The electromagnetic collapse problem

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We consider the electric field generation and charged particle acceleration mechanism due to the transverse oscillations of a current-carrying plasma. The proposed mechanism is responsible for the origin of accelerated particles under laboratory and space conditions.

Acceleration of charged particles to energies many times larger than the applied potential difference has been observed in powerful discharges in z-pinch type devices. The mechanism for the corresponding accelerating electromagnetic fields is apparently connected with the development of two-dimensional plasma instabilities. We show in the present paper that a dynamic acceleration effect exists also for one-dimensional transverse waves of plasma configurations with a longitudinal current.

We use the formalism of two-fluid relativistic electromagnetic gas dynamics. (REMGD) to study one-dimensional motions of plane and cylindrical currents. In the case of equal masses ($m_+ = m_-$) and equal temperatures ($T_+ = T_-$), the problem can be simplified considerably and becomes a quasi-one-fluid one. In the linear approximation it reduces to the solution of two ordinary second-order differential equations for the transverse velocity and the longitudinal electric field.

In the nonrelativistic approximation one reaches a similar simplification also for $m_+ \neq m_-$ for a constant temperature ratio, $T_+/T_- = \text{const}$. Exact self-similar solutions can then be found, describing the collapse of a planar current layer and the radial oscillations of a cylindrical current. Of most interest for the problem of the dynamic particle acceleration are the transverse motions of self-contained configurations with a conserved longitudinal current. The self-similar solution for a layer does not satisfy this condition, and it is of interest for the problem of charged particle acceleration in the case of an outside build-up of the current. In the self-similar homogeneous waves of a cylinder the current is conserved but there is no acceleration. In the general case the transverse motions of self-contained configurations have a wave nature where for the basic transverse wave modes the particle acceleration in the inner region is accompanied by a deceleration in the outer region in compression and vice versa in expansion. The acceleration occurring in that case is connected with redistribution of the current density over its cross-section, and the effective accelerating electric field is proportional to the square of the current.

The basic problem of the acceleration of charged particles up to the superhigh energies characteristic for cosmic rays is the generation of accelerating electric fields arising a result of plasma dynamics.[1–3] As z-pinch experiments show, the amplitude of the electric field and the energy of the accelerated particles increase proportionally to the square of the current in the discharge. Extrapolating the experimental scaling to the high-current region one easily obtains the superhigh energies for the accelerated particles observed in cosmic rays. In conclusion one should note that the dynamic particle acceleration for the case of radial oscillations of the z-pinch does not occur in the framework of ideal MHD since by definition the accelerating longitudinal field,

$$E_z{}^* = E_z + v_r B_\varphi / c$$

vanishes.

1. EQUATIONS OF DISSIPATIONLESS TWO-FLUID REMGD

If there are no dissipative processes the set of REMGD equations has the form[4]

$$\operatorname{div}\mathbf{B}=0,\quad \operatorname{div}\mathbf{E}=4\pi e\left(\frac{n_+}{\Gamma_+}-\frac{n_-}{\Gamma_-}\right),\quad \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=-\operatorname{rot}\mathbf{E},$$

$$\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\operatorname{rot}\mathbf{B}-\frac{4\pi e}{c}\left(\frac{n_+\mathbf{v}_+}{\Gamma_+}-\frac{n_-\mathbf{v}_-}{\Gamma_-}\right), \tag{1}$$

$$\frac{\rho}{\Gamma}\frac{d}{dt}\frac{W\mathbf{v}}{\Gamma}=-\nabla p\pm\frac{en}{\Gamma}\mathbf{E}^*,\quad \frac{\partial}{\partial t}\frac{n}{\Gamma}+\operatorname{div}\frac{n\mathbf{v}}{\Gamma}=0,$$

$$\frac{dS}{dt}=0. \tag{2}$$

Here Eqs. (1) are a complete set of Maxwell equations and the relativistically invariant gas dynamic equations (2) are a double set of equations for the ion and electron gases with masses $m_\pm$ and charges $e_\pm = \pm e$. The quantities $\rho = mn, p = nT$, and T are, respectively, the invariant densities, pressures, and temperatures,

$$\Gamma=(1-v^2/c^2)^{1/2},\quad \mathbf{E}^*=\mathbf{E}+\frac{1}{c}[\mathbf{vB}],$$

and W and S are the enthalpy and the entropy, given by the relations

$$W=1+\frac{\gamma}{\gamma-1}\frac{p}{\rho c^2},\quad T\,dS=c^2dW-\frac{dp}{\rho},$$

where γ is the adiabatic index.

A consequence of Eqs. (1) and (2) is the energy conservation law,

$$\frac{\partial}{\partial t}\left\{\Sigma\left(\frac{\rho W c^2}{\Gamma^2}-p\right)+\frac{E^2+B^2}{8\pi}\right\}+\operatorname{div}\left(\Sigma\frac{\rho W c^2}{\Gamma^2}\mathbf{v}+\frac{c}{4\pi}[\mathbf{EB}]\right)=0.$$

The total energy density is the sum of the rest-mass, kinetic, thermal, and electromagnetic energy densities,

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$$\varepsilon=\Sigma\left(\frac{\rho Wc^2}{\Gamma^2}-p\right)+\frac{E^2+B^2}{8\pi}=\Sigma(\varepsilon_0+\varepsilon_K+\varepsilon_T)+\varepsilon_{EB},$$

where

$$\varepsilon_0=\frac{\rho c^2}{\Gamma},\quad \varepsilon_K=\varepsilon_0\left(\frac{1}{\Gamma}-1\right),\quad \varepsilon_T=\frac{p}{\Gamma^2}\left(\frac{1}{\gamma-1}+\frac{v^2}{c^2}\right),$$
$$\varepsilon_{EB}=\frac{E^2+B^2}{8\pi}.$$

We can write the gas dynamic equations (2) for the variables $N=n/\Gamma$, $M=mW/\Gamma$, $c_S^2=\gamma p/\rho W$, where the N are the densities, the M are the effective particle masses, and c_S is the sound speed, in evolutionary form:

$$\frac{dN}{dt}=-N\operatorname{div}\mathbf{v},\quad \frac{dp}{dt}=-\frac{\gamma p}{\lambda}\left(\operatorname{div}\mathbf{v}-\frac{\mathbf{qv}}{MNc^2}\right),\quad (2a)$$
$$\frac{d\mathbf{v}}{dt}=-\frac{\nabla p}{MN}+\frac{e}{M}\mathbf{E}^*+\frac{\mathbf{v}}{\lambda MNc^2}(\gamma p\operatorname{div}\mathbf{v}+\Gamma_S^2\mathbf{qv}).$$

Here we have

$$\mathbf{q}=\nabla p-eN\mathbf{E},\quad \lambda=1-\beta^2\beta_S^2,\quad \beta=v/c,\quad \beta_S=c_S/c,$$
$$\Gamma=(1-\beta^2)^{1/2},\quad \Gamma_S=(1-\beta_S^2)^{1/2}.$$

2. ONE-DIMENSIONAL REMGD EQUATIONS

For the one-dimensional problem in the coordinates $x^i=x^1, x^2, x^3, \partial/\partial x^2=\partial/\partial x^3=0$ the components of the electromagnetic field are given by the formulae

$$E^1=-\frac{\partial\Phi}{\partial x^1},\quad E^2=-\frac{1}{c}\frac{\partial A^2}{\partial t},\quad E^3=-\frac{1}{c}\frac{\partial A^3}{\partial t},$$
$$B_1=0,\quad B^2=-\frac{\partial A_3}{\partial x^1},$$
$$B^3=\frac{1}{g^{1/2}}\frac{\partial A_2}{\partial x^1},\quad A_1=0,$$

where g is the determinant of the metric tensor, and the set of the two-fluid REMGD equations can be written in the form

$$\Delta\Phi=-4\pi e(N_+-N_-),\quad \Box^*A_2=-\frac{4\pi e}{c}(N_+v_2^+-N_-v_2^-),$$
$$\Box A_3=\frac{4\pi e}{c}(N_+v_3^+-N_-v_3^-),$$
$$\left(\frac{dM\mathbf{v}}{dt}\right)^1=-\frac{1}{N}\frac{\partial p}{\partial x^1}\pm e\left(E^1+\frac{v^i}{c}\frac{\partial A_i}{\partial x^1}\right),$$
$$\frac{d}{dt}\left(M\mathbf{v}\pm\frac{e}{c}\mathbf{A}\right)_{2,3}=0,\quad (3)$$
$$\frac{dN}{dt}=-\frac{N}{g^{1/2}}\frac{\partial g^{1/2}v^1}{\partial x^1},\quad \frac{d}{dt}pn^{-\gamma}=0,$$

where

$$\Box=\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2},\quad \Box^*=\Delta^*-\frac{1}{c^2}\frac{\partial^2}{\partial t^2},\quad \Delta=\frac{1}{g^{1/2}}\frac{\partial}{\partial x^1}g^{1/2}\frac{\partial}{\partial x^1},$$
$$\Delta^*=g^{1/2}\frac{\partial}{\partial x^1}\frac{1}{g^{1/2}}\frac{\partial}{\partial x^1}.$$

This set describes one-dimensional motions of a plane layer with $x^i=(x,y,z)$, $g=1$, or of a cylinder with $x^i=(r,\varphi,z)$, $g=r^2$.

One sees easily that in the case of equal masses, $m_+=m_-$, the set (3) has solutions in the case of equal temperatures and oppositely directed longitudinal velocities, $v_{2,3}^+=-v_{2,3}^-$, in which coupled transverse motion ($v_+^1=v_-^1$) occurs. In the case considered we put $m_\pm=m$, $v_\pm^1=v$, $v_{2,3}^\pm=\pm V_{2,3}$, $N_\pm=N$, $p_\pm=0$, $E_1=0$, and the set (3) reduces to a set of seven quasi-one-fluid equations for the functions N, p, v, $V_{2,3}$, and $A_{2,3}$:

$$\Box^*A_2=-\frac{8\pi e}{c}NV_2,\quad \Box A_3=-\frac{8\pi e}{c}NV_3,$$
$$\frac{d}{dt}\left(MV+\frac{e}{c}A\right)_{2,3}=0,$$
$$\left(\frac{d}{dt}Mv\right)^1=-\frac{1}{N}\frac{\partial p}{\partial x^1}+\frac{e}{c}V^i\frac{\partial A_i}{\partial x^1},\quad \frac{dN}{dt}=-N\operatorname{div}\mathbf{v},\quad (3a)$$
$$\frac{d}{dt}pn^{-\gamma}=0.$$

In the nonrelativistic approximation, $\beta^2\ll 1, \beta_S^2\ll 1$, the difference in the densities is relativistically small and one needs to take it into account only when determining the charge density. The set of Eqs. (3) for the one-dimensional motion then takes the form

$$\operatorname{div}\mathbf{E}=4\pi e(N_+-N_-),\quad \Delta^*A_2=-\frac{4\pi e}{c}Nu_2,$$
$$\Delta A_3=-\frac{4\pi e}{c}Nu_3,$$
$$\frac{d}{dt}\left(mv\pm\frac{e}{c}A\right)_{2,3}=0,\quad (4)$$
$$m\left(\frac{d\mathbf{v}}{dt}\right)^1=-\frac{1}{N}\frac{\partial p}{\partial x^1}\pm e\left(E^1+\frac{v^i}{c}\frac{\partial A_i}{\partial x^1}\right),$$
$$\frac{dN}{dt}=-N\operatorname{div}\mathbf{v},\quad \frac{d}{dt}pN^{-\gamma}=0,$$

where $\mathbf{u}=\mathbf{v}_+-\mathbf{v}_-$ is the current velocity.

The set obtained here has a solution with coupled transverse motion, $v_+^1=v_-^1$, for the ions and electrons with $T_+/T_-=\text{const}$ when the center of mass is conserved, $(m_+v^++m_-v^-)_{2,3}=0$. The corresponding generalization of the quasi-one-fluid equations (3a) to the case of unequal masses, $m_+\neq m_-$, leads to the system

$$\Delta^*A_2=-\frac{4\pi e}{c}Nu_2,\quad \Delta A_3=-\frac{4\pi e}{c}Nu_3,$$
$$m_\Sigma\left(\frac{d\mathbf{v}}{dt}\right)^1=-\frac{1}{N}\frac{\partial p_\Sigma}{\partial x^1}+\frac{e}{c}u^i\frac{\partial A_i}{\partial x^1},\quad (4a)$$
$$\frac{d}{dt}\left(\bar{m}u+\frac{e}{c}A\right)_{2,3}=0,\quad \frac{dN}{dt}=-\frac{N}{g^{1/2}}\frac{\partial}{\partial x^1}g^{1/2}v,$$
$$\frac{d}{dt}p_\Sigma N^{-\gamma}=0.$$

Here we have

$$m_\Sigma=m_++m_-,\quad \bar m^{-1}=m_+^{-1}+m_-^{-1},\quad p_\Sigma=p_++p_-,\quad v_\pm{}^1=v,$$

$$v_{2,3}^\pm=V_{2,3}^\pm,\qquad \mathbf{u}=\mathbf{V}_+-\mathbf{V}_-.$$

In the case considered there is for $m_+\neq m_-$ a transverse electric field. The charge density determined by the equation $\operatorname{div}\mathbf{E}=4\pi e(N_+-N_-)$ is relativistically small and vanishes in the case of equal masses and equal temperatures.

In what follows we restrict ourselves to transverse motions of a purely longitudinal current, $V=\mathbf{e}_3V^3$, in the case where there is no longitudinal magnetic field, $B_3=0$, when we have $A_2=0$, $B=\mathbf{e}_2B^2$, $A=\mathbf{e}_3A^3$. Linearization of the quasi-one-fluid REMGD set (3a) in the vicinity of the equilibrium state $B'=8\pi eNV/c, p'=-eNVB/c$ for perturbations proportional to $e^{-i\omega t}$ then yields a set of two second-order ordinary differential equations for the transverse velocity $v=v^1$ and the longitudinal electric field $E=E^3$:

$$\begin{aligned}&(g^{1/2}F')'+\left(\frac{\omega^2}{c^2}-\frac{2\varkappa^2\Gamma^2}{\lambda}\right)g^{1/2}F\\&\quad=-\frac{j\Gamma_s^2}{\lambda}f'-\left(j'-\frac{2\varkappa^2}{\lambda}\Gamma^2H\right)f,\\&\left(\frac{\gamma p}{\lambda g^{1/2}}f'\right)'+\left[\frac{MN\omega^2}{g^{1/2}}-\frac{j\Gamma_s^2}{2\lambda}\left(\frac{H}{g^{1/2}}\right)'-\frac{H}{g^{1/2}}Q\right]f\\&\quad=\frac{j\Gamma_s^2}{2\lambda}F'+QF.\end{aligned}\tag{5}$$

Here the prime indicates differentiation with respect to x^1, and we have

$$H=B/(4\pi)^{1/2},\quad F=cE/(4\pi)^{1/2},\quad f=g^{1/2}v,\quad g^{1/2}j=(g^{1/2}H)',$$
$$p'=-jH/2,\quad \varkappa^2=4\pi e^2N/Mc^2,\quad Q=\varkappa^2\Gamma^2H/\lambda-(j\Gamma^2\beta_s^2/2\lambda)'.$$

The perturbations of the other variables $\tilde N, \tilde p, \tilde B$, and $\tilde V$ can be expressed in terms of v and E through the relations

$$\begin{aligned}&\frac{\partial N}{\partial t}=-\frac{1}{g^{1/2}}(g^{1/2}Nv)',\quad \frac{dp}{dt}=-\frac{\gamma p}{\lambda}\left(v'+\frac{eV}{Mc^2}E^*\right),\\&\frac{\partial B}{\partial t}=cE',\quad \frac{dV}{dt}=\frac{\Gamma^2}{\lambda}\left(\beta_s^2Vv'+\frac{e}{M}E^*\right),\end{aligned}\tag{5a}$$

where $E^*=E+vB/c$ is the accelerating longitudinal field.

3. SELF-SIMILAR SOLUTIONS

1. In the case of a uniform longitudinal current, $N=N(t)$, $u=u(t)$, the set of quasi-one-fluid nonrelativistic equations (4a) has an exact symmetric solution satisfying the condition $p(x)=0$ at the free boundaries $x=\pm a$:

$$\begin{aligned}&v=\frac{\dot a}{a}x,\quad N=N_0\frac{a_0}{a},\quad u=u_0\frac{a_0}{a},\quad p=p_0\left(\frac{a_0}{a}\right)^\gamma\left(1-\frac{x^2}{a^2}\right),\\&eA_z=-\bar m cu\left(1+\frac{\varkappa_\Sigma^2x^2}{2}\right),\quad 2\ddot a=a_0u_0^2\varkappa_0^2\left[\left(\frac{a_0}{a}\right)^\gamma-\frac{a_0^2}{a^2}\right],\end{aligned}\tag{6}$$

where we have $\varkappa_0^2=8\pi e^2N_0/m_\Sigma c^2$, $\varkappa_\Sigma^2=\varkappa_+^2+\varkappa_-^2$. Calculating the electric and magnetic fields leads to

$$E_z=-\frac{\bar mu\dot a}{ea}(1+\varkappa_\Sigma^2x^2),\quad B_y=\frac{\bar mcu}{e}\varkappa_\Sigma^2x,\quad E_z^*=-\frac{\bar mu\dot a}{ea}.$$

In the present nonrelativistic approximation, when the current layer is compressed we find an unlimited uniform particle acceleration, $m_\pm\dot V_\pm=\pm E^*(t)$. The total current increases in that case, $J=J_6a_0/a$, and to obtain collapse one therefore needs an external action leading to an increase of the current according to the relation $Ja=\text{const}$.

The integral of the last equation (6) describes the motion of the boundaries $x=\pm a(t)$ of the layer in the potential field,

$$\begin{aligned}&\ddot a=-U'(a),\quad \frac{\dot a^2}{2}+U(a)=\frac{\dot a_0^2}{2},\\&U=\frac{1}{2}a_0^2u_0^2\varkappa_0^2\left\{\frac{1}{\gamma-1}\left[\left(\frac{a_0}{a}\right)^{\gamma-1}-1\right]+1-\frac{a_0}{a}\right\}.\end{aligned}\tag{7}$$

The impossibility of exceeding the speed of light in the relativistic formulation of the problem restricts the collapse to the magnitude $a/a_0\sim\beta_0$.

2. If the condition $p'(r)/rN(r)=\text{const}$ is satisfied in the equilibrium state for a cylinder with a longitudinal current the set (4a) has an exact solution describing uniform pulsations of the cylinder:

$$\begin{aligned}&v=\frac{\dot a}{a}r,\quad V^\pm=V_0^\pm(\xi),\quad A=A_0(\xi),\\&N=\frac{a_0^2}{a^2}N_0(\xi),\quad p_\Sigma=\left(\frac{a_0}{a}\right)^{2\gamma}p_{0\Sigma}(\xi),\end{aligned}\tag{8}$$

$$p_{0\Sigma}'(\xi)=\frac{e}{c}N_0u_0A_0'(\xi),\quad [\xi A_0'(\xi)]'=-\frac{4\pi e}{c}a^2Nu_0(\xi)\xi,$$
$$\ddot a=-\frac{p_{0\Sigma}'(\xi)}{am_\Sigma\xi N_0(\xi)}\left[\left(\frac{a_0}{a}\right)^{2\gamma-2}-1\right].$$

Here we have $\xi=r/a$, the longitudinal field $E_z=-vB_\varphi/c$ can be expressed by the same formula as in MHD, the total current does not change, $dJ/dt=0$, and there is no acceleration, $dV_\pm/dt=0$.

In the special case of a uniform current we have, according to (8),

$$\begin{aligned}&v=\frac{\dot a}{a}r,\quad u=u_0,\quad N=N_0\left(\frac{a_0}{a}\right)^2,\quad p=p_0\left(\frac{a_0}{a}\right)^{2\gamma}\left(1-\frac{r^2}{a^2}\right),\\&A=-\frac{\pi eNu}{c}r^2,\quad \ddot a=\frac{1}{4}\varkappa_0^2u^2a_0\left[\left(\frac{a_0}{a}\right)^{2\gamma-1}-\frac{a_0}{a}\right].\end{aligned}\tag{8a}$$

The square of the frequency of the small oscillations is $\omega^2=(\gamma-1)\varkappa_0^2u^2/2>0$. The integral of the last equation (8a) gives an expression for the potential which determines the nonlinear oscillations of the boundary of the cylinder, $r=a(t)$:

$$\begin{aligned}&\ddot a=-U'(a),\quad \frac{\dot a^2}{2}+U(a)=0,\quad U(a)=-\frac{1}{4}\varkappa_0^2u^2a_0^2\left\{\ln\frac{a_{1,2}}{a}\right.\\&\left.\quad-\frac{1}{2\gamma-2}\left[\left(\frac{a_0}{a}\right)^{2\gamma-2}-\left(\frac{a_0}{a_{1,2}}\right)^{2\gamma-2}\right]\right\},\end{aligned}\tag{9}$$

where a_1 and a_2 are, respectively, the maximum and minimum radii of the cylinder, which are connected through the relation

$$(a_0/a_1)^{2\gamma-2}-(a_0/a_2)^{2\gamma-2}=\ln(a_2/a_1)^{2\gamma-2}.$$

In contrast to a layer, the self-similar solutions for a cylindrical current thus describe oscillations satisfying the conditions of current conservation and the absence of particle acceleration.

4. NONRELATIVISTIC LINEAR THEORY

In the nonrelativistic approximation the set of linear equations following from (4a) is analogous to (5) for the transverse motions of a longitudinal current and it has the form

$$\begin{cases} (g^{1/2}F')'-\varkappa_\Sigma^2 g^{1/2}F=\varkappa_\Sigma^2 Hf-(jf)', \\ \left(\dfrac{\gamma p_\Sigma}{g^{1/2}}f'\right)'+\left[m_\Sigma N\omega^2-g^{1/2}j\left(\dfrac{H}{g^{1/2}}\right)'-\varkappa_\Sigma^2H^2\right]\dfrac{f}{g^{1/2}}=\varkappa_\Sigma^2 HF+jF'. \end{cases} \tag{10}$$

In the equilibrium state we have here

$$g^{1/2}j=(g^{1/2}H)',\quad p_\Sigma'=-jH,\quad p(a)=0.$$

The boundary condition at the free boundary $x^1=a$ for F, corresponding to a self-contained current, follows from the first equation (10), if we require that the derivative of the external longitudinal electric field vanishes, $F_e'=0$. This implies the condition $F'=-jf/g^{1/2}$ for $x^1=a$ which, according to (5a), is equivalent to the conservation of the total current, $Ja=\text{const}$.

The introduction of a new variable ε which is proportional to the acting field E^* and the acceleration,

$$\varepsilon=F+fH/g^{1/2}=cE^*/(4\pi)^{1/2}=mc\dot{V}/(4\pi)^{1/2}e,$$

changes (10) to the set

$$\begin{cases} (g^{1/2}\varepsilon')'-\varkappa_\Sigma^2 g^{1/2}\varepsilon=\left[gH\left(\dfrac{f}{g}\right)'\right]', \\ \left[(\gamma p_\Sigma+H^2)\dfrac{f'}{g^{1/2}}\right]'+\left[m_\Sigma N\omega^2-H\left(\dfrac{g'H}{g}\right)'\right]\dfrac{f}{g^{1/2}}=\dfrac{1}{g}(gH\varepsilon')'. \end{cases} \tag{10a}$$

The boundary condition for ε corresponding to current conservation will clearly be $\varepsilon'=g^{1/2}H(f/g)'$ for $x^1=a$.

To transform (10a) to standard form we introduce a "potential function" defined by the equation

$$\varphi=\int_0^{x_1}\bar{N}\varepsilon g^{1/2}\,dx^1,$$

where $\bar{N}=N/N_0$ is the normalized density.

As a result we get a set of equations for φ and f:

$$\begin{cases} \left(\dfrac{\varphi'}{g^{1/2}\bar{N}}\right)'-\dfrac{\varkappa_\Sigma^2}{g^{1/2}\bar{N}}\varphi=g^{1/2}H\left(\dfrac{f}{g}\right)', \\ \left(\dfrac{\gamma p_\Sigma}{g^{1/2}}f'\right)'+\left(m_\Sigma N\omega^2+\dfrac{g'}{g}jH\right)\dfrac{f}{g^{1/2}}=\dfrac{\varkappa_\Sigma^2}{g\bar{N}}(g^{1/2}H\varphi)', \end{cases} \tag{10b}$$

where the boundary condition corresponding to the condition that the solution is self-contained is $\varphi(a)=0$. The boundary condition for φ, in conjunction with the condition $p(a)=0$ for the equilibrium pressure on the boundary with the vacuum, is sufficient to find a unique solution of the set (10b). The eigenvalue problem obtained in this way has a solution in the form of a discrete set of eigenfunctions for the different wave modes.

The solution of the set of Eqs. (10) can be simplified in the limiting cases of low and high running densities when we have $\varkappa_\Sigma^2a^2\ll1$ and $\varkappa_\Sigma^2a^2\gg1$. The equation for f then becomes independent of F and the equation for F reduces to a quadrature. Indeed, using (10a) and (10b) we get

a) $\varkappa_\Sigma^2a^2\ll1$:

$$\left(\frac{\gamma p_\Sigma}{g^{1/2}}f'\right)'+\left(m_\Sigma N\omega^2+\frac{g'}{g}jH\right)\frac{f}{g^{1/2}}=0,\quad \varepsilon'=g^{1/2}H\left(\frac{f}{g}\right)'. \tag{11a}$$

b) $\varkappa_\Sigma^2a^2\gg1$:

$$\left[(\gamma p_\Sigma+H^2)\frac{f'}{g^{1/2}}\right]'+\left[m_\Sigma N\omega^2-H\left(\frac{g'H}{g}\right)'\right]\frac{f}{g^{1/2}}=0,$$

$$\varphi=-\frac{\bar{N}gH}{\varkappa_\Sigma^2}\left(\frac{f}{g}\right)'. \tag{11b}$$

In the limit as $\varkappa_\Sigma^2a^2\to\infty$ we have $\varphi\to0$ so that $E^*\to0$ and Eq. (11b) for f is the same as the corresponding equation in classical one-fluid MHD.

ONE-DIMENSIONAL SYMMETRIC MOTIONS OF A PLANE LAYER

In the case of a uniform longitudinal current when we have $N=\text{const}$, $V=\text{const}$, $H=hx$, $j=h$, $p_\Sigma=h^2(1-x^2)/2$ the first two modes of the solutions of the set (10) can be expressed by the formulae

$$f_1=\nu x,\quad \varepsilon_1=-\frac{\nu h}{\varkappa_\Sigma^2},$$

$$f_2=\nu(x+\alpha x^3),\quad \varepsilon_2=-\frac{\nu h}{\varkappa_\Sigma^2}\left[1+9\alpha\left(\frac{2}{\varkappa_\Sigma^2}+x^2\right)\right],$$

$$\alpha=\frac{5}{3}\frac{2-\gamma}{\gamma+12/\varkappa_\Sigma^2}>0, \tag{12}$$

where we have

$$a=1,\quad \Omega^2=m_\Sigma N\omega^2/h^2=l(2l-1)(\gamma-2)<0,\quad l=1,2,\ldots$$

When the first mode is realized the particles are accelerated uniformly, while in the higher modes the acceleration is an increasing function of x. The solutions are unstable and do not satisfy the constant current condition. The growth rates for the development of the instability increase with the mode number.

a) For $\varkappa_\Sigma^2a^2\ll1$ the solutions of Eq. (11a) for $f(x)$ are the Legendre polynomials $P_l(x)$. Hence, the eigenfunctions of the system (11a) which satisfy the condition for self-containment can be written in the form

$$f_1=\nu x,\quad \varepsilon_1=-\frac{h\nu}{6}(1-3x^2),$$

$$f_3=\nu\left(x-\frac{5x^3}{3}\right),\quad \varepsilon_3=\frac{h\nu}{12}(1+6x^2-15x^4),$$

$$f_5=\nu\left(x-\frac{14x^3}{3}+\frac{21x^5}{5}\right), \tag{13}$$

$$\varepsilon_5=\frac{h\nu}{30}(1+15x^2-105x^4+105x^6).$$

Here we have

$$a=1,\quad \Omega^2=\gamma(l+1)l/2>0,\quad l=1,3,5\ldots$$

The solutions describe stable oscillations with frequencies which increase with the number l of the mode. The acceleration is an alternating function $\varepsilon_l(x)$ with a vanishing average value:

$$\int_0^1 \varepsilon_l(x)dx=0.$$

The function $f_l(x)$ satisfies for $l>1$ the relation

$$\int_0^1 f_l(x)x^2\,dx=0.$$

b) To obtain analytical solutions of the set (11b) for $\varkappa_\Sigma^2 a^2 \gg 1$ we restrict ourselves to the $\gamma=2$ case. In that case

$$f=\nu\sin\Omega x,\quad \varepsilon=-\frac{h\nu\Omega}{\varkappa_\Sigma^2}(\cos\Omega x-\Omega x\sin\Omega x), \tag{14}$$

where we have $\Omega^2=m_\Sigma N\omega^2/h^2$. The expression for the square of the frequency, obtained from the requirement that the self-containment condition $f'=(a)=0$ be satisfied, has the form

$$\Omega^2=\pi l(2l-1)/2a,\quad l=1,2,3\ldots$$

The solution thus has an oscillatory nature and the accelerating field E^* decreases when the parameter $\varkappa_\Sigma^2 a^2$ increases.

In dimensional variables the accelerating field and the frequency of the oscillations for the first mode (13) are given by the formulae

$$E^*(x)=-\frac{\dot aB(a)}{6c}\left(1-\frac{3x^2}{a^2}\right),\quad \omega=\frac{\gamma^{1/2}B(a)}{(4\pi\rho_\Sigma)^{1/2}a}.$$

During the compression, when we have $\dot a<0$, the particles are accelerated in the central $x<a/3^{1/2}$ region while they are slowed down for $a/3^{1/2}<x<a$. Putting $\dot a\approx\omega a$ we get the following expression:

$$E^*\approx-\frac{\gamma^{1/2}eB^2(a)}{6\varkappa_0 m_\Sigma c^2}\left(1-\frac{3x^2}{a^2}\right),$$

where we have $\varkappa_0^2=4\pi e^2N/m_\Sigma c^2$. From this it is clear that the accelerating field is proportional to the square of the total current, $E^*\propto J^2/N^{1/2}$.

ONE-DIMENSIONAL RADIAL PULSATIONS OF A CYLINDER

Again restricting ourselves to the case of a uniform longitudinal current,

$$N=\text{const},\quad V=\text{const},\quad H=hr,\quad j=2h,\quad p_\Sigma=h^2(1-r^2),$$

we get an analytical solution of the set (10) for the $l=1,2$ modes in the form

$$f_1=\nu r^2,\quad \varepsilon_1=0,$$

$$f_2=\nu(r^2+\alpha r^4),\quad \varepsilon_2=-\frac{8\alpha h\nu}{\varkappa_\Sigma^2}\left(\frac{4}{\varkappa_\Sigma^2}+r^2\right),$$

$$\alpha=-\frac{3}{2}\frac{\gamma-1}{\gamma+8/\varkappa_\Sigma^2}<0. \tag{15}$$

Here we have

$$a=1,\quad \Omega^2=4l^2(\gamma-1)>0,\quad l=1,2,3\ldots$$

The current conservation condition is satisfied for the first mode and the particles are not accelerated. In the second mode we have acceleration but the current is not conserved. The solutions (15) have an oscillatory nature.

a) The eigenfunctions $f_l(r)$ and $\varepsilon_l(r)$ of the set (11a) which satisfy the self-containment condition $\varphi(a)=0$ for the current are for $\varkappa_\Sigma^2 a^2\ll 1$ given by the formulae

$$\begin{aligned} &f_1=\nu r^2,\quad \varepsilon_1=0,\\ &f_2=\nu\left(r^2-\frac{3r^4}{2}\right),\quad \varepsilon_2=\frac{h\nu}{4}(1-3r^4),\\ &f_3=\nu\left(r^2-4r^4+\frac{10r^6}{3}\right),\quad \varepsilon_3=\frac{h\nu}{9}(1-18r^4+20r^6).\end{aligned} \tag{16}$$

Here we have

$$a=1,\quad \Omega^2=4(\gamma l^2-1)>0,\quad l=1,2,3\ldots$$

The ground-state mode corresponds to uniform waves without acceleration. In the higher modes there is acceleration and they are described by alternating functions $\varepsilon_l(r)$ and $f_l(r)$ satisfying the relations

$$\int_0^1 \varepsilon_l r\,dr=0,\quad \int_0^1 f_l r\,dr=0.$$

b) We can obtain analytical solutions of the set (11a) for $\varkappa_\Sigma^2 a^2\gg 1$ only for $\gamma=1$. In that case we have

$$f=\nu rJ_1(\Omega r),\quad \varepsilon=-\frac{h\nu}{\varkappa_\Sigma^2}\Omega rJ_1(\Omega r), \tag{17}$$

where $\Omega^2=m_\Sigma N\omega^2/h^2$. In order that the self-containment condition $(f/r^2)'=0$ be satisfied for $r=1$ it is necessary that Ω is a root of the equation $\Omega J_0(\Omega)-2J_1(\Omega)=0$. The solution has an oscillatory nature and the accelerating field decreases when the parameter $\varkappa_\Sigma a$ increases.

In dimensional variables the accelerating field and the frequency of the oscillations for the second mode can be written in the form

$$E^*(r)=-\frac{\dot aB(a)}{2c}\left(1-\frac{3r^4}{a^4}\right),\quad \omega=\frac{(4\gamma-1)^{1/2}B(a)}{(\pi\rho_\Sigma)^{1/2}a}.$$

When the cylinder is compressed, $\dot a<0$, the particles are accelerated in the central $r<a/3^{1/2}$ region and slowed down for $a/3^{1/2}<r<a$. Putting $\dot a\approx\omega a$ we find the expression

$$E^*(r)\approx-\frac{(4\gamma-1)^{1/2}eB^2(a)}{\varkappa_0 m_\Sigma c^2}\left(1-\frac{3r^4}{a^4}\right).$$

As for the oscillations of a layer we have $E^*\propto J^2/N^{1/2}$.

Problems of the two-dimensional dynamics of the plasma and of z-pinch type configurations, in the framework of two-fluid REMGD, were considered, in particular, in Refs. 5 and 6.

CONCLUSION

Up to now the cause of the occurrence of anomalously accelerated charged particles in z-pinch type devices had not been given an unambiguous theoretical explanation. The mechanism for the generation of accelerating electromagnetic fields is usually connected with the development of two-dimensional plasma instabilities and the effect of the anomalous resistivity. However, in view of the complex geometry there are considerable mathematical difficulties for an analytical study of the corresponding dynamical problems. In the present paper we considered the very simple problems of one-dimensional transverse motions of a layer and of a cylinder with a longitudinal current, assuming the plasma to be ideally conducting. We showed that in the case of equal particle masses, $m_+ = m_-$, the equations of two-fluid relativistic EMGD have a class of solutions satisfying a system of quasi-one-fluid equations. In the nonrelativistic approximation a similar result obtains for arbitrary mass ratios m_+/m_-.

We obtained self-similar solutions of the corresponding nonrelativistic set of equations which describe the collapse of the layer and nonlinear oscillations of the cylinder. The use of a developed mathematical formalism enabled us to explain the presence of the dynamical charged particle acceleration effect which has no analog in the framework of classical one-fluid MHD. A study of the quasi-one-fluid system of EMGD equations shows that there are solutions satisfying the self-containment condition ($J = \text{const}$) in which the electrons and the ions are accelerated in opposite directions while the acceleration is proportional to the square of the total current and inversely proportional to the plasma density, in agreement with the experimental results. The accelerating electromagnetic field is in this case an alternating function of the radius with a number of nodes which is determined by the number of the mode of the radial oscillations.

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