Anyon gas on a lattice

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An anyon gas on a quadratic lattice ($\mathbf{Z}^2$) is studied in the low-density limit. An exact expression is derived for the second virial coefficient $B(\theta,\beta)$ (β is the inverse temperature), which is a smooth function of the statistics parameter θ and, in contrast to the case of anyons in a plane, it does not have any cuspoid singularities. In the limit $\beta\to\infty$ the function $B(\theta,\beta)$ asymptotically approaches its continuous analog, having linear dependence on β and piecewise-parabolic dependence on θ.

1. INTRODUCTION

A number of new results concerning the physics of anyons—particles with fractional statistics—have recently been obtained.[1-4] Interest in this field of research has been significantly stimulated by the hypothesis that anyons play a key role in the mechanism of high-T_c superconductivity.[5-9] However the lattice problems arising in RVB (resonant-valence-bond) models are extremely complicated, and only limited progress beyond the mean-field approximation has been made. This has shifted attention to anyons in a continuous $(2+1)$-dimensional space, where, thanks to the interaction of different fields of physics (topological models, conformal theory, ...) and mathematics (knot theory, braid groups, ...), impressive achievements have indeed been made.

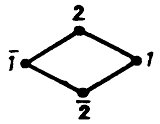
There are comparatively few exact results concerning anyons on a lattice[10] and, as will become evident from what follows, significant (and sometimes insurmountable) combinatorial difficulties are encountered when attempts are made to obtain analogous results, entirely trivial in the continuous case, for lattice models.

The aim of this work is, first of all, to develop an effective technique for calculating Green's functions that would make it possible to transfer the problem from trajectory space on a square lattice ($\mathbf{Z}^2$) to some appropriate Hilbert space. For definiteness, we shall concentrate our presentation on the problem of calculating the second virial coefficient of an anyon lattice gas in the low-density limit.

In application to the present problem, the program indicated above can be implemented with the help of a combination of the covering-space method[11] and the block-resolvent expansion.[12] It will be shown in Sec. 2 that this problem essentially reduced to the calculation of the Green's function of a particle undergoing a random walk on $\mathbf{Z}^2$ and avoiding some particular elementary cell K times. We draw diagonal rays from the center of the elementary cell singled out (see Fig. 1). Consider random walks that start on one of the four rays, proceed for some time in the half-plane adjoining this ray, and then intersect (for the first time) one of the neighboring rays (Fig. 2). We denote the set of such paths by Γ_a (a is the number of the ray). It is obvious that an arbitrary path γ, starting and ending on the rays, can be represented in the form

$$\gamma=\gamma_{a_1}\gamma_{a_2}\dots\gamma_{a_k}$$

for some k, where $\gamma_a\in\Gamma_a$. In addition, the generating function for the number of paths of prescribed length from the set Γ_a can be represented as a product of two one-dimensional random walks on $\mathbf{Z}$: a free random walk and a random walk with a first-arrival condition. After the summation over the intermediate coordinates of the points of intersection of the random-walk trajectories and the rays is performed the problem reduces to a random walk on the finite graph



with an effective transition matrix $\mathbf{F}$, representing an operator in $L_2(0,1)$—the space of states of a node of this graph. The problem then reduces to studying the spectral properties of a relatively simple integral operator.

This analysis can be easily performed numerically, which makes it possible to find the second virial coefficient $B(\theta,\beta)$ for a fixed temperature (β^{-1}) and statistics parameter θ (see Fig. 3). Comparing with the results of Ref. 13, where the second virial coefficient is calculated for the case of anyons in a plane, shows that the dependence on θ is qualitatively similar, the difference being that in the discrete case there are no cuspoid singularities. This result is natural, since cusps in this case are an indirect manifestation of ultraviolet divergences in the continuous limit,[11] which are removed by the lattice. In addition, it should be noted that in the lattice case the temperature dependence of $B(\theta,\beta)$ is complicated.

2. FORMULATION OF THE MODEL

We study an ensemble of anyons undergoing hops on $\mathbf{Z}^2$. In the low-density limit, to which we confine our attention here, anyons can be interpreted as a "charged particle + string" composite formation. We are interested in the second virial coefficient $B(\theta,\beta)$, which can be represented in terms of the partition function as follows:

$$B(\theta,\beta)-B(\theta_0,\beta)=-[Z_2(\theta,\beta)-Z_2(\theta_0,\beta)]/Z_1(\beta),\quad(1)$$

where θ is a parameter determining the fractional statistics of the particles, θ_0 is some reference statistics for which we choose boson statistics ($\theta_0=0$), $Z_1(\beta)$ is the single-particle partition function of random walks with a distinguished point, and $Z_2(\theta,\beta)$ is the single-partition function of the relative motion:

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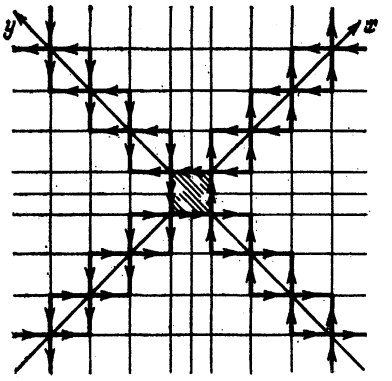


FIG. 1. The four-string gauge. To each rib with an arrow there is associated the phase factor $\exp(i\bar{\theta})$, where $\bar{\theta} = \theta/4$.

$$Z_2(\theta,\beta)=\frac{1}{2}\sum_{j\in\mathbf{Z}^2}\{\langle j|\exp(-\beta H^\theta)|j\rangle+\langle j|\exp(-\beta H^\theta)|-j\rangle\}, \tag{2}$$

with the Hamiltonian

$$H^\theta = -\sum_{\langle ij\rangle} c_i{}^+ c_j \exp(i\theta_{ij}), \tag{3}$$

(c_i^+ and c_j are creation and annihilation operators of particles with zero statistics and the summation extends over the nearest neighbors), in which the phase factors are determined by the condition

$$\theta_{ij}=\begin{cases}\pm 2\bar{\theta}, & i,\ j\in\Lambda,\\ \pm\bar{\theta}, & i \ \text{ or } \ j\in\Lambda,\\ 0, & i,\ j\notin\Lambda,\end{cases} \tag{4}$$

where Λ is the set of "strings," attached to the elementary cell singled out, through which the flux $\varphi = 2\theta$ passes. For what follows it will be convenient to choose a gauge in which

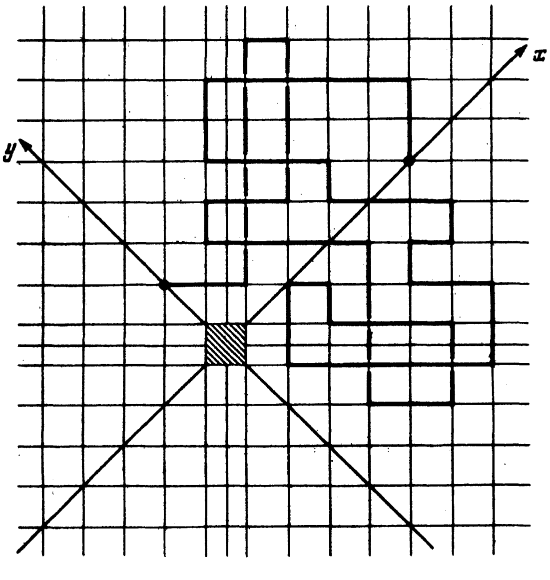


FIG. 2. A typical random walk in the half-plane $x > 0$. The path starts on the ray $(y = 0, x > 0)$ and intersects the straight line $x = 0$ first.

the string set consists of "ribs" attached to the four diagonal rays emanating from the central elementary cell (Fig. 1). In this case $\bar{\sigma} = \theta/4$, for which the total flux for one pass around any closed contour making one complete circuit around the central plaquette is equal to 2θ and the sign of the phase picked up in the transition $i \to j$ is determined by the relative orientation of the transition vector and the arrow in Fig. 1.

The partition function $Z_1(\beta)$ can be calculated in an elementary fashion. Indeed, the Green's function of free random walks on $\mathbf{Z}^2$ is

$$G(j\to l|z)=\sum_{\gamma:j\to l}{}' z^{|\gamma|+1}, \qquad l\in\mathbf{Z}^2, \tag{5}$$

where $|\gamma|$ is the length of the path γ starting at the jth site and ending at the lth site (number of hops) and $1/z = E$ is the spectral parameter. The trace of this Green's function is

$$\mathrm{tr}_* G(z)=\sum_{\gamma:j\to j} z^{|\gamma|+1}=\sum_{L=0}^{\infty} z^{L+1}\binom{2L}{L}^2 = z\frac{2}{\pi}K(4z), \tag{6}$$

where $K(k)$ is the complete elliptic integral of the first kind and the asterisk indicates that random walks with a distinguished point are being studied, with eliminates the infinite volume of the lattice. Performing the inverse Laplace transform we find

$$Z_1(\beta)=\mathcal{L}^{-1}[\mathrm{tr}_* G(z)](\beta)=I_0{}^2(2\beta). \tag{7}$$

We now study the partition function $Z_2(\theta, \beta)$ of the relative motion. In order to obtain this function it is necessary to calculate the Green's function of random walks with return to the distinguished point $(j \to j)$ or random walks with reflection into the diametrically opposite point $(j \to -j)$. We denote the set $\{j, -j\}$ by J. We introduce the Green's function of random walks on $\mathbf{Z}^2$ in the field of a vortex with flux 2θ:

$$G^\theta(j\to l|z)=\sum_{\gamma:j\to l} z^{|\gamma|+1}\exp[2i\nu(\gamma)\theta], \tag{8}$$

where ν is the number of times the path γ winds around the vortex in the lattice.

The trace of the Green's function diverges. This makes it necessary to regularize G^θ by subtracting from it the Green's function G^{θ_0} for some reference statistics θ_0, for which we choose Bose statistics [just as in Eq. (1)]. In Secs. 3 and 4 we calculate $\mathrm{tr}\,\tilde{G}^\theta(z)$, where $\tilde{G}^\theta = G^\theta - G^0$. The result is expressed in the form of the trace of a function of some integral operator. The subsequent calculations are performed numerically, which makes it possible to obtain the dependence $B(\theta, \beta)$.

3. CALCULATION OF THE GREEN'S FUNCTION OF RANDOM WALKS ON A LATTICE IN THE FIELD OF A VORTEX

After the problem of the relative motion of two anyons is reduced to random walks in the field of a vortex we need to calculate the following trace:

$$\widehat{\mathrm{tr}}\,\tilde{G}^\theta(z)=\frac{1}{2}\sum_{\gamma:j\to J} z^{|\gamma|+1}\{\exp[2i\nu(\gamma)\theta]-1\}, \tag{9}$$

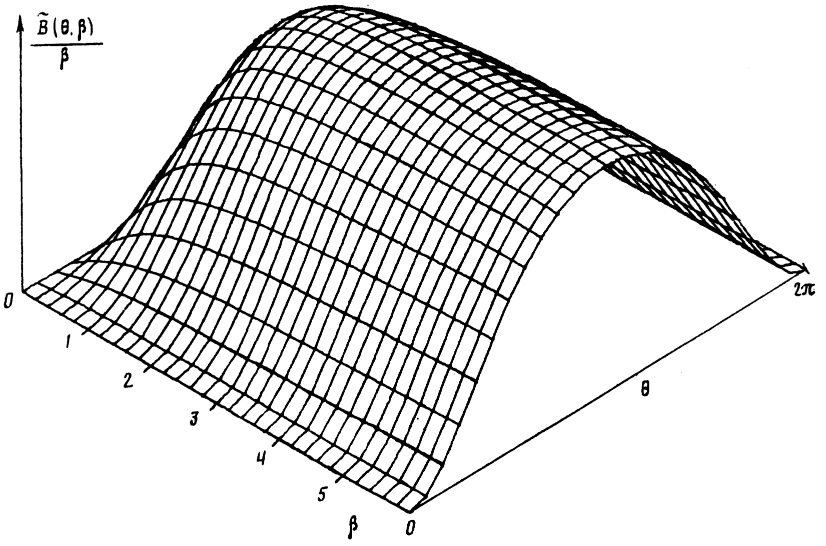


FIG. 3. The second virial coefficient as a function of the inverse temperature β and the statistics determining parameter θ.

which is related to the required partition function (2) by the Laplace transform:

$$\hat{\mathrm{tr}}\, G^\theta\left(\frac{1}{E}\right)=\int_0^\infty e^{-\beta E} Z_2(\theta,\beta)\,d\beta. \tag{10}$$

To calculate tr $\tilde{G}^\theta(z)$ we expand $\tilde{G}^\theta(z)$ in a sum over different homotopic classes, which we enumerate by means of ν:

$$\hat{\mathrm{tr}}\, G^\theta(z)=\frac{1}{2}\sum_{\nu=-\infty}^{\infty} z\, \hat{\mathrm{tr}}\, R_\nu(z)\,(e^{i\nu\theta}-1), \tag{11}$$

where

$$\hat{\mathrm{tr}}\, R_\nu(z)=\sum_{j\in Z^2}\sum_{\substack{\gamma:j\to J\\ \nu(\gamma)=\nu}} z^{|\gamma|}. \tag{12}$$

To perform the summation in Eq. (12) over the random paths on the lattice we employ a variant of the block-resolvent expansion: For each path γ we study an enlarged path $\Gamma(\gamma)$ consisting of the coordinates of successive first passages from one coordinate axis to another.

The initial problem (11) and (12) of summation over all paths $\gamma: j\to J$ splits, in the process, into a sequence of two simpler problems: 1) calculation of the generating function for random walks in the half-plane $x>0$ which start from the ray ($y=0$, $x>0$) and intersecting the straight line $x=0$ first and 2) calculation of some partition function for a finite graph with the isotopic space $L_2(0,1)$. We start with the solution of the first problem. After relatively simple combinatorial analysis we find the generating function, which we are seeking, of the first passages:

$$G(x,y|z)=\sum_{l=0}^{\infty} g_l(x,y)\, z^{2l+1}, \tag{13}$$

where $g_l(x,y)$ is the number of paths γ starting from the point $(2x+1, 0)$ and arriving at the pint $(0, 2y+1)$ in the lattice, $x\geqslant 0$, $|\gamma|=2l+1$. The function $G(x,y|z)$ is the product of two propagators of certain one-dimensional random processes-the free process (along y) and the process with the condition of first passage through zero (along x):

$$g_l(x,y)=\binom{2l+1}{y+l+1}\binom{2l+1}{x+l+1}\frac{2x+1}{2l+1}. \tag{14}$$

In order to transfer from random walks on the lattice to random walks on the graph we study the K-fold convolution

$$G_K(x_1,x_{K+1}|z)=\sum_{L,l_i}{}'\sum_{0\leqslant x_i\leqslant l_i}\prod_{i=1}^{K} g_{l_i}(x_i, x_{i+1})\, z^L, \tag{15}$$

where the primed sum denotes summation over all L and l_i such that $\Sigma_{j=1}^{K}(2l_j+1)=L$. For what follows we require only

$$\mathrm{tr}\, G_K(z)=\sum_{x\in \mathbf{Z}_+} G_K(x,x|z), \tag{16}$$

where $\mathbf{Z}_+$ is the set of positive integers. Summing over the coordinates of the first intersections x_i we find

$$\mathrm{tr}\, G_K(z)=\sum_{L,l_i}{}'\prod_{j=1}^{K}\frac{2l_j+1}{l_j+l_{j+1}+1}\binom{2l_j}{l_j}^2 z^L. \tag{17}$$

The problem of summing over the intermediate times of the first intersections l_i can be expressed in terms of the integral operator

$$\mathrm{tr}\, G_K(z)=\mathrm{Tr}\,\mathbf{F}(z)^K, \tag{18}$$

where tr is the trace in $l_2(Z^2)$ and $l_2(\mathbf{Z}_+)$ while Tr is the trace in $L_2(0,1)$. The operator $\mathbf{F}(z)$ operates on functions from the Hilbert space $L_2(0,1)$ and has the kernel

$$F(X,Y|z)=\sum_{l=0}^{\infty} z^{2l+1}(XY)^l(2l+1)\binom{2l}{l}^2, \quad X, Y\in[0,1], \tag{19}$$

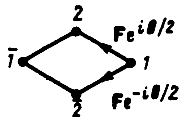
which is derived directly from Eq. (17) after the corresponding integral representation is constructed. The kernel (19) is essentially a complete elliptic integral:

$$F(X,Y|z)=\frac{2z}{\pi}\int_0^{\pi/2}\frac{d\alpha}{(1-16z^2XY\sin^2\alpha)^{1/2}}=\frac{2z}{\pi}\frac{E(4z(XY)^{1/2})}{1-[4z(XY)^{1/2}]^2}, \tag{20}$$

where $E(k)$ is a complete elliptic integral of the second kind. It can be shown that the kernel (20) indeed determines the integral operator, which is a Hilbert–Schmidt operator.

4. RANDOM WALKS ON A FINITE GRAPH WITH THE ISOSPACE $L_2(0,1)$

We now study the second part of the problem of calculating tr $\widetilde{G}^{(\theta)}(z)$: summation over all possible sequences of first intersections of the four rays $(x=y,\ x>0)$, $(x=y,\ x<0)$, $(x=-y,\ x>0)$ and $(x=-y,\ x<0)$. This problem can be formulated in the form of random walks on a graph consisting of four vertices, to each of which there is associated a Hilbert space $L_2(0,1)$. The transition matrix in this case is the integral operator $\mathbf{F}(z)$ operating in the layer. We assign the phase factor $\exp(i\theta/2)$ for counterclockwise motion and the phase factor $\exp(-i\theta/2)$ for clockwise motion:



(21)

We now calculate the weighted generating function of random walks on the graph (21) which start at the vertex a and end at one of the vertices a or $\bar{a}$ (here $a,\bar{a}\in\{1,2,\bar{1},\bar{2}\}$ and $\bar{\bar{a}}\equiv a$):

$$\widetilde{Q}^{\theta}(z,\omega)=\sum_{\hat{\Gamma}:a\to\{a,\bar{a}\}}\operatorname{Tr}\mathbf{F}(z)^{|\hat{\Gamma}|}\{\exp[2i\nu(\hat{\Gamma})\theta]-1\}\omega^{|\hat{\Gamma}|}, \tag{22}$$

where to each transition there is associated the factor $\mathbf{F}(z)\cdot\omega\cdot\exp(\pm i\theta/2)$, depending on the direction of the transition in the graph; $|\hat{\Gamma}|$ is the number of transitions; and, $\nu(\hat{\Gamma})$ is the number of times the path $\hat{\Gamma}$ in the graph is traversed. It can be shown that the generating function (22) is simply related to the desired Green's function (11):

$$\widehat{\operatorname{tr}}\,\widetilde{G}^{\theta}(z)=-\frac{1}{2}z^2\frac{\partial}{\partial z}\int_0^1\frac{d\omega}{\omega}\widetilde{Q}^{\theta}(z,\omega). \tag{23}$$

Indeed, a factor $1/2z$ is associated with the representation of (9) in terms of the weighted generating function

$$\tilde{q}^{\theta}(z)=\sum_{\gamma:j\to J}z^{|\gamma|}\{\exp[2i\nu(\gamma)\theta]-1\}, \tag{24}$$

This function differs in two ways from the function (22): First, the generating function (24) controls all paths $\gamma:j\to J$ and not only paths starting on rays. In order to take into account the fact that any point of the path γ can indeed be chosen as the starting point, the operator $z\partial/\partial z$, which multiplies the contribution of each path γ by its length $|\gamma|$, i.e., the number of possible methods of choosing the starting point, must be applied to the function (22). Second, on the other hand, some paths γ under this enlargement $(\gamma\to\Gamma\to\gamma)$ of random walks are counted n times: The same path γ has $|\hat{\Gamma}|$ different representations according to the number of intersections of the rays; these intersections can be chosen as the initial points of the enlarged random walk $\hat{\Gamma}$. In order to prevent this double counting we introduce the operator $\int_0^1(d\omega/\omega)(\ldots)$. Thus the relation (23) is proved. To calculate the generating function (22) we study the operator corresponding to transitions over two steps:

$$\hat{\mathbf{A}}^{\theta}(z,\omega)=[2\sigma_3\cos\theta+2]\,\omega^2\mathbf{F}^2(z), \tag{25}$$

where σ_3 is the classifying parameter for the number of transitions to the opposite ray $(a\to\bar{a})$ and has the obvious property $\sigma_3^2=1$ (whence follows the notation). The complete generating function is obtained from Eq. (25) by summing a geometric progression:

$$\widetilde{Q}^{\theta}(z,\omega)=4\operatorname{Sp}\operatorname{Tr}P_+\frac{1}{1-\hat{A}^{\theta}(z,\omega)}, \tag{26}$$

where $P_+(10/00)$, Sp is the trace in the two-dimensional space of σ-matrices, and the factor of 4 takes into account the different possible positions of the starting ray. After simple transformations we find

$$\widetilde{Q}^{\theta}(z,\omega)=4\operatorname{Tr}\left[1-4\omega^2\mathbf{F}^2(z)\cos^2\frac{\theta}{2}\right]^{-1}, \tag{27}$$

whence, after Eq. (23) is taken into account, we obtain finally

$$\widehat{\operatorname{tr}}\,\widetilde{G}^{\theta}(z)=-2z^2\frac{\partial}{\partial z}\operatorname{Tr}\ln\left[1-4\mathbf{F}^2(z)\cos^2\frac{\theta}{2}\right]. \tag{28}$$

Taking the inverse Laplace transform (10), from Eq. (28) we find $\widetilde{Z}_2(\theta,\beta)$, after which, using Eq. (1) and the expression (7) for $Z_1(\beta)$, we obtain a representation of $\widetilde{\beta}=(\theta,\beta)=B(\theta,\beta)-B(0,\beta)$ in the form of a trace of some compact operator, which can be easily calculated numerically. As a result we obtain the function $\widetilde{B}(\theta,\beta)$ represented in Fig. 3.

Let us now discuss the differences between the function obtained above and its continuous analog.[13] In the case of an anyon gas in a plane cuspoid singularities are observed near the point $\theta=0$. These singularities are an indirect result of ultraviolet divergences[11] and they smear out when all go over to the lattice; this is evident from Fig. 3.

Note that as β increases the dependence $B(\theta,\beta)$ asymptotically approaches the linear law $\propto\beta$, and in the process the sinusoidal dependence on θ gradually transforms into a piecewise-parabolic dependence; the maximum with respect to θ of the function $B(\theta,\beta)/\beta$ rapidly approaches its asymptotic value, equal to 2π (even for $\beta\approx 6$ we have $\widetilde{B}(\theta,\beta)/\beta\approx 6.1$).

5. CONCLUSIONS

Thus in this paper we studied an anyon lattice gas in the low-density limit. We obtained an exact expression for the second virial coefficient, which, is contradistinction to the continuous case, does not contain unphysical cuspoid singularities.

We are certainly aware that the nontrivial physics of anyon systems occurs at relatively high densities, and in this connection the present results must be regarded only as an intermediate stage, which is necessary for the development of the diagrammatic technique for anyon systems in the language of integral operators in the appropriate Hilbert space.

We hope to develop the technique proposed here for solving many-body problems. The main difficulty here is the problem of calculating the partition function for random walks of a charged particle in an arbitrary vortex configuration. We have been able to find an exact solution in the case when all vortices lie on the same straight line.

Note that the technique examined here can also be used in the physics of polymers which get caught on an obstacle.

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[1)] Moscow Institute of the Theory and Prediction of Earthquakes.

[1]Y. Kitazawa and H. Murayama, Nucl. Phys. **338**, 777 (1990).
[2]M. Stone, Int. J. Mod. Phys. B **4**, 1465 (1990).
[3]T. H. Hansson and A. Kathede, Mod. Phys. Lett. A **4**, 1937 (1989).
[4]A. M. Din, Nucl. Phys. B **330**, 757 (1990).
[5]B. I. Halperin, J. March-Russell, and F. Wilczek, Phys. Rev. B **40**, 8726 (1989).
[6]J. Sonnenschein, Preprint UCLA/89/TEP/51.
[7]J. D. Lykken, Preprint Fermilab-conf-90/72-T.
[8]Y. Hosotani and S. Chakravarty, Phys. Rev. B **42**, 342 (1990).
[9]Y. L. Chen, F. Wilczek, E. Witten, and B. I. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989).
[10]J. Fröhlich and P. A. Marchetti, Comm. Math. Phys. **121**, 177 (1989).
[11]E. M. Serebryanyĭ, Teor. Matem. Fiz. **64**, 299 (1985).
[12]J. Fröhlich and T. Spenser, Comm. Math. Phys. **88**, 151 (1983).
[13]D. Arovas *et al.*, Nucl. Phys. B **251**, 117 (1985).

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