Gravitational field of rotating bodies

S. L. Parnovskiĭ

Astronomical Observatory, Kiev State University
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Two vacuum solutions of the Einstein equations are constructed and investigated and can be regarded as candidates for the role of the exterior solution for axially symmetric uniformly rotating bodies of prolate and oblate shape, respectively. Both solutions contain two infinite sets of arbitrary constants, corresponding to the multipoles of the mass and angular momentum. In the absence of rotation and for a particular set of multipole moments for the mass the first solution goes over into the Zipoy–Voorhees metric, while the second goes over into the known solution with a ring singularity. The relationship of the first solution to the van Stockum metric is also investigated, and clarifies the interpretation of the latter.

1. INTRODUCTION

In the framework of the general theory of relativity the question of the form of the gravitational field of rotating bodies has not yet been solved. As is well known, the Kerr metric, which describes rotating black holes, cannot be the exterior solution for a real rotating object. In the present paper we find suitable candidates for the role of such a solution for the axially symmetric steady-rotation case, both for prolate and for oblate rotating bodies. The metric obtained contains two infinite sets of arbitrary constants, corresponding to the multipole moments for the distribution of mass and angular momentum. It is clear that this gives the possibility of joining this metric with any interior solution describing a source with a physically acceptable equation of state.

In Secs. 2–7 of this paper we consider the solution for prolate bodies. In the absence of rotation and for a particular set of multipoles it goes over into the well-known Zipoy–Voorhees metric,[1,2] sometimes called the γ solution:

$$ds^2=\left(\operatorname{th}\frac{v}{2}\right)^{2\mu}dt^2-\frac{L^2}{4}\left(\operatorname{th}\frac{v}{2}\right)^{-2\mu}\operatorname{sh}^2 v \times\left[\left(1+\frac{\cos^2 u}{\operatorname{sh}^2 v}\right)^{1-\mu^2}(du^2+dv^2)+\cos^2 u\, d\varphi^2\right]. \quad (1)$$

An example of the solution of interest to us is found in Sec. 3 by means of iterations. The most general case is considered in Sec. 6. The properties of the space-time described by this solution are investigated in Sec. 5, in which, in particular, it is shown that in the region near the rotating singularity (the source of the gravitational field) closed timelike geodesics are admissible. However, a source of finite size can occupy this region without permitting any violation of causality.

In Sec. 7 we consider the relationship of our solution to the van Stockum metric that describes the space-time around an infinitely long rotating dust cylinder. Here it is shown that the condition $0<\mu<1/2$ imposed on the metric is not accidental. When this condition is violated, the exterior solution for the prolate infinitely thin or long rotating body either does not exist or else changes its form qualitatively. In the analysis of the van Stockum metric it is also shown that two of the three admissible exterior solutions [(60) with (62), and (60) with (63), below] correspond to the case when, in accordance with the Mach principle, an infinitely long and massive source drags the entire space-time in such a way that relative rotation is absent.

In Sec. 8 the approach used in Secs. 2–6 is applied to the analysis of the exterior solution for oblate axially symmetric rotating bodies. It is shown that the equations for the self-consistent solutions in this case differ only slightly from the analogous equations obtained in Sec. 6 for the case of prolate bodies. However, because of the qualitatively different structure of the singularity, in this case perturbation-theory methods cannot be applied near the singularity.

2. PROLATE BODIES. INITIAL CONSIDERATIONS

In the case of axially symmetric steady rotation the space-time metric can be brought to the Lewis–Papapetrou form

$$ds^2=e^{\nu}(dt-\omega d\varphi)^2-e^{-\nu}(e^{\gamma}(d\rho^2+dz^2)+\rho^2 d\varphi^2). \quad (2)$$

The functions $\nu(\rho,z)$ and $\omega(\rho,z)$ appearing in this metric are connected by the relations

$$\Delta\nu=e^{2\nu}\rho^{-2}(\nabla\omega)^2, \quad (3)$$

$$\nabla(e^{2\nu}\rho^{-2}\nabla\omega)=0. \quad (4)$$

Here the vector operations are performed in a conventional auxiliary flat coordinate space with cylindrical coordinates ρ, φ, z. If ν and ω are known, the function $\gamma(\rho,z)$ can be found (to within a constant determined from the condition that there be no conic singularities) from the relations

$$2\gamma_{,\rho}=\rho[\nu_{,\rho}{}^2-\nu_{z}^2-e^{2\nu}\rho^{-2}(\omega_{,z}{}^2-\omega_{,\rho}{}^2)],$$
$$\gamma_{,z}=\rho(\nu_{,\rho}\nu_{,z}-e^{2\nu}\rho^{-2}\omega_{,\rho}\omega_{,z}). \quad (5)$$

We obtain the desired solution with rotation by iterations from the initial static ($\omega=0$) metric (2), which belongs to the Weyl class. It can be seen from (3) that ν in this case is a harmonic function. Since it should tend to zero far from the source (the space is asymptotically flat), this function (except for the trivial case $\nu=0$) should have singularities. In the analysis of prolate sources of the gravitational field an interesting case is that in which this singularity lies on the rotation axis $\rho=0$ and has the form of a segment of finite length L in coordinate space (point singularities with $\rho=0$ correspond to singularities with respect to direction; see Ref. 3). This does not mean that the singularities in the

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space-time (2) are also line singularities. They can also be point singularities, or can belong to the new type of singularities (impossible in a space of finite curvature) that is considered in Ref. 3. Later, in Ref. 4, such singularities were called paradoxical singularities.

In constructing the solution it is most convenient to use the coordinate system of a prolate ellipsoid of revolution (u, v, φ in coordinate space), setting

$$\rho=\tfrac{1}{2}L\cos u\,\mathrm{sh}\,v,\quad z=\tfrac{1}{2}L\sin u\,\mathrm{ch}\,v. \tag{6}$$

In this case the singularity corresponds to $v=0$.

The solution of the equation $\Delta^2\nu=0$ which satisfies the condition that it be asymptotically Galilean and also regular on the rotation axis $u=\pm\pi/2$ away from the singularity is

$$\nu=\sum_{i=0}^{\infty}\alpha_i P_i(\sin u)Q_i(\mathrm{ch}\,v), \tag{7}$$

where $P_i(x)$ and $Q_i(x)$ are Legendre polynomials of the first and second kind. The set of numbers α_1 is determined by the distribution μ of the line density of the mass of the source with respect to z or u in coordinate space, by the relation

$$2\mu(u)=\sum_{i=0}^{\infty}\alpha_i P_i(\sin u) \tag{8}$$

and is related to the set of multipole moments. For $v\to\infty$ each of the terms of the sum (7) falls off as r^{-i-1}, where $r^2=\rho^2+z^2$.

We separate out the first term on the right-hand side of (7) and set

$$\nu=\nu_0+\varkappa,\quad \nu_0=2\mu\ln[\mathrm{th}(v/2)]. \tag{9}$$

Then the metric (2) takes the form

$$ds^2=e^{\varkappa}\left(\mathrm{th}\frac{v}{2}\right)^{2\mu}(dt-\omega\,d\varphi)^2-\frac{L^2}{4}e^{-\varkappa}\left(\mathrm{th}\frac{v}{2}\right)^{-2\mu}$$
$$\times\mathrm{sh}^2 v\left[\left(1+\frac{\cos^2 u}{\mathrm{sh}^2 v}\right)^{1-\mu^2}e^{\xi}(du^2+dv^2)+\cos^2 u\,d\varphi^2\right], \tag{10}$$

where the function $\xi(u,v)$ is related to γ from (2) by the relation

$$\gamma=\xi-\mu^2\ln(1+\cos^2 u\,\mathrm{sh}^{-2}v) \tag{11}$$

and is determined from $\varkappa(u,v)$ $\omega(u,v)$ in accordance with (5), (9), and (11). For $\varkappa=\omega=0$ we have $\xi=0$, and (10) goes over into the metric (1) investigated in Ref. 3. Depending on the value of μ (the constant line density of the mass of the source in coordinate space), the singularity $v=0$ belongs to different types. For $\mu<0$ the space-time (1) has a point singularity, for $0<\mu<1$ it has a line singularity, and for $\mu>1$ it has a paradoxical singularity.

In the general case (10), the functions $\varkappa$ and ω are related by (3) and (4), which, with allowance for (9), have in the coordinates (6) the form

$$\varkappa_{,uu}+\varkappa_{,vv}+\varkappa_{,v}\,\mathrm{cth}\,v-\varkappa_{,u}\,\mathrm{tg}\,u$$
$$=4L^{-2}e^{2\varkappa}\,\mathrm{th}^{4\mu}(v/2)\cos^{-2}u\,\mathrm{sh}^{-2}v\,(\omega_{,u}^2+\omega_{,v}^2), \tag{12}$$

$$\omega_{,uu}+\omega_{,vv}+\omega_{,u}(\mathrm{tg}\,u+2\varkappa_{,u})+\omega_{,v}(4\mu\,\mathrm{sh}^{-1}v-\mathrm{cth}\,v+2\varkappa_{,v})=0. \tag{13}$$

In addition, from the condition that the solution be asymptotically Galilean we have

$$\varkappa(u,v)\xrightarrow[v\to\infty]{}0,\quad \omega(u,v)\xrightarrow[v\to\infty]{}0. \tag{14}$$

Next, we shall seek the solution in the form (10) with $\varkappa$ and ω in the form of series:

$$\varkappa=\varkappa_1+\varkappa_2+\ldots,\quad \omega=\omega_1+\omega_2+\ldots. \tag{15}$$

Here we take as the initial solution $\varkappa_0=\omega_0=0$, i.e., the metric (1).

3. CONSTRUCTION OF THE SIMPLEST GENERALIZATION OF THE γ METRIC

We shall study the extension of (1) by means of the perturbation-theory series (15). Having set $\varkappa=0$, after separation of the coordinates we obtain from (13)

$$\omega_1=\cos u\sum_{n=1}^{\infty}P_n^1(\sin u)q_n(v), \tag{16}$$

$$q_n(v)=\beta_n F(n,-1-n,-2\mu,\mathrm{ch}^2(v/2))$$
$$-\gamma_n\,\mathrm{sh}^2 v\,\mathrm{th}^{-4\mu}(v/2)F(1-n,n+2,2+2\mu,\mathrm{ch}^2(v/2)), \tag{17}$$

where $F(a,b,c,x)$ is the hypergeometric function, and $P_n^1(x)$ are associated Legendre functions. The solution (17) is general, except for the case $2\mu=k$ (where $|k|\leqslant n$ is an integer), which will be considered as a special case in the next section.

Near the singularity $v=0$ we have

$$q_n(v)\xrightarrow[v\to 0]{}\beta_n'+\gamma_n'v^{2-4\mu}. \tag{18}$$

Only for $\mu<1/2$ does the right-hand side in (12) have a lower power of v^{-1} than does the left-hand side, and Eq. (3) is fulfilled in the leading terms. Therefore, further analysis is possible only for $\mu<1/2$, since, as we shall see below, the numbers β_n and γ_n are related to each other, and only for $\omega=0$ is it possible to set γ_n equal to zero for all n. We recall that the case $\mu=1/2$ will be considered as a special case.

Each of the terms on the right-hand side of (17) diverges as $v\to\infty$. However, for

$$\beta_n n(n+1)\Gamma(1+n+2\mu)\Gamma(-2\mu)=4\gamma_n\Gamma(2+2\mu)\Gamma(n+1-2\mu) \tag{19}$$

these divergences cancel, and, by virtue of relations between Kummer series,[5] we obtain

$$q_n(v)\propto\mathrm{ch}^{-2\mu}(v/2)F(n+1+2\mu,n,2n+2,\mathrm{ch}^{-2}(v/2)),$$
$$q_n\xrightarrow[v\to\infty]{}O(r^{-n}). \tag{20}$$

This is the solution of interest to us. The numbers β_n and the numbers γ_n related to them by (19) determine a set of multipole moments $\omega_1(u,v)$. In this section we shall consider the case $\beta_n=\gamma_n=0$ for $n>1$, by retaining, from the entire series (16), only the first term, which is directly related to the angular momentum J of the central body:

$$\omega_1 = C\cos^2 u\cdot q(v),$$
$$q(v) = \mathrm{sh}^2 v\,\mathrm{th}^{-4\mu}(v/2) + 1 - 4\mu^2 - (2\mu + \mathrm{ch}\, v)^2. \qquad (21)$$

Here we have used the notation $C = -\gamma_1$.

Starting from this expression and using Eqs. (12) and (13) alternately, we can find all the terms of the series (15) for $\varkappa$ and ω, and, consequently, the metric (10) for the case (21). However, here the following problem arises. In the process of solving Eq. (12) for $\varkappa_n$ this function is determined nonuniquely. To a particular solution found we can add any number of terms of the series in the right-hand side of (7), with arbitrary coefficients. In this section we shall construct one possible solution, characterized by a particular set of these coefficients, and leave the treatment of the more general case to Sec. 6. Specifically, we shall require that in the expressions for $\varkappa_n$ terms of the form (7) are absent, i.e., there is no logarithmic divergence as $v \to 0$.

Analogously, in the solving of Eq. (13) the functions ω_n are determined to within the addition of terms (taken with arbitrary coefficients) from the expansion (16). In this section we shall set equal to zero all terms of this kind that are not needed in the solution of Eq. (13). The proposed choice of coefficients leads to the preservation of the original set of numbers α_i in (7) ($\alpha_0 = 2\mu$, $\alpha_i = 0$) (a set related to the multipole set for the mass of the source), and to a unique determination of the angular-momentum multipoles, related to β_n by (16), (17), and (19). As a result, the solution obtained will also be found uniquely.

Thus, by substituting the expression (21) for ω_2 into the right-hand side of (12), we obtain an equation for $\varkappa_1$:

$$\varkappa_{1,uu} + \varkappa_{1,vv} + \varkappa_{1,v}\,\mathrm{cth}\, v - \varkappa_{1,u}\,\mathrm{tg}\, u$$
$$= 4C^2L^{-2}\,\mathrm{th}^{4\mu}(v/2)\,\mathrm{sh}^{-2} v\,(4\sin^2 u\cdot q(v)^2 + \cos^2 u\cdot q_{,v}{}^2), \qquad (22)$$

the solution of which can be sought naturally in the form

$$\varkappa_1 = p_{11}(v) + p_{12}(v)\cos^2 u. \qquad (23)$$

Calculations lead to the expression

$$p_{12} = 4C^2L^{-2}\{\mathrm{sh}^2 v\,\mathrm{th}^{-4\mu}(v/2) - \mathrm{th}^{4\mu}(v/2)$$
$$\times[\mathrm{sh}^2 v + 4(\mathrm{ch}\, v + 2\mu)^2]$$
$$+ 4\,\mathrm{ch}^2 v - 8\mu\,\mathrm{ch}\, v - 16\mu^2\} + C_1\,(\mathrm{ch}^2 v - 1/3) \qquad (24)$$
$$+ C_2\{(\mathrm{ch}^2 v - 1/3)\ln[\mathrm{th}(v/2)] + \mathrm{ch}\, v\},$$

where C_1 and C_2 are arbitrary constants. From the condition (14), $C_1 = 0$. In addition, we should set $C_2 = 0$ in order not to change the set of numbers α_i in (7). Analogously, in the expression

$$p_{11} = 16C^2L^{-2}[4\mu^2 - \mathrm{ch}^2 v + (\mathrm{ch}\, v + 2\mu)^2\mathrm{th}^{4\mu}(v/2)]$$
$$+ C_1 + C_2\ln[\mathrm{th}(v/2)] \qquad (25)$$

we should also set $C_1 = C_2 = 0$. For $\mu < 0$ the expression for $\varkappa_1$ has a power divergence at the singularity $v = 0$, while ν_0 from (9) diverges only logarithmically. Therefore, the method that we are considering is applicable only for $\mu > 0$.

We now substitute the expression for $\varkappa_1$ into (13) and replace ω by $\omega_1 + \omega_2$, neglecting the terms $\varkappa_{1,u}\omega_{2,u}$ and $\varkappa_{1,v}\omega_{2,v}$, which will appear later in the equation for ω_3, and obtain an equation for ω_2. Its solution has the form

$$\omega_2 = \lambda_{21}(v)\cos^2 u + \lambda_{22}(v)\cos^4 u, \qquad (26)$$
$$\lambda_{22} = 4C^3L^{-2}[\Lambda_1(v) + C_1\Lambda_2(v)],$$
$$\Lambda_1 = {}^4/_5(36\mu^2 - 1)\,\mathrm{sh}^2 v\,\mathrm{th}^{-4\mu}(v/2) - \mathrm{sh}^4 v\,\mathrm{th}^{-8\mu}(v/2)$$
$$+ 4\,\mathrm{sh}^2 v\,(\mathrm{ch}\, v + 2\mu)^2\mathrm{th}^{4\mu}(v/2) - 3\,\mathrm{ch}^4 v + 8\mu\,\mathrm{ch}^3 v$$
$$+ [(96\mu^2 + 14)/5]\,\mathrm{ch}^2 v$$
$$- (8\mu/5)(3 + 32\mu^2)\,\mathrm{ch}\, v + {}^1/_5 - 352\mu^2/15 - 896\mu^4/15, \qquad (27)$$
$$\Lambda_2 = \mathrm{ch}^4 v + {}^8/_3\mu\,\mathrm{ch}^3 v + {}^1/_5(16\mu^2 - 6)\,\mathrm{ch}^2 v$$
$$+ {}^{32}/_{15}(\mu^3 - \mu)\,\mathrm{ch}\, v + 32\mu^4/45$$
$$- 56\mu^2/45 + {}^1/_5 - \mathrm{sh}^2 v\,\mathrm{th}^{-4\mu}(v/2)\,(\mathrm{ch}^2 v - {}^4/_3\mu\,\mathrm{ch}\, v + 8\mu^2/15 - {}^1/_5),$$

$$\lambda_{21} = {}^{16}/_5 C^3L^{-2}\{\mathrm{sh}^2 v\,\mathrm{th}^{-4\mu}(v/2)\,\mathrm{ch}\, v\,(\mathrm{ch}\, v + 12\mu)$$
$$- 5\,\mathrm{sh}^2 v\,(2\mu + \mathrm{ch}\, v)^2\mathrm{th}^{4\mu}(v/2) + 4\,\mathrm{ch}^4 v - 16\mu\,\mathrm{ch}^3 v$$
$$+ (224\mu^3 + 24\mu)\,\mathrm{ch}\, v - 4 + 32\mu^2/3 + 1408\mu^4/3$$
$$- {}^1/_3 C_1[3\,\mathrm{ch}^4 v + 8\mu\,\mathrm{ch}^3 v$$
$$+ 8\mu(1 - 4\mu^2)\,\mathrm{ch}\, v - 3 + 8\mu^2(13 - 28\mu^2)/3$$
$$- \mathrm{sh}^2 v\,\mathrm{th}^{-4\mu}(v/2)\,\mathrm{ch}\, v\,(3\,\mathrm{ch}\, v - 4\mu)]$$
$$- C_2(\mathrm{sh}^2 v + 4\mu\,\mathrm{ch}\, v + 8\mu^2) - C_3\,\mathrm{sh}^2 v\,\mathrm{th}^{-4\mu}(v/2)\}. \qquad (28)$$

Here, C_1, C_2, and C_3 are constants. In order not to complicate further the expressions obtained, one of these constants will be chosen in such a way that Λ_1 and Λ_2 tend to zero as $v \to \infty$. Both these functions, and with them λ_{22}, fall off as $\cosh^{-3} v$:

$$\lambda_{22} \xrightarrow[v\to\infty]{} \frac{256}{4725}\,\mu(4\mu^2 - 1)C^3L^{-2}(6(16\mu^2 - 1)(9 - 44\mu^2)$$
$$- C_1(\mu^2 - 1)(4\mu^2 - 9)]\,\mathrm{ch}^{-3} v = O(r^{-3}). \qquad (29)$$

For

$$C_1 = 6(16\mu^2 - 1)(9 - 44\mu^2)(\mu^2 - 1)^{-1}(4\mu^2 - 9)^{-1} \qquad (30)$$

this expression vanishes and

$$\lambda_{22} \xrightarrow[v\to\infty]{} O(r^{-4}).$$

This damping power corresponds to the power obtained from an analysis of the asymptotic behavior of the equation for λ_{22}, which, because of the unwieldiness of its free term, is not given here. The appearance of the power r^{-4} in (29) is due to the contribution of the third term in the series (16). Therefore, in the analysis in this section, this term should be eliminated by imposing the condition (30). We note that the values $\mu^2 = 1$ and $\mu^2 = 9/4$ are special and will be considered below. For other values of μ the quantity C_1 is finite.

When the expression (28) is analyzed, it follows from the condition $\lambda_{21} \to 0$ as $v \to \infty$ that

$$C_2 + C_3 = 76\mu^2 + 4 + C_1(8\mu^2/3 - 1). \qquad (31)$$

In addition, it follows from the above considerations that we must set equal to zero the coefficient in the expression

$$\lambda_{21} \underset{v\to\infty}{\longrightarrow} -\frac{128}{15}\mu(4\mu^2-1)C^3L^{-2} \times\left[C_3+C_1\frac{8\mu^2-3}{15}-\frac{6(14\mu^2+1)}{5}\right]\text{ch}^{-1}v \qquad (32)$$

and ensure that

$$\lambda_{21}\underset{v\to\infty}{\longrightarrow} O(r^{-2}).$$

As a result, we obtain

$$C_2=2(4624\mu^6-3280\mu^4+783\mu^2-27)(\mu^2-1)^{-1}(4\mu^2-9)^{-1},$$
$$C_3=10\mu^2(232\mu^4-166\mu^2+39)(\mu^2-1)^{-1}(4\mu^2-9)^{-1}. \qquad (33)$$

Next, one could substitute $\omega=\omega_1+\omega_2$ and $\varkappa=\varkappa_1+\varkappa_2$ into (12) and seek $\varkappa_2$ in the form

$$\varkappa_2=p_{21}(v)+p_{22}(v)\cos^2 u+p_{23}(v)\cos^4 u,$$

having obtained even more-unwieldy expressions for $p_{2i}(v)$, etc. This unwieldiness suggests that it is scarcely possible to sum the series (15) for arbitrary μ.

Thus, the solution (10), (15) has the form

$$\omega_i=C^{2i-1}L^{2-2i}\sum_{k=1}^{i}\lambda_{ik}(v)\cos^{2k}u, \qquad (34)$$

$$\varkappa_i=C^{2i}L^{-2i}\sum_{k=1}^{i+1}p_{ik}(v)\cos^{2k-2}u. \qquad (35)$$

The equations for λ_{ik} and p_{ik} are obtained from (12) and (13) by expanding in powers of $\cos^2 u$ (it is convenient to change to the variable $x=\cosh v$):

$$(x^2-1)p_{ik,xx}+2xp_{ik,x}-2(k-1)(2k-1)p_{ik} =-4kp_{i,k+1}+4\,\text{th}^{4\mu}(v/2)\,\text{sh}^{-2}v\cdot Q_{ik}(v),$$
$$(x^2-1)\lambda_{ik,xx}+4\mu\lambda_{ik,x}-2k(2k-1)\lambda_{ik}=F_{ik}(v)-4k(k+1)\lambda_{i,k+1}, \qquad (36)$$

where

$$F_{ik}(v)=-2\sum_{m=1}^{i-1}\sum_{n=1}^{m}\{\lambda_{mn}'p_{i-m,k+1-n}' +4n\lambda_{mn}[(k+1-n)p_{i-m,k+2-n}-(k-n)p_{i-m,k+1-n}]\}. \qquad (37)$$

Here, a prime denotes a derivative with respect to v. To determine the function $Q_{ik}(v)$ appearing in (36), we must expand the expression

$$\exp\left(2\sum_{m=1}^{i-1}\alpha^m\sum_{n=1}^{m+1}p_{mn}\beta^{n-1}\right)\left[\left(\sum_{m=1}^{i-1}\alpha^m\sum_{n=1}^{m}\lambda_{mn}'\beta^n\right)^2 +4\beta^2(1-\beta^2)\left(\sum_{m=1}^{i-1}\alpha^m\sum_{n=0}^{m-1}n\lambda_{m,n+1}\beta^n\right)^2\right]$$

in a series in powers of α and β. The function that appears in this series with the powers $\alpha^{i-1}\beta^k$ will be $Q_{ik}(v)$. We note that, both in Q_{ik} and in F_{ik} from (37), only the already determined functions λ_{ik} and p_{ik} appear. When they are calculated we must set $\lambda_{ik}=0$ for $k>i$ and $p_{ik}=0$ for $k>i+1$.

The solution of Eqs. (36) and (37) is determined to within the addition of terms of the form

$$p_{ik}=p_{ik}^{(0)}+C_1P_{2k-1}(\text{ch}\,v)+C_2Q_{2k-2}(\text{ch}\,v), \qquad (38)$$

$$\lambda_{ik}=\lambda_{ik}^{(0)}+C_3F(2k-1,-2k,-2\mu,\text{ch}^2(v/2)) +C_4\,\text{sh}^2 v\,\text{th}^{-4\mu}(v/2)F(2-2k,1+2k,2+2\mu,\text{ch}^2(v/2)). \qquad (39)$$

The coefficients C_i are uniquely determined from the conditions $p_{ik}\to 0$ as $v\to\infty$ (C_1), from the absence of a logarithmic divergence in p_{ik} at $v=0(C_2)$, and from the conditions $\lambda_{ik}\to O(x^{-2k-2})$ as $v\to\infty$ (C_3 and C_4).

In Sec. 6 we obtain the general solution with an arbitrary set of multipoles. But first we shall consider the special values of μ.

4. THE SPECIAL VALUES OF μ

If $2\mu=k$, where $|k|$ is an integer and $|k|\leqslant n$, the two functions on the right-hand side of (17) coincide. In this case, as the solution (16) we must take[5]

$$q_n(v)=\gamma_n\,\text{ch}^{-n}v\cdot F(n+1+k,n,2n+2,\text{ch}^{-1}v). \qquad (40)$$

This hypergeometric function reduces to polynomials and logarithms:

$$q_n=\frac{(2n+1)!}{(n-1)!(n+1)!(n-k)!(n+k)!}\gamma_n\,\text{ch}^{-n}v\cdot W(\text{ch}^{-1}v), \qquad (41)$$

$$W(\tau)=\begin{cases}\dfrac{d^{n+k}}{d\tau^{n+k}}\left\{(\tau-1)^{n+1}\dfrac{d^{n-k}}{d\tau^{n-k}}[\tau^{-1}\ln(1-\tau)]\right\}, & 0<k\leqslant n,\\[2ex] \dfrac{d^{n-1}}{d\tau^{n-1}}\left\{(\tau-1)^{n-k}\dfrac{d^{n+1}}{d\tau^{n+1}}[\tau^{-1}\ln(1-\tau)]\right\}, & -n\leqslant k<0.\end{cases} \qquad (42)$$

It is not difficult to show that for $k\geqslant 2$ it diverges at $v=0$ in a power-law manner, and for $\mu=1/2$ gives

$$\omega_0\propto\cos^2 u\cdot[1+2\,\text{ch}\,v-2\,\text{ch}^2 v\cdot\ln(1-\text{ch}^{-1}v)] \qquad (43)$$

with a logarithmic divergence at $v=0$. For $k<0$ there is no divergence in q_n, but in this case, as always for $\mu<0$, a divergence appears in $\varkappa_1$. Thus, consideration of the special values of μ gives nothing new, and, when examining our solution in the entire space-time, including the region near the singularity, we should confine ourselves to the interval

$$0<\mu<{}^1/_2. \qquad (44)$$

5. ANALYSIS OF THE SOLUTION

The initial solution (1) was investigated in detail in Ref. 3. When the condition (44) is fulfilled the singularity $v=0$ is a line singularity. Its mass, determined from the form of g_{00} at large v, is equal to

$$M=\mu L/2. \qquad (45)$$

In the construction of the solution (10) it is important to ensure that the series (15) converges. To judge from (34) and (35), this is possible when the quantity C/L is bounded—in particular, when $C/L \ll 1$. With this assumption we shall consider the properties of the space-time described by the metric obtained above. It is not difficult to convince oneself that it has no singularities other than $v = 0$, and the function ξ turns out to be substantially smaller than the second term on the right-hand side of (11).

The expression (23)–(25) found for $\varkappa_1$ makes it possible to determine, in the first approximation, the correction to Eq. (45) due to allowance for rotation:

$$\Delta M^{(1)} = {}^{16}/_3 C^2 \mu (1-4\mu^2) L^{-1}. \qquad (46)$$

This quantity is positive and much smaller than M. The next correction, associated with $\varkappa_2$, will be smaller in order of magnitude by a further factor of C^2L^{-2}.

The angular momentum of the field source, found from ω_1, is equal to

$$J = CL\mu(4\mu^2-1)/6, \quad a = J/M = C(4\mu^2-1)/3. \qquad (47)$$

The next terms of the expansion of ω_i do not contribute to this quantity, since we require that, for them, $\lambda_{i1} \to o(r^{-1})$ as $v \to \infty$. Thus, for example, in (32) we set equal to zero the term that could have had an effect on J.

The dragging of space-time by the rotating source (the singularity $v = 0$) falls off both with distance from the source and with distance from the center of the singularity in the direction of its ends. The latter can be seen from the fact that

$$\omega_1 \underset{v \to 0}{\longrightarrow} -4\mu(1+2\mu^2) C \cos^2 u. \qquad (48)$$

Near the singularity, because of this drag, there exists a region in which $g_{\varphi\varphi}$ becomes positive, implying the possibility that closed timelike geodesics appear, and, consequently, that the principle of causality is violated. The boundaries of this region in coordinate space have the shape of a spindle with its end points at the ends of the singularity. This can be seen from the fact that near these ends at $\rho = 0$, $z = z_0 = \pm L/2$ the surface $g_{\varphi\varphi} = 0$ satisfies the condition

$$\frac{\partial \rho}{\partial (z - z_0)} \sim v^{2\mu} \underset{v \to 0}{\longrightarrow} 0.$$

The region of violation of causality cannot extend to infinity. For $C \ll L$ and $u = 0$ the boundary of this region reaches the greatest values of the coordinate v:

$$v_{max} \approx [2^{-2\mu} 8\mu(1+2\mu) CL^{-1}]^{1/(2\mu-1)}. \qquad (49)$$

The solution constructed is of interest to us as a possible exterior solution for prolate, uniformly rotating, axially symmetric bodies. The existence of the region of violation of causality cannot impede this interpretation, which can be preserved by requiring simply that this region be situated entirely inside the central body. Here, the minimum length that the "equator" of the central body must have for this increases with increase of the angular momentum of the body.

6. GENERALIZATION OF THE SOLUTION OBTAINED FOR PROLATE BODIES

We return to the question of the free choice of the set of multipoles for $\varkappa$ and ω, our analysis of which was interrupted in the third section. As a first step we shall show that in the framework of the scheme (34)–(39) for construction of the solution by the method of perturbation theory it is possible to introduce just half of these multipoles, i.e., the terms of the series (7) and (16), (20), taken with arbitrary coefficients. Here, in order to avoid repeated introduction of the same multipoles, we must adhere to certain rules. Terms with the function $Q_{2n}(\cosh v)$ appear in the expansion for $\varkappa$, with arbitrary coefficient α_n, at the first opportunity, i.e., in the terms $p_{n,n+1}(v)$. In all the subsequent terms of the series (35) [p_{ik} with $i \neq n$ or $k \neq n+1$], the coefficient with which they appear is fixed from the condition that the corresponding logarithmic divergences are absent and

$$p_{ik} \underset{v \to \infty}{\longrightarrow} 0 \quad \text{for} \quad k > n+1.$$

The first term of the series (7), which determines the mass, is specified from the beginning in (9) and does not change further. Therefore, in $\varkappa_1$ terms proportional to ln tanh$(v/2)$ should be absent.

Analogously, terms of the form

$$\text{ch}^{2-4n}(v/2) F(2n+2\mu, 2n-1, 4n, \text{ch}^{-2}(v/2))$$

should appear, with arbitrary coefficient β_n, only in the terms λ_{nn} from (34). For $k \neq i$ in (39) we should fix the coefficients using the condition

$$\lambda_{ik} \underset{v \to \infty}{\longrightarrow} O(x^{-2k-2}).$$

The sequences α_n and β_n [or the coefficients C_2 for $k = i+1$ in (38) and C_4 for $k = i$ in (39)] ensure two infinite sets of arbitrary quantities. In the analysis of the space-time around a prolate rotating body they should ensure the possibility of joining with the interior solution, if the latter is symmetric under reflections $u \to -u$. This requirement stems from the fact that the solution (10), (15), (34), (35) includes only the even terms of the series (7) and the odd terms of the series (16). As a result, it possesses the above-mentioned symmetry and can be joined only with the corresponding interior solution.

The series (15) for our solution will converge only for restricted values of the multipole moments. In view of the fact that the parameter C becomes one of many, one may think of abandoning the expansion (34), (35) in powers of C and going over directly to the self-consistent solution. Making use of the form of the dependence (34) and (35), we set

$$\omega = \sum_{i=1}^{\infty} A_i(v) \cos^{2i} u, \quad \varkappa = \sum_{i=0}^{\infty} B_i(v) \cos^{2i} u. \qquad (50)$$

Substituting this into Eqs. (12) and (13) and expanding the latter in powers of $\cos^2 u$, we obtain the conditions

$$A_i'' + A_i'(4\mu\, \text{sh}^{-1} v - \text{cth}\, v) - 2i(2i-1)A_i + 4i(i+1)A_{i+1}$$
$$+2\sum_{l=1}^{i}\{B'_{l-1}A'_{i-l+1} + 4lA_l[(i-l)B_{i-l} - (i+1-l)B_{i+1-l}]\} = 0, \qquad (51)$$

$$B_i''+B_i' \operatorname{cth} v-2i(2i+1)B_i-4(i+1)B_{i+1}$$

$$=\Lambda(v)\sum_{k=0}^{i}D_{i-k}\Big\{4(k+1)A_1A_{k+1}$$

$$+\sum_{l=1}^{k}[A_l'A_{k+1-l}'+4(k+1-l)A_{k+1-l}((l+1)A_{l+1}-lA_l)]\Big\},$$

$$\Lambda(v)=4L^{-2}\operatorname{th}^{4\mu}(v/2)\operatorname{sh}^{-2}v, \tag{52}$$

where we have used the notation

$$\sum_{i=0}^{\infty}D_i\tau^i=\exp\Big(2\sum_{j=0}^{\infty}B_j\tau^j\Big). \tag{53}$$

These conditions are extremely complicated. Even the first of them have the form

$$A_1''+A_1'(4\mu \operatorname{sh}^{-1}v-\operatorname{cth} v)+2B_0'A_1'+8A_2=2A_1+8A_1B_1,$$

$$B_0''+B_0'\operatorname{cth} v+4B_1=4\Lambda(v)A_1^2 \tag{54}$$

and do not enable us to find an equation for any one of the functions A_i and B_i.

By including in the analysis all the terms of the series (7) and (16), we shall seek the solution in the most general form

$$\omega=\cos^2 u\sum_{i=0}^{\infty}A_i(v)\sin^i u,\qquad \varkappa=\sum_{i=0}^{\infty}B_i(v)\sin^i u, \tag{55}$$

which follows from (16). The equations (12) and (13), after substitution of (55) and expansion in powers of sin u, will give the conditions

$$A_i''-A_{i-2}''+(A_i'-A_{i-2}')(4\mu \operatorname{sh}^{-1}v-\operatorname{cth} v)+2(1+4i-i^2)A_i$$
$$+(i^2-11i+16)A_{i-2}+(i+1)(i+2)A_{i+2}+2(i+1)B_1A_{i+1}$$
$$+2\sum_{k=0}^{i}\{B_{i-k}'(A_k'-A_{k-2}')+A_{i-k}[2(1+k-i)kB_k$$
$$+(k-2)(i+k-2)B_{k-2}$$
$$+(k+2)(i-k)B_{k+2}]\}=0, \tag{56}$$

$$B_i''+B_i'\operatorname{cth} v+(i+1)(i+2)B_{i+2}-i(i+1)B_i$$
$$=\Lambda(v)\sum_{k=0}^{i}D_{i-k}\Big\{A_1(k+1)A_{k+1}+\sum_{l=0}^{k}[A_{k-l}'(A_l'+A_{l-2}')$$
$$+A_{k-l}(l(2+l-k)(2A_l-A_{l-2})+(l+2)(k-l)A_{l+2})]\Big\}, \tag{57}$$

where $\Lambda(v)$ is the same as in (52). D_i is defined in (53), and functions with negative indices should be assumed to be equal to zero. Naturally, this system of equations cannot be solved either. Only in the case of small multipole moments can be obtain $\varkappa$ and ω in the form of the series (15) by means of perturbation theory. Here, we must start from

$$\omega_1=\cos^2 u(Cq(v)+C_2q_2(v)\sin u), \tag{58}$$

where $q(v)$ must be taken from (21), and

$$q_2(v)=\operatorname{ch}^3 v+3\mu \operatorname{ch}^2 v+(4\mu^2-1)\operatorname{ch} v+\mu(8\mu^2-5)/3$$
$$+(\mu-\operatorname{ch} v)\operatorname{sh}^2 v \operatorname{th}^{-4\mu}(v/2). \tag{59}$$

Then, using (12) and (13) alternately, we can obtain the next terms of the series. The principle of the choice of the coefficients in the solution of these equations remains as before: Upon the first appearance of terms of the form (7) in $\varkappa_i$ or of the form (16) in ω_i these coefficients are taken to be arbitrary, and upon their next appearance they are found from the same considerations as for the mirror-symmetric case. The mass of the singularities for $\omega=0$, the first correction to this quantity, and the angular momentum of the source will be determined, as before, by Eqs. (45)–(47).

7. RELATIONSHIP OF THE SOLUTION OBTAINED TO THE VAN STOCKUM METRIC

In this section we consider the relationship of the solution obtained to the singularity in the form of a filament of finite length with an exterior solution for an infinitely long rotating dust cylinder, obtained by van Stockum.[6] The van Stockum metric has the form

$$ds^2=F(r)dt^2-2M(r)dtd\varphi-L(r)d\varphi^2-H(r)(dz^2+dr^2), \tag{60}$$

where the functions F, M, L, and H depend also on the parameters R and a, regarded as the radius and angular velocity of the rigidly rotating dust cylinder. For $r\leqslant R$ the components of the metric (60) and the density ρ and four-velocity u^i of the dust $[x^i=(t,r,\varphi,z)]$ have the form

$$H=\exp(-a^2r^2),\ L=r^2(1-a^2r^2),\ M=ar^2,\ F=1,$$
$$2\pi\rho=a^2\exp(a^2r^2),\ u^i=\delta_0^i. \tag{61}$$

Outside, at $r>R$, the exterior vacuum solution that joins with (61) at $r=R$ takes one of the following three forms, depending on the value of aR:

1) For $aR<1/2$,

$$H=\exp(-a^2R^2)(R/r)^{2a^2R^2},$$
$$L={}^1/_2 rR\operatorname{sh}(3\varepsilon+\theta)\operatorname{sh}^{-1}\varepsilon\operatorname{ch}^{-1}\varepsilon, \tag{62}$$
$$M=r\operatorname{sh}(\varepsilon+\theta)\operatorname{sh}^{-1}2\varepsilon,\ F=(r/R)\operatorname{sh}(\varepsilon-\theta)\operatorname{sh}^{-1}\varepsilon,$$
$$\theta=(1-4a^2R^2)^{1/2}\ln(r/R),\ \operatorname{th}\varepsilon=(1-4a^2R^2)^{1/2};$$

2) for $aR=1/2$,

$$H=e^{-1/4}(R/r)^{1/2},\ L={}^1/_4 rR[3+\ln(r/R)],$$
$$M={}^1/_2 r[1+\ln(r/R)],\ F=(r/R)[1-\ln(r/R)]; \tag{63}$$

3) for $aR>1/2$,

$$H=\exp(-a^2R^2)(R/r)^{2a^2R^2},\ L={}^1/_2 rR\sin(3\varepsilon+\theta)\operatorname{cosec}\varepsilon\sec\varepsilon,$$
$$M=r\sin(\varepsilon+\theta)\operatorname{cosec}2\varepsilon,\ F=(r/R)\sin(\varepsilon-\theta)\operatorname{cosec}\varepsilon, \tag{64}$$
$$\theta=(4a^2R^2-1)^{1/2}\ln(r/R),\ \operatorname{tg}\varepsilon=(4a^2R^2-1)^{1/2}.$$

We shall consider these solutions in more detail. Since they all depend only on one spatial variable r, each of them can be transformed, by the transformation

$$x=x(r),\ t^*=\alpha(R,a)t+\beta(R,a)\varphi,\ \varphi^*=\gamma(R,a)t+\delta(R,a)\varphi, \tag{65}$$

into one of the three well-known solutions (dependent on the spatial variable x) described in Ref. 7. These are the spatial Kasner solution

$$ds^2=x^{2p_1}dt^2-dx^2-x^{2p_2}dy^2-x^{2p_3}dz^2, \tag{66}$$

$$p_1+p_2+p_3=p_1^2+p_2^2+p_3^2=1,$$

the spatial Kasner solution with complex exponents ($p_{1,2}=p'+ip''$)

$$ds^2=x^{2p'}[(du^2-dv^2)\cos\psi-2\sin\psi du dv]-dx^2-x^{2p_3}dz^2,$$

$$\psi=2p''\ln(x/\alpha),\ \alpha=\text{const},\ 2p'+p_3=2p'^2-2p''^2+p_3^2=1, \tag{67}$$

and the spatial Kasner solution with equal exponents [$p_1=p_2$; (p_1,p_3) is equal to (0,1) or (2/3, − 1/3); α is a constant]

$$ds^2=-2x^{2p_1}du dv-dx^2\pm x^{2p_1}\ln(x/\alpha)dv^2-x^{2p_3}dz^2. \tag{68}$$

Apart from these, there exist only two other metrics that depend only on the spatial coordinate x and have a nonzero component g_{xx}. They are regular in the entire space-time and have the form

$$ds^2=-dx^2+e^{px}[\cos(3^{1/2}px)(du^2-dv^2) -2\sin(3^{1/2}px)du dv]-e^{-2px}dz^2,\quad p=\text{const}, \tag{69}$$

$$ds^2=-dx^2+2dx_1dx_3-dx_2^2+2xdx_2dx_3 -(C+x^2/2)dx_3^2,\quad C=\text{const}. \tag{70}$$

The solution (64) for $aR\neq 1$ goes over into (62), with

$$x=\text{const}\cdot r^{1/(1-p_3)}=\text{const}\cdot r^{1-a^2R^2}. \tag{71}$$

Therefore, for $aR>1$ the only (and true) singularity $x=0$ goes over into $r=\infty$ (this was noted by Bonnor in Ref. 8), and this points to the incorrectness of the usual interpretation of (64) as the exterior solution around a rotating cylinder for this case. For $aR=1$ this solution goes over into (69) with $p=\exp(1/2)R^{-1}$.

The solution (62) goes over into (66). But in the interpretation of the latter a problem arises, as was pointed out by Bonnor in Ref. 8. The metric (62) possesses a timelike Killing field orthogonal to the hypersurface $r=\text{const}$, and consequently, is static. In fact, by the transformation (65) it reduces to the explicitly static solution (66), which describes space-time around an infinitely long line singularity with constant mass density.[9] An analogous problem also arises in the interpretation of the solution (63). Although, for this solution, the Killing field is null or lightlike, it is equivalent to the metric (68) [with $p_1=2/3$ (Ref. 8)] that was investigated in Ref. 10. This solution describes a strong zero-frequency gravitational wave in the field of an infinitely long filament with $p_1=2/3$. In this case too, rotation is absent. This is manifested also in the absence of rotation of the geodesics in the space-time (63).

We shall show that the absence of rotation for $aR\leqslant 1/2$ is a manifestation of Mach's principle. In other words, the rotation of an infinitely long and massive source leads to drag of the entire space into rotation with the same angular frequency. For this we shall examine into what our solution (10) goes over as the source is lengthened. After the introduction of new coordinates

$$2\rho=L^\alpha\,\mathrm{sh}\,v,\quad 2z=L^\alpha\sin u,\quad t^*=L^{-\alpha\mu}t,$$

$$\varphi^*=\varphi L^\beta$$

with

$$\alpha=(1-\mu+\mu^2)^{-1},\quad \beta=\alpha\mu^2$$

in the limit $L\to\infty$ it tends to the metric

$$ds^2=\rho^{2\mu}dt^{*2}-\rho^{2\mu^2-2\mu}(d\rho^2+dz^2)-\rho^{2-2\mu}d\varphi^{*2}. \tag{72}$$

In fact, for the first term on the right-hand side of (10),

$$\mathrm{th}^{2\mu}(v/2)(dt-\omega\,d\varphi)^2\xrightarrow[L\to\infty]{}\rho^{2\mu}(dt^*-L^{-\mu\alpha(1+\mu)}\omega\,d\varphi^*)^2. \tag{73}$$

If the ratio J/M (and hence the angular velocity of rotation of the singularity) does not change as the singularity is lengthened, then, according to (47), the quantities C and ω do not change either. In view of this, when the condition (44) is fulfilled the coefficient of $d\varphi^*$ in (73) tends to zero as $L\to\infty$. Therefore, for any angular velocity of the source the solution (10) tends to the static metric (72), which is equivalent to the Kasner spatial metric (66) (Ref. 9). The metrics (62) and (72) are connected by the transformation (65), with

$$\mu=\tfrac{1}{2}[1-(1-4a^2R^2)^{1/2}] \tag{74}$$

and the condition $0<aR<1/2$ leads to the condition (44).

Thus, for $0<\mu<1/2$, in the rotation of an infinitely long line source the entire space-time is dragged by it into rotation with the same angular frequency. Therefore, in this case the space-time is described by the static solution (66), (72). But in the van Stockum metric (62) it is considered in the rotating system of coordinates (65). The formally obtained quantity J/M (Refs. 6 and 8) reflects only the angular velocity of this system and has no relation to the angular velocity of the source.

For $\mu=\frac{1}{2}$ ($p_1=\frac{2}{3}$) the solution (66) joins with (67) and (68). With increase of μ a qualitative change of the exterior solution occurs at this point. To investigate this change, it would be extremely useful to find the solution (with a line source of finite length L) that goes over into the metric (67) as $L\to\infty$. Unlike (67), this solution should not belong to the Lewis class, since, for the latter, sources of finite length are not admissible. It is not difficult to convince oneself of this by making use of the results of Ref. 11.

Thus, we have established the relationship of the solution obtained to the well-known van Stockum metric. The difficulties that appear in the interpretation of the latter have been overcome. At the same time, a number of results associated with our solution have been found not to be correct. These are the formulas for the specific angular momentum of the source and the consideration of the exterior solution for $aR\gg 1$.

On further conclusion, important for the analysis of the exterior solution obtained in this paper for prolate bodies, is that the conditions (44) for its existence are not accidental and do not decrease the region of its applicability.

8. OBLATE ROTATING BODIES

Earlier in the paper we considered the case of prolate bodies. But, perhaps, the approach used is also applicable to oblate, axially symmetric, uniformly rotating bodies. In this case, it is natural to perform the analysis in the coordinates of an oblate ellipsoid of revolution in a conventional flat space with

$$\rho=R\cos u\,\mathrm{ch}\,v,\quad z=R\sin u\,\mathrm{sh}\,v. \qquad (75)$$

For $\omega=0$ the solution of Eq. (3) will be

$$\nu=\sum_{n=0}^{\infty}\alpha_n P_n(\sin u)Q_n(i\,\mathrm{sh}\,v). \qquad (76)$$

We separate out the first term of the series, the coefficient μ of which is related to the mass of the source:

$$\nu_0=\mu[\pi-2\,\mathrm{arctg}(\mathrm{sh}\,v)]. \qquad (77)$$

Using Eqs. (5) we can find the related function

$$\gamma_0=\mu^2\ln(1-\cos^2 u\,\mathrm{ch}^{-2}v). \qquad (78)$$

Introducing the notation

$$\nu=\nu_0+\varkappa,\quad \gamma=\gamma_0+\xi, \qquad (79)$$

we bring the metric (2) to the form

$$\begin{aligned} ds^2&=\Phi(u,v)[dt-\omega(u,v)d\varphi]^2-R^2\Phi(u,v)^{-1}\mathrm{ch}^2 v\\ &\times[e^{\xi(u,v)}(1-\cos^2u\,\mathrm{ch}^{-2}v)^{1+\mu^2}(du^2+dv^2)+\cos^2u\cdot d\varphi^2],\\ \Phi(u,v)&=\exp[\varkappa(u,v)]\exp[\mu\pi-2\mu\,\mathrm{arctg}(\mathrm{sh}\,v)]. \end{aligned} \qquad (80)$$

For $\omega=\varkappa=0$ it reduces to the well-known solution investigated in Ref. 2. Its source is the ring $v=u=0$, which has radius R in the conventional coordinate space and mass $M=\mu R$.

For the metric (82) the conditions (3) and (4) take the form

$$\begin{aligned} &\varkappa_{,uu}+\varkappa_{,vv}+\varkappa_{,v}\,\mathrm{th}\,v-\varkappa_{,u}\,\mathrm{tg}\,u\\ &=R^{-2}\Phi(u,v)^2\cos^{-2}u\,\mathrm{ch}^{-2}v(\omega_{,u}{}^2+\omega_{,v}{}^2), \end{aligned} \qquad (81)$$

$$\omega_{,uu}+\omega_{,vv}+\omega_{,u}(2\varkappa_{,u}+\mathrm{tg}\,u)+\omega_{,v}(4\mu\,\mathrm{ch}^{-1}v-\mathrm{th}\,v+2\varkappa_{,v})=0. \qquad (82)$$

For $\varkappa=0$ Eq. (82) has a (regular at $u=\pm\pi/2$) solution of the form (16) with

$$\begin{aligned} q_n(v)&=\beta_nF(n,-1-n,-2i\mu,0{,}5+i\,\mathrm{sh}\,v)\\ &-\gamma_n(1+4\,\mathrm{sh}^2v)\exp[-4\mu\,\mathrm{arctg}(2\,\mathrm{sh}\,v)]\\ &\times F(1-n,n+2,2+2i\mu,0{,}5+i\,\mathrm{sh}\,v). \end{aligned} \qquad (83)$$

The solution that falls off as $v\to\infty$ is real. Thus, for $n=1$ we have

$$\begin{aligned} q_1=C\{&(1+4\,\mathrm{sh}^2v)\exp[-4\mu\,\mathrm{arctg}(2\,\mathrm{sh}\,v)]\\ &-\exp(-2\pi\mu)[4(\mathrm{sh}\,v+\mu)^2+4\mu^2+1]\}. \end{aligned} \qquad (84)$$

It is easy to convince oneself, on the basis of the form of the dependences of (76) and (16) on u, that for the case of oblate bodies as well the solution must be sought in the form (55). The relations that are then obtained between A_i and B_i from the conditions (81) and (82) will differ from (56) and (57) only in the replacement $\sinh v\leftrightarrow\cosh v$ and in the different function

$$\Lambda(v)=R^{-2}\mathrm{ch}^{-2}v\cdot\exp[\mu\pi-2\mu\,\mathrm{arctg}(\mathrm{sh}\,v)]. \qquad (85)$$

In the case of mirror symmetry with respect to the plane $u=0$ the solution can be written in the form (50). The relations between A_i and B_i will be obtained from (51) and (52) after the analogous replacements.

The difference between the cases of prolate and oblate bodies is associated with the possibility of applying perturbation theory. By substituting (84) into the right-hand side of (81), it is not difficult to convince oneself that the latter will have a lower power of v than does the abbreviated expression on the left-hand side for Δv_0. Thus, no arbitrarily small rotation can be regarded, for $v\to 0$, as a perturbation of the original static metric. However, if the central oblate body has finite size, application of perturbation theory is possible, provided that not only $C\ll R$ but also

$$C^2\ll R^2 v_{\min}\mu\exp(-2\pi\mu),$$

where $v_{\min}$ is the smallest value of the coordinate v on the surface of the body. In this case, the analysis does not differ fundamentally from that in the case of prolate singularities, and requires cumbersome calculations.

From the form of the dependence (16) or (84) it is not difficult to find

$$\begin{aligned} J&=2\mu RC(4\mu^2+1)\exp(-2\pi\mu)/3,\\ J/M&=2C(4\mu^2+1)\exp(-2\pi\mu)/3. \end{aligned} \qquad (86)$$

The quantity J/M increases with increase of μ, and J has a maximum at $\mu\approx 0.67$. It is also possible to show that, when the condition for applicability of perturbation theory is fulfilled, the space-time cannot possess closed timelike geodesics.

9. CONCLUSION

The solution constructed in the paper for prolate sources is a generalization of the Zipoy–Voorhees metric (1) for the case of rotation. Written in the form (10), (55), it is, when the conditions (12)–(14) are fulfilled, the exact vacuum solution of the Einstein equations. However, the resulting system of ordinary differential equations (56), (57) is too complicated to be solved. At the same time, in the case of small multipole moments of the mass (starting from the dipole moment) and of the angular momentum, the desired metric can be obtained in the form of a series of successive approximations (15). However, even in the simplest case of a source that is symmetric under reflection $u\to -u$ the resulting terms of the series are described by extremely unwieldy expressions.

If we consider our solution in the entire space-time, it will possess a rotating bare line singularity of finite length. Around this singularity will lie a region of possible violation of causality.

The metric obtained is a candidate for the role of the exterior solution for prolate, axially symmetric, uniformly rotating bodies, under the condition that no part of the region of violation of causality lies outside the body. The possibility of joining this metric with an appropriate interior metric should be ensured by the circumstance that it contains two infinite series of arbitrary parameters. However, even in the simplest central-body variants, the determination of both solutions is a very difficult problem. In particular, the difficulties of solving this problem are great because the solution is obtained in the form of a series and because of the complexity of the resulting expressions. However, until at least one example of a complete solution has been constructed, the metric found in this paper can be regarded only as a candidate for the role of the exterior solution around rotating bodies.

The same can also be said of the exterior solution for prolate bodies, considered in Sec. 8. While the formulas obtained are similar, the types of the two singularities are very different. This is manifested, in particular, in the fact that the exterior solution with rotation can be constructed by means of perturbation theory from the original static metric only in the case of a central oblate source of finite size.

We note that for a prolate source of finite size violation of the condition (44) is possible. Since negative values of μ are related to a negative mass of the source, they do not have physical meaning. The violation can be associated only with values $\mu \geqslant 1/2$. However, with decrease of the width of the source to zero or increase of its length to infinity the condition (44) is obligatory.

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