# A new $1/n$-expansion technique

S. S. Stepanov and R. S. Tutik

*Dnepropetrovsk State University*

(Submitted 15 February 1991)
Zh. Eksp. Teor. Fiz. **100**, 415–421 (August 1991)

An efficient method is developed for calculating the coefficients of the $1/N$ expansion of arbitrarily high orders both for the ground states and for the radially excited states of the discrete spectrum of the Schrödinger equation. The method is based on a semiclassical interpretation of the $1/N$ expansion. The explicit use of expansion in Planck's constant clarifies the reason for the complementarity of the $1/N$ approach and the WKB approximation. Going over to the Riccati equation and using the $\hbar$ expansion make it possible to apply the quantization conditions to take account of the nodes of the wave function, and this leads to simple recursion relations. For the example of the funnel potential calculations are given of the first ten coefficients in the $1/n$-expansion scheme for the energy for different values of the orbital and radial quantum numbers.

## 1. INTRODUCTION

Recently, one of the most widely used methods for investigating the bound states of the Schrödinger equation

$$-\frac{\hbar^2}{2m}U''(r)+\left[V(r)+\frac{\hbar^2 l(l+1)}{2mr^2}-E\right]U(r)=0 \qquad (1)$$

has been the $1/N$ expansion, the present state and the history of the development of which are reflected in the reviews in Refs. 1 and 2. At the basis of this semiclassical method is the analysis of the classical motion of a particle at the bottom of the potential well formed by the effective potential

$$V_{\text{eff}}(r)=V(r)+\frac{\hbar^2 l(l+1)}{2mr^2}$$

of Eq. (1), with subsequent allowance for quantum fluctuations and anharmonicity effects.

There exist two equivalent approaches to the construction of $1/N$ expansions. In one of them the passage to the classical limit is executed by letting the dimensionality of space tend to infinity ($D\to\infty$). The expansion is performed in the small parameter $1/(l+D/2)$ (Ref. 3) or $1/(l-a+D/2)$ (Ref. 4) ($l$ is is the orbital angular momentum and $a$ is a shift parameter that improves the convergence). The physical dimensionality of space is restored in the final formulas by the substitution $D=3$.

Another approach, proposed in Refs. 5, uses the passage to the classical limit as the orbital angular momentum tends to infinity. But, for a fixed radial quantum number $n_r$, if $l\to\infty$ the principal quantum number $n=(n_r+l+1)\to\infty$ as well. The resulting small parameter $1/n$ is chosen as the expansion parameter, and this is why this method is called the $1/n$ expansion.

Both these methods, which differ in essence only in the choice of the expansion parameter, take anharmonicity effects into account by using the logarithmic,[3,4,6,7] the Rayleigh-Schrödinger,[8–10] or the hypervirial,[11–13] perturbation-theory scheme. The resulting $1/N$ expansions often diverge asymptotically,[3] and, for the application of the latest methods of summation of divergent series, high orders of the expansion have to be evaluated. However, corrections of high orders have been considered only for a narrow class of potentials,[2,3,7,8,11,14] in view of the difficulties that arise in their calculation. As noted in Ref. 1, when Rayleigh-Schrödinger perturbation theory is used it is difficult to advance beyond the first few terms of the $1/N$ expansion. Logarithmic perturbation theory, based on the Riccati equation, leads to simple recursion relations in the case of the ground states, but becomes very unwieldy in the description of the radial excitations.[3,4] Here, complications arise because of the separation of the zeros of the wave function in the form of an individual factor.

There is, however, another way of taking the zeros of the wave function into account—namely, by introducing the quantization condition[15,16] by analogy with the WKB approximation. This approach requires the explicit use of expansions in Planck's constant. Together with the results of Ref. 17, it enables one to understand better the semiclassical nature of $1/N$ expansions. In addition, it should clarify why the $1/N$ expansion and the WKB approximation, as semiclassical methods, are complementary to each other.

The present paper is devoted, first, to a semiclassical interpretation of $1/N$ expansions that explicitly uses expansion in $\hbar$, and, second, to an account of a method of simple recursion relations that permit one to calculate $1/N$ corrections of arbitrary order for the energies of both the ground states and the radially excited bound states.

The proposed technique is a further development of the $\hbar$-expansion method recently applied[18] to an investigation of bound states in the $(l, E)$ plane.

## 2. THE $\hbar$-EXPANSION METHOD

Because of the nonuniqueness of the passage to classical mechanics, following Ref. 14 we write

$$\hbar^2 l(l+1)\equiv\Lambda^2+\hbar A\Lambda+\hbar^2 B, \qquad (2)$$

which brings the Schrödinger equation (1) to the form

$$-\frac{\hbar^2}{2m}U''(r)+\left[\frac{\Lambda^2}{2mr^2}\left(1+\hbar\frac{A}{\Lambda}+\hbar^2\frac{B}{\Lambda^2}\right)+V(r)-E\right]U(r)=0. \qquad (3)$$

Next, after certain simplifying substitutions, the solution is usually constructed in the form of a series in powers of $1/\Lambda$. Here, the specific variant of the $1/N$ method is determined by the choice of the parameters $A$ and $B$; in particular,

 0038-5646/91/080227-04$03.00 © 1991 American Institute of Physics 

$$A=1-D, \qquad B=\frac{D(D-2)}{4},$$
$$\Lambda=\hbar\left(l+\frac{D}{2}\right) \text{ [Ref. 3]},$$
$$A=1-D+2a, \quad B=\frac{(D-2a)(D-2a-2)}{4},$$
$$\Lambda=\hbar\left(l-a+\frac{D}{2}\right) \text{ [Ref. 4]},$$
$$A=-1, \quad B=0, \quad \Lambda=\hbar(l+1) \text{ [Ref. 5]},$$
$$A=0, \quad B=0, \quad \Lambda=[\hbar^2 l(l+1)]^{1/2} \text{ [Refs. 14,19]}. \tag{4}$$

However, as can be seen from Eq. (3), $\Lambda$ appears in the expansion parameter in the combination $\hbar/\Lambda$. Consequently, it is possible to choose another route and to perform the expansions in powers of Planck's constant.

Using logarithmic perturbation theory, after the substitution $\mathbb{C}(r)=\hbar U'(r)/U(r)$ we go over from Eq. (3) to the Riccati equation

$$\hbar\mathbb{C}'(r)+\mathbb{C}^2(r)=\frac{\Lambda^2}{r^2}+2m(V(r)-E)+\frac{1}{r^2}(\hbar A\Lambda+\hbar^2 B). \tag{5}$$

We represent the function $\mathbb{C}(r)$ and the energy $E$ in the form of asymptotic series in $\hbar$:

$$\mathbb{C}(r)=\sum_{k=0}^{\infty}\mathbb{C}_k(r)\hbar^k, \tag{6}$$

$$E=\sum_{k=0}^{\infty}E_k\hbar^k. \tag{7}$$

In the classical limit ($\hbar\to 0$), which specifies the zeroth approximation, for the energy we have

$$E_0=V(r_0)+\frac{\Lambda^2}{2mr_0^2}, \tag{8}$$

which corresponds to the motion of a classical particle in a stable circular orbit. The radius $r_0$ of this orbit is determined by the position of the minimum of the effective potential, and is found from the equation

$$mr_0^3V'(r_0)=\Lambda^2. \tag{9}$$

Substituting the expansions (6) and (7) into (5), we arrive at the chain of equations

$$\begin{aligned}
&\mathbb{C}_0^2(r)=2m[V(r)-E_0]+\frac{\Lambda^2}{r^2},\\
&\mathbb{C}_0'(r)+2\mathbb{C}_0(r)\mathbb{C}_1(r)=\left(\frac{r_0}{r}\right)^2\gamma_1-2mE_1,\\
&\qquad\qquad\cdots\cdots\cdots\\
&\mathbb{C}_{k-1}'(r)+\sum_{i=0}^{k}\mathbb{C}_i(r)\mathbb{C}_{k-i}(r)=\left(\frac{r_0}{r}\right)^2\gamma_k-2mE_k,
\end{aligned} \tag{10}$$

where

$$\gamma_1=A\Lambda/r_0^2, \quad \gamma_2=B/r_0^2, \quad \gamma_3=\gamma_4=\ldots=0.$$

To take anharmonicity effects into account it is convenient to move the coordinate origin to the point $r=r_0$ and to expand the effective potential in a Taylor series in $x=(r-r_0)/r_0$. Then the first equation of the system (10) gives

$$\mathbb{C}_0(x)=-\omega_0 x(1+a_1x+a_2x^2+\ldots)^{1/2}, \tag{11}$$

where the minus sign ensures fulfillment of the boundary conditions and we have introduced the notation

$$\omega_0^2=2m(V_2+{}^3/_2V_1), \tag{12}$$

$$a_k=\frac{2m}{\omega_0^2}\left(V_{k+2}+(-1)^k\frac{k+3}{2}V_k\right), \quad V_k=r_0^kV^{(k)}(r_0)/k!. \tag{13}$$

At the same time, for $\mathbb{C}_k(x)$ we have

$$\mathbb{C}_k(x)=-\frac{1}{2\mathbb{C}_0(x)}\left[2mE_k-\frac{\gamma_k}{(1+x)^2}+\frac{\mathbb{C}_{k-1}'(x)}{r_0}+\sum_{i=1}^{k-1}\mathbb{C}_i(x)\mathbb{C}_{k-i}(x)\right]. \tag{14}$$

It is clear that the recursion relations (14) coincide with the relations obtained by means of the standard $1/N$-expansion technique for ground states.[3,4] In the standard approach, however, complications appear in the description of the radial excitations. Application of the $\hbar$ expansion makes it possible to circumvent these difficulties using the quantization conditions.

In the case of a radially excited state the wave function $U(r)$ of Eq. (1) has exactly $n_r$ real zeros. Therefore, its logarithmic derivative $\mathbb{C}(r)$ has $n_r$ simple poles at these points. Taking into account that the residue at each of these poles is equal to $\hbar$, we arrive at the well known Zwaan-Dunham quantization conditions[15,16]

$$\frac{1}{2\pi i}\oint\mathbb{C}(r)dr=n_r\hbar, \qquad n_r=0,1,2,\ldots, \tag{15}$$

where the integration contour encloses only the above-mentioned nodes of the wave function.

The conditions (15) are exact, and are used to find corrections to the WKB approximation.[16,20] But in the WKB method the passage to the classical limit is implemented using the rule

$$\hbar\to 0, \quad n_r\to\infty, \quad \hbar n_r=\text{const}. \tag{16}$$

Being complementary to the WKB approximation, the $1/N$-expansion method under consideration requires that the rule for passage to the classical limit be

$$\hbar\to 0, \quad n_r=\text{const}, \quad \hbar n_r\to 0. \tag{17}$$

In this case, after substitution of the expansion (6), the quantization conditions (15) give

$$\begin{aligned}
&\frac{1}{2\pi i}\oint\mathbb{C}_1(r)dr=n_r,\\
&\frac{1}{2\pi i}\oint\mathbb{C}_k(r)dr=0, \qquad k\neq 1.
\end{aligned} \tag{18}$$

A further application of the theorem of residues to the explicit form (14) of the functions $\mathbb{C}_k(r)$ solves the problem of describing the radially excited states.

From the expression (11) it can be seen that for the function $\mathbb{C}_0(x)$ the point $x=0$ will be a simple zero. But then the function $\mathbb{C}_k(x)$ has a pole of order $2k-1$ at this point, and consequently can be represented in the neighborhood of this point by a Laurent series:

$$\mathbb{C}_k(x)=x^{1-2k}\sum_{\alpha=0}^{\infty}\mathbb{C}_\alpha^{\,k}x^\alpha. \tag{19}$$

According to the theorem of residues, the quantization conditions (18), expressed in terms of the coefficients of the Laurent series (19), take the form

$$\operatorname{Res}\mathbb{C}_1(0)=\mathbb{C}_0^{\,1}=\frac{n_r}{r_0}, \tag{20}$$

$$\operatorname{Res}\mathbb{C}_k(0)=\mathbb{C}_{2k-2}^{\,k}=0, \quad k\neq 1. \tag{21}$$

To unify the notation, we write the expansion (11) of the function $\mathbb{C}_0(x)$ as

$$\mathbb{C}_0(x)=x\sum_{\alpha=0}^{\infty}\mathbb{C}_\alpha^{\,0}x^\alpha, \tag{22}$$

where the coefficients $\mathbb{C}_\alpha^{\,0}$ are related to $\omega_0$ and $a_\alpha$ by

$$\begin{aligned}&\mathbb{C}_0^{\,0}=-\omega_0, \quad \mathbb{C}_1^{\,0}=-\omega_0 a_1/2,\\ &\mathbb{C}_\alpha^{\,0}=\frac{1}{\omega_0}\Big(\sum_{j=1}^{\alpha-1}\mathbb{C}_j^{\,0}\mathbb{C}_{\alpha-j}^{\,0}-\omega_0^2 a_\alpha\Big).\end{aligned} \tag{23}$$

Substituting the expansions (19) and (22) into (14) and equating coefficients of equal powers of $x$, for $\alpha\neq 2k-2$ we have

$$\begin{aligned}&\frac{3-2k+\alpha}{r_0}\mathbb{C}_\alpha^{\,k-1}+\sum_{j=0}^{k}\sum_{\beta=0}^{\alpha}\mathbb{C}_\beta^{\,j}\mathbb{C}_{\alpha-\beta}^{\,k-j}\\ &=\theta(\alpha-2k+2)(-1)^\alpha(\alpha-2k+3)\gamma_k,\end{aligned} \tag{24}$$

where we have introduced the Heaviside function

$$\theta(\alpha)=\begin{cases}1, & \alpha\geqslant 0,\\ 0, & \alpha<0.\end{cases} \tag{25}$$

From this we obtain a recursion formula for $\mathbb{C}_\alpha^{\,k}$:

$$\begin{aligned}\mathbb{C}_\alpha^{\,k}=\frac{1}{2\mathbb{C}_0^{\,0}}\Big[&\theta(\alpha-2k+2)(-1)^\alpha(\alpha-2k+3)\gamma_k-\frac{3-2k+\alpha}{r_0}\mathbb{C}_\alpha^{\,k-1}\\ &-\sum_{j=1}^{k-1}\sum_{\beta=0}^{\alpha}\mathbb{C}_\beta^{\,j}\mathbb{C}_{\alpha-\beta}^{\,k-j}-2\sum_{\beta=1}^{\alpha}\mathbb{C}_\beta^{\,0}\mathbb{C}_{\alpha-\beta}^{\,k}\Big].\end{aligned} \tag{26}$$

Using the coefficients $\mathbb{C}_\alpha^{\,0}$, $\mathbb{C}_0^{\,1}$, and $\mathbb{C}_{2k-2}^{k}$, given by the expressions (20), (21), and (23), from Eq. (26) we find all the remaining coefficients $\mathbb{C}_\alpha^{\,k}$ for $\alpha\neq 2k-2$.

If, however, $\alpha=2k-2$, from (14) we have, for $k=1$,

$$\operatorname{Res}\mathbb{C}_1(0)=-\frac{1}{2\omega_0}\Big[\gamma_1+\frac{\omega_0}{r_0}-2mE_1\Big], \tag{27}$$

which, after application of the condition (20), gives

$$2mE_1=\gamma_1+(2n_r+1)\frac{\omega_0}{r_0}. \tag{28}$$

In the case $k\neq 1$, after application of the condition $\mathbb{C}_{2k-2}^{k}=0$, it follows from the expression (14) that

$$2mE_k=\gamma_k-\frac{1}{r_0}\mathbb{C}_{2k-2}^{\,k-1}-\sum_{j=1}^{k-1}\sum_{\beta=0}^{2k-2}\mathbb{C}_\beta^{\,j}\mathbb{C}_{2k-2-\beta}^{\,k-j}-2\sum_{\beta=1}^{2k-2}\mathbb{C}_\beta^{\,0}\mathbb{C}_{2k-2-\beta}^{\,k}. \tag{29}$$

Thus, the expressions (26), (28), and (29) fully solve the problem of constructing the recursion formulas and of the expansion of the energies of the bound states in Planck's constant.

## 3. CALCULATION OF THE ENERGY

We write out in explicit form the first terms of the $\hbar$ expansion for the energy $E=E_0+\hbar E_1+\hbar^2E_2+\ldots$:

$$\begin{aligned}2mE_0&=2mV(r_0)+\Lambda^2/r_0^2,\\ 2mE_1&=\gamma_1+(2n_r+1)\,\omega_0/r_0,\\ 2mE_2&=\gamma_2-\Big(\frac{\gamma_1}{\omega_0}\Big)^2+\frac{3}{2r_0}\Big(\frac{\gamma_1}{\omega_0}\Big)(2n_r+1)(a_1+1)\\ &+\frac{1}{16r_0^2}[12(2n_r^2+2n_r+1)a_2-(30n_r^2+30n_r+11)a_1^2].\end{aligned} \tag{30}$$

The passage to any variant (4) of the $1/N$ expansion

$$E=\varepsilon^{(0)}+\varepsilon^{(1)}/\Lambda+\varepsilon^{(2)}/\Lambda^2+\ldots \tag{31}$$

is implemented using the formula

$$\varepsilon^{(k)}=(\hbar\Lambda)^kE_k. \tag{32}$$

As an example, we shall consider the $1/n$ expansion of the eigenvalues of the Schrödinger equation in the case of a funnel-type potential

TABLE I. Accuracy of the $1/n$ expansion for the funnel potential.

| $k$ | $\xi_{nl}^{(k)}$ | | | | | |
|---|---|---|---|---|---|---|
| | 1S | 1P | 1D | 2S | 2P | 3S |
| 1 | 0,996345 | 2,617172 | 3,698773 | 3,313040 | 4,321043 | 4,943314 |
| 2 | 0,979324 | 2,611439 | 3,695779 | 3,264759 | 4,302749 | 4,894418 |
| 3 | 0,979854 | 2,611013 | 3,695573 | 3,246268 | 4,298707 | 4,876274 |
| 4 | 0,980751 | 2,611125 | 3,695595 | 3,237220 | 4,297366 | 4,864382 |
| 5 | 0,980289 | 2,611142 | 3,695600 | 3,232818 | 4,296917 | 4,856608 |
| 6 | 0,980245 | 2,611130 | 3,695599 | 3,230711 | 4,296769 | 4,851525 |
| 7 | 0,980515 | 2,611128 | 3,695599 | 3,229716 | 3,296720 | 4,848205 |
| 8 | 0,980338 | 2,611131 | 3,695599 | 3,229249 | 4,296704 | 4,846040 |
| 9 | 0,980217 | 2,611132 | 3,695599 | 3,229031 | 4,296699 | 4,844633 |
| 10 | 0,980580 | 2,611130 | 3,695599 | 3,228932 | 4,296698 | 4,843722 |
| $\xi_{nl}^{\text{exact}}$ | 0,980366 | 2,611131 | 3,695599 | 3,228853 | 4,296697 | 4,842092 |

*Note.* $\xi_{nl}^{(k)}$ are successive sums of terms of the $1/n$ expansion: $\xi_{nl}^{(k)}=\varepsilon^{(0)}+\varepsilon^{(1)}n^{-1}+\ldots\varepsilon^{(k)}n^{-k}$; $\xi_{nl}^{\text{exact}}$ is the numerical solution of the Schrödinger equation.[21]

$$V(r)=-\varkappa/r+r/a^2. \tag{33}$$

Using the notation $v(r) = -V(r)$,

$$v_k=\frac{2r_0^2}{\Lambda^2(k+1)}V_k, \quad \omega=\omega_0\frac{r_0}{\Lambda}, \quad \varepsilon^{(k)}=2\Lambda^{k+2}E_k$$

and setting $1/n = h/\Lambda = 1/(p+l+1)$, where $p = n_r$, we convince ourselves that for $m = \hbar = 1$ the coefficients of the $\hbar$ expansion lead to the expressions for the $1/n$ expansion that were obtained in Ref. 2.

The rate of convergence of the $1/n$ expansion can be seen from the table, in which successive sums of terms of the series in powers of $1/n$ are presented. Here, the Schrödinger equation with the potential (33) was first brought to the standard form

$$\frac{d^2u}{d\rho^2}+\left[\xi+\frac{\lambda}{\rho}-\rho-\frac{l(l+1)}{\rho^2}\right]U=0, \tag{34}$$

where $\lambda=\varkappa(2ma)^{2/3}$, $\xi=(2ma^4)^{1/3}E$, and the calculations were performed with the following values of the parameters: $2M = 1.84$ GeV, $\varkappa = 0.52$, and $a = 2.34$ GeV, which corresponds to $\lambda = 1.37623$.

It can be seen from the table that the accuracy of the description of the energy spectrum of the Schrödinger equation by means of a sum of low-order terms of the $1/n$ expansion increases with increase of the orbital quantum number and decreases slightly with increase of the radial quantum number. The states with $l = 0$ are described least accurately. We note that the energy values calculated to order $(1/n)^2$ inclusive, and given in the second row of the table, coincide with the values obtained in Ref. 2.

Thus, a semiclassical interpretation (using expansion in Planck's constant) of the $1/N$ method has made explicit the reason why this method is complementary to the WKB approximation. The reason lies in the difference in the passages to the classical limit. The WKB approximation, using the rule $\hbar\to 0$, $n_r\to\infty$, $\hbar n_r = \text{const}$, becomes more accurate with increase of the radial quantum number, whereas for the $1/N$ expansion the rule for passage to the classical limit is $\hbar\to 0$, $\hbar n_r\to 0$, $n_r = \text{const}$. Here, the classical limit, which gives the principal contribution to the expansion, is reached more rapidly for states with small radial quantum numbers, which explains the more accurate description of the nodeless states.

In addition, application of the $\hbar$ expansion has made it possible to construct a new, effective algorithm for determining the coefficients of the $1/N$ expansion, based on the use of the quantization conditions. The recursion formulas obtained can be programmed easily, giving in analytical or numerical form the coefficients of any order for the $1/N$ expansion of the energy of both ground states and radially excited bound states. We note that the proposed algorithm is quite universal. Passage from one variant of the $1/N$ expansion to another reduces entirely to replacing the initial coefficients in the recursion relations.

The authors are grateful to G. M. Zinov'ev for his interest in the work and for useful discussions.

[1] A. Chatterjee, Phys. Rep. **186**, 249 (1990).
[2] V. D. Mur, V. S. Popov, and A. V. Sergeev, Zh. Eksp. Teor. Fiz. **97**, 32 (1990) [Sov. Phys. JETP **70**, 16 (1990)].
[3] L. D. Mlodinow and M. P. Shatz, J. Math. Phys. **25**, 943 (1984).
[4] U. Sukhatme and T. Imbo, Phys. Rev. D **28**, 418 (1983); T. Imbo, A. Pagnamenta, and U. Sukhatme, Phys. Rev. D **29**, 1669 (1984).
[5] V. S. Popov, V. M. Vaĭnberg, and V. D. Mur, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 439 (1985) [JETP Lett. **41**, 539 (1985)]; Yad. Fiz. **44**, 1103 (1986) [Sov. J. Nucl. Phys. **44**, 714 (1986)].
[6] S. Hikami and E. Brézin, J. Phys. A **12**, 759 (1979).
[7] A. D. Dolgov, V. L. Eletskiĭ, and V. S. Popov, Zh. Eksp. Teor. Fiz. **79**, 1704 (1980) [Sov. Phys. JETP **52**, 861 (1980)].
[8] A. V. Kudinov and M. A. Smondyrev, Teor. Mat. Fiz. **56**, 357 (1983) [Theor. Math. Phys. (USSR) **56**, 871 (1983)].
[9] A. Chatterjee, J. Phys. A **18**, 1193, 2403 (1985); Phys. Rev. A **34**, 2470 (1986).
[10] C. H. Lai, J. Math. Phys. **28**, 1801 (1987).
[11] S. A. Maluendes, F. M. Fernández, E. A. Mesón, and E. A. Castro, Phys. Rev. D **34**, 1835 (1986).
[12] A. Chatterjee, Phys. Rev. A **35**, 2772 (1987).
[13] R. Sever and C. Tezcan, Phys. Rev. A **37**, 3158 (1988).
[14] V. M. Vaĭnberg, V. D. Mur, V. S. Popov, A. V. Sergeev, and A. V. Shcheblykin, Teor. Mat. Fiz. **74**, 399 (1988) [Theor. Math. Phys. (USSR) **74**, 269 (1988)].
[15] A. Zwaan, Thesis, Utrecht (1929).
[16] J. L. Dunham, Phys. Rev. **41**, 713 (1932).
[17] V. D. Mur and V. S. Popov, Zh. Eksp. Teor. Fiz. **97**, 1729 (1990) [Sov. Phys. JETP **70**, 975 (1990)].
[18] N. A. Kobylinsky, S. S. Stepanov, and R. S. Tutik, Phys. Lett. B **235**, 182 (1990); Z. Phys. C **47**, 469 (1990); J. Phys. A **23**, L237 (1990).
[19] B. Baumgartner, H. Grosse, and A. Martin, Nucl. Phys. **B254**, 528 (1985).
[20] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations* (Clarendon, Oxford, 1946) [Russ. transl., IL, Moscow, 1960].
[21] A. M. Badalyan, D. I. Kitoroagé, and D. S. Pariĭskiĭ, Yad. Fiz. **46**, 226 (1987) [Sov. J. Nucl. Phys. **46**, 139 (1987)].

Translated by P. J. Shepherd