(E)|^2 = (2\pi\omega)^{-1}[B_1(E) S_1(E)]^2,$$

$$|f_r(E)|^2 = (2\pi\omega)^{-1}[B_1(E) S_1(E)]^2, \tag{50}$$

$$\mathrm{Re}[f_l(E) f_r(E)] = (2\pi\omega)^{-1}[B_1(E) S_1(E)]^2 \times \cos\left[\frac{1}{\omega} \int_{-\infty}^{+\infty} E(\tau')\, d\tau' \right],$$

$$\mathrm{Re}\{[f_l(E) + f_r(E)] f_c(E)\} = (2\pi\omega)^{-1} B_1(E) B_2(E) S_1(E) S_2(E),$$

$$2\cos\left\{ \left[\Phi_2 - \frac{1}{\omega} \int_{-\infty}^{+\infty} E(\tau')\, d\tau' \right] \Big/ 2 \right\} \times \cos\left\{ \Phi_1 + \left[\Phi_2 + \frac{1}{\omega} \int_{-\infty}^{+\infty} E(\tau')\, d\tau' \right] \Big/ 2 \right\}.$$

The interpretation of these results is quite clear. The squares of the moduli take into account the contribution made to the ionization by each part of the pulse separately, while the crossing germs show the interference of particles ionized by different parts of the pulse.

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