Large-scale cosmological structure and topologically stable states of a scalar field which are periodic in the radial coordinate

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(Submitted 13 October 1990)
Zh. Eksp. Teor. Fiz. **100**, 5–19 (July 1991)

A quasistatic stable state of a scalar field in a closed Friedmann universe with a periodic distribution of energy density along the radial coordinate is described. The classical solution is stable by virtue of the angular nature of the pseudo-Goldstone scalar field (the self-effect of this field is described by a cosinusoidal potential). As a result, there is a conserved topological number (the winding number). The solution found here is linked with a possible periodicity in the distribution of galaxies [T. J. Broadhurst *et al.*, Nature **343**, 726 (1990)]. Physically, it appears that this may not be a complete solution, applying over the entire closed universe. It may instead be only a linear metastable fragment of this entire solution, which decays because of smearing at the ends after a sufficiently long time.

1. INTRODUCTION

A recent study of the redshifts of distant galaxies has revealed a surprising periodic structure (with a period of $128h^{-1}$ Mpc) in the distribution of these galaxies.[1] This structure was so unexpected that the first attempts to explain it invoked a hypothesis of a periodic time variation in the fundamental constants.[2,3]

The existence of a large-scale cosmological structure in the distribution of matter, with a period on the order of 100 Mpc, has been debated in the literature for several years now.[4–7] Two theoretical ideas which have been advanced to explain possible large-scale irregularities have been brought into this debate.

The first is the hypothesis of a late phase transition ($z \lesssim 10^3$, $T \lesssim 3000$ K), which resulted in the formation of irregularities in the distribution of galaxies.[5,6] Since by assumption the transition occurs at an epoch at which photons have already separated from matter, the appearance of irregularities in the distribution of matter does not contradict the severe limitation on the absence of irregularities in the microwave background radiation.

The second idea is that a scalar field is responsible for the irregularities in the distribution of matter. At a certain temperature T_0 below the temperature at which radiation and matter separate ($T_0 < 3000$ K), say $T_0 \sim 30$ K ($z \sim 10$), this scalar field separates out into a condensate. The characteristic dimension of the structure at the time of the phase transition in this case is the Compton wavelength of the corresponding particles, e.g., $m^{-1} \sim 10$ Mpc ($m \sim 10^{-30}$ eV). During the expansion of the universe, this dimension then increases to ~ 100 Mpc. If the phase transition occurs at T_0, the height of the barrier in the Higgs potential, V_0, must evidently be $\sim T_0^4$. On the other hand, we have $V_0 \approx m^2 v^2$, where v is the vacuum expectation value. We thus find the estimate $v \approx 10^{16}$ GeV for $T_0 \approx 30$ K and $m \approx 10^{-30}$ eV. This value corresponds to the Grand Unification scale. It is assumed that the irregularities in the distribution of matter arise from a gravitational coupling of matter with the scalar field.

Opinion is divided on just how the irregularities of the scalar field arise. Wasserman,[5] who originally introduced the hypothesis of a late phase transition, discusses the formation and growth of bubbles of the new phase which arise from perturbative fluctuations. Hill *et al.*,[6] who described the dynamics of the phase transition in detail, discuss domain walls between different vacuums of the scalar field. Finally, Press *et al.*[7] discuss the large-scale structure forming as a result of the dynamics of wave packets made up of soft bosons, with a wavelength on the order of tens of megaparsecs. None of these mechanisms predicts the regular periodic structure which is apparently observed experimentally.

We will not examine here the dynamics of the process by which the irregularities develop. Our purpose is instead to show that it is possible in principle that there exists a quasi-steady scalar field of a pseudo-Goldstone type, periodic in the proper distance and having a stable energy distribution, which has not been described previously.

We begin with a description of a model whose solution applies to the entire universe. In this model, a closed universe is "wrapped up" by a scalar field with a number of periods which does not change in the course of the expansion. This model makes particularly clear the topological reason for the stability of the solution (this stability is also tested explicitly below), but it ignores the horizon problem and also the pronounced distortion of the solution by cosmological expansion at large distances. We will be discussing the possible existence of "fragments" of the solution, stable with respect to field fluctuations in the range of the radial variable but unstable with respect to variations of the field at the ends of the interval under consideration. Such fragments might survive a fairly long time; they might give rise to a periodic structure in the distribution of galaxies and then decay.

For strict stability both the closure of the universe and the angular nature of the field of the Goldstone type under consideration are important. Specifically, we will be discussing a model with a self-effect of the scalar field described by the potential

$$V = V_0 \cos \frac{\Phi}{v}. \quad (1)$$

The angular nature of the field Φ is built into (1). It has also

 0038-5646/91/070001-08$03.00

been demonstrated that the corresponding solution for a $\lambda\Phi^4$ interaction is not stable.

The period in the radial distribution of the energy density of the field Φ is proportional to the Compton wavelength of the particle, m^{-1}. The proportionality factor changes (specifically, it increases) in the course of the expansion. If we assume that its value is of order unity at the time of the phase transition, then we can set, for example, $m^{-1}\approx 10$ Mpc, as discussed above. Expansion might have increased the distance between galactic shells to ~ 100 Mpc by the present time. In turn, one might attempt to identify the average energy density of this classical solution with the dark matter. In order of magnitude, we can estimate the density ρ at the time of the phase transition to be $\rho\approx m^2v^2\approx 2\cdot 10^{-29}$ g/cm^3 for $m=10^{-30}$ eV and $v=10^{16}$ GeV. This value agrees with the critical density.

Simply explaining the existence of a particle with a mass $\sim 10^{-30}$ eV requires invoking some specific mechanism. We will mention one possible scenario for the appearance of such a vanishingly small mass. For several years now we have been discussing the possible existence of a "massless axion": an arion, i.e., a Goldstone boson which is unrelated to the axial QCD anomaly.[8] Despite the word "massless," an arion might have a very small mass by virtue of a contribution from a weak anomaly through a triangle diagram with an emission of W bosons. One might imagine that, as a result of instanton effects, the mass of an arion would turn out to be

$$m^2=Cm_W^2\left(\frac{8\pi^2}{g_W^2}\right)^4\exp(-8\pi^2/g_W^2), \tag{2}$$

where the dimensionless coefficient C is definitely smaller than unity (it includes, in particular, the small coupling constant describing the coupling of the arion with quarks and leptons). Substituting numerical values of m_W and g_W into (2), we find

$$m=C^{1/2}\cdot 1{,}7\cdot 10^{-27}\ \text{eV}. \tag{2$'$}$$

Although the resulting mass is probably slightly smaller than that required, it is in qualitative agreement with the vanishingly small value which we need.

2. CLASSICAL STATE OF THE SCALAR FIELD

We consider the self-interacting scalar field (1) in a closed universe with a metric

$$ds^2=dt^2-a^2(t)\left[(d\chi)^2+\sin^2\chi((d\theta)^2+\sin\theta d\varphi^2)\right]. \tag{3}$$

As usual, the radial coordinate is $r=a\sin\chi$, and the "proper distance" is $d=a\chi$. We seek steady-state solutions $\Phi=\Phi(\chi)$ which depend on the one coordinate χ. Actually, of course, we can speak only in terms of quasisteady solutions, since the scale factor $a=a(t)$ depends on the time. For the time being we ignore the term with the second time derivative of the field Φ, although we will see later on that this approximation is, strictly speaking, legitimate only for distances smaller than the cosmological horizon. The formal extension of the solution found below to the entire universe leads to a better understanding of the topological nature of its stability. In discussing the physical content of the model, on the other hand, we will for the most part restrict the discussion to fragments of the solution no bigger than the distance to the horizon. The equation for $\Phi(\chi)$ is

$$-\frac{1}{a^2}\frac{1}{\sin^2\chi}\frac{d}{d\chi}\left[\sin^2\chi\left(\frac{d\Phi}{d\chi}\right)\right]+m^2v\sin\frac{\Phi}{v}=0, \tag{4}$$

where m is the mass of the particle. The parameter V_0 in Eq. (1) is related to m by $V_0=-m^2v^2$.

Below we discuss the case $ma\gg 1$. It is natural to assume that the Compton wavelength m^{-1} is much smaller than a; in fact, it is necessary to make this assumption. Otherwise, irregularities with dimensions $\sim m^{-1}$ in the distribution of the scalar field would disrupt the overall uniformity and isotropy which we need for the validity of a model with metric (3). The solutions described below, with a period of order unity in the variable $\rho=ma\chi$, correspond to a period of order m^{-1} in terms of the proper distance $d=a\chi$.

Under the condition $ma\gg 1$, Eq. (4) simplifies, reducing to the planar case:

$$\frac{d^2\psi}{d\rho^2}+\sin\psi=0,\qquad \psi=\frac{\Phi}{v}+\pi. \tag{5}$$

Equation (5) holds over the entire range of the variable χ, except in small neighborhoods of the poles, $0\leqslant\chi<\delta$ and $\pi>\chi>\pi-\delta$, $\delta\sim(ma)^{-1}$, in which $\sin\chi$ tends toward zero. In this region, however, we have either $\sin\chi\approx\chi$ or $\sin\chi\approx\pi-\chi$, and Eq. (4) becomes the radial sine-Gordon equation:

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d\psi}{d\rho}\right)+\sin\psi=0,\qquad \rho=ma\chi,\qquad \chi\ll 1, \tag{6a}$$

$$\frac{1}{\rho'^2}\frac{d}{d\rho'}\left(\rho'^2\frac{d\psi}{d\rho'^2}\right)+\sin\psi=0,\qquad \rho'=ma(\pi-\chi),\ \pi-\chi\ll 1. \tag{6b}$$

Equations (6a) and (6b) are obviously the same as (5) under the conditions $\rho\gg 1$ and $\rho'\gg 1$. We first discuss a solution of Eq. (5), and then match it with the solution of the refined equations (6). Introducing the half-angle $\varphi=\psi/2$, we easily find a first integral of Eq. (5):

$$\left(\frac{d\varphi}{d\rho}\right)^2=C-\sin^2\varphi=\frac{1}{k^2}-\sin^2\varphi,$$
$$C=\frac{1}{k^2},\qquad \varphi=\frac{\psi}{2}=\frac{\Phi}{2v}+\frac{\pi}{2}, \tag{7}$$

where the integration constant C is chosen to satisfy $C\geqslant 1$, $0\leqslant k\leqslant 1$. From (7) we immediately find

$$\varphi=\arcsin\text{sn}\left(\frac{\rho-\rho_0}{k}\right),\qquad \rho=ma\chi, \tag{8}$$

where φ and sn are the elliptic amplitude and elliptic sine, respectively. We have chosen the origin of coordinates in such a way that $\varphi=0$ at $\rho=\rho_0$. We now take account of the fact that the solution (8) is not valid near the values $\chi=0$ and $\chi=\pi$. As we mentioned above, we should use Eqs. (6) in this region. As can be seen from Eq. (6a), in the limit $\rho\to 0$ we have $\psi=2\varphi\sim a+b/\rho$. We can evidently restrict the analysis to the nonsingular solution with $b = 0$, since the singular case corresponds to an infinite total energy. The solution which we need then depends on only one integration constant, so the matching of the solution (8) with the small-ρ region should determine the relationship between ρ_0 and k. Fortunately, this relationship can be found without explicitly solving Eq. (6a). It can easily be seen from Eq. (6a) that a nonsingular solution which tends toward a constant value in the limit $\rho \to 0$ is an even function of ρ (it can be expanded in a series in even powers of ρ). Since it must become (8) in the limit $\rho \gg 1$, we have the condition

$$\mathrm{sn}\frac{\rho-\rho_0}{k} = \mathrm{sn}\frac{-\rho-\rho_0}{k} = -\mathrm{sn}\frac{\rho+\rho_0}{k}. \tag{9}$$

We then find

$$\frac{\rho_0}{k} = \pm K(k), \tag{10}$$

where $K(k)$ is the complete elliptic integral of the first kind. Under condition (10), Eq. (9) is satisfied, since the elliptic sine changes sign over a displacement equal to the half-period $2K$. The refined version of solution (8) is thus

$$\varphi = \pm \arcsin\mathrm{sn}\left(\frac{\rho}{k} - K(k)\right). \tag{11}$$

Correspondingly, under the condition $\rho' = (\pi - \chi)ma \ll 1$ we find from Eq. (6b) $\psi \sim A + B/\rho'$. Again, we have to restrict the discussion to the solution with $B = 0$. The requirement of even parity in ρ' then gives us, in place of (9),

$$\mathrm{sn}\left[\frac{-\rho'}{k} + \frac{\pi ma}{k} - K(k)\right] = \mathrm{sn}\left[\frac{\rho'}{k} + \frac{\pi ma}{k} - K(k)\right]. \tag{12}$$

From this result we find the "quantization" condition on k:

$$\frac{\pi ma}{k_N} = 2NK(k_N), \tag{13}$$

where N is an integer.

The solution (11) is periodic not for the field φ itself but only for $\sin \varphi$. However, this is the physical quantity, by virtue of the Goldstone nature of the field φ. For example, it is a simple matter to calculate the energy density ε corresponding to solution (11):

$$\begin{aligned} \varepsilon &= \frac{1}{2a^2}\left(\frac{d\Phi}{d\chi}\right)^2 - m^2v^2\cos\frac{\Phi}{v}, \\ \varepsilon - \varepsilon_0 &= 2m^2v^2\left[\frac{1+k^2}{k^2} - 2\,\mathrm{sn}^2\left(\frac{\rho}{k} - K(k)\right)\right]. \end{aligned} \tag{14}$$

Here ε_0 is the energy density of the vacuum, which corresponds to $\Phi = 0$ for our choice of the potential [see (1)], and which has the value $\varepsilon_0 = -m^2v^2$.

We see from (14) that the period in the distribution of the energy density in the coordinate ρ is $2K(k)k$. This corresponds to a period

$$\Delta\chi = \frac{2K(k)k}{ma} \tag{15a}$$

in the angular variable χ or to a period

$$\Delta d = a\Delta\chi = \frac{2K(k)k}{m} \tag{15b}$$

in the proper distance.

The period $\Delta\chi$ (and also Δd) increases as a function of the modulus of the elliptic sine k. In the limit $k \to 1$ we have

$$K(k) \sim \ln(4/k'), \quad k'^2 = 1 - k^2,$$

i.e., $K(k) \to \infty$ as $k \to 1$. At $k = 1$, we are left with only one kink, instead of an oscillatory solution. This case corresponds to a situation in which the total energy of this classical solution differs from the vacuum energy by a finite amount, and the solution itself is a soliton (a domain wall):

$$\varphi = \mathrm{arcsinth}(\rho - \mathrm{const}). \tag{16}$$

The average energy density (11) for the solution under consideration here can be found from (14) easily, by integrating over the period of the elliptic sine:

$$\varepsilon - \varepsilon_0 = 2m^2v^2\left[1 - \frac{1}{k^2} + \frac{2E(k)}{k^2K(k)}\right]. \tag{17}$$

Here $E(k)$ is the complete elliptic integral of the second kind.

3. STABILITY OF THE CLASSICAL SOLUTION

The classical solution which has been found is an extremum of the action. We can show that this solution realizes a local minimum of the energy in functional space. To do this, we begin with an explicit calculation of the second variation of the energy, and we show that it is positive. We then turn to the topological reasons for the stability of the solution (11).

It can be seen from (14) that the variation of the energy stored in the interval $\chi_0, \chi_0 + \Delta\chi$ is proportional to the integral

$$\begin{aligned} \delta^2E &= \int_{\chi_0}^{\chi_0+\Delta\chi} d\chi\, \delta\Phi(\chi)\left[-\frac{1}{a^2}\frac{d}{d\chi^2} + V''(\Phi)\right]\delta\Phi(\chi), \\ V''(\Phi) &= m^2\cos\frac{\Phi}{v} = -m^2(1 - 2\,\mathrm{sn}^2 u), \quad u = \frac{\rho}{k} - K(k). \end{aligned} \tag{18}$$

In (18), we are left with the variations of $\Phi(\chi)$ which depend only on the variable χ, since the dependence of $\delta\Phi$ on the variables θ and φ simply increases the value of δ^2E: The corresponding terms are proportional to $(\partial(\delta\Phi)/\partial\theta)^2$ and $(\partial(\delta\Phi)/\partial\varphi)^2$. In addition, we are assuming $\delta\Phi(\chi_0) = \delta\Phi(\chi_0 + \Delta\chi) = 0$. We discuss this condition below.

To study the sign of the quantity δ^2E, we need to solve the eigenvalue problem for the operator in Eq. (18). The pertinent differential equation is

$$\frac{d^2\psi_n}{du^2} + (\varepsilon_n + k^2 - 2k^2\,\mathrm{sn}^2 u)\psi_n = 0, \tag{19}$$

where the eigenvalues ε_n are proportional to the eigenvalues λ_n of the operator $-d^2/a^2d\chi^2 + V''(\Phi)$:

$$\varepsilon_n = \frac{\lambda_n k^2}{m^2}.$$

Equation (19) is analyzed in detail in the Appendix. Here we write out the results derived there. The most important of these results is that all the eigenvalues satisfy $\varepsilon_n \geqslant 0$. This makes the solution stable.

Equation (19) is a Schrödinger equation with a periodic potential. Accordingly, we replace the discrete variable n by a natural continuous variable which numbers the energy levels, the quasimomentum p. We have to follow the changes in $\varepsilon = \varepsilon(p)$ as the quasimomentum increases from 0 to ∞. The solution of this problem is based on two equations derived in the Appendix, which express the quasimomentum p and the energy ε in terms of the complex auxiliary parameter α:

$$p=\frac{1}{iK(k)}\left[\alpha\zeta(\omega)-\omega\zeta(\alpha)\right],$$
$$\varepsilon={}^1/_3(2-k^2)-\frac{\wp(\alpha)}{e_1-e_3}. \tag{20}$$

The notation used in (20) is standard in the theory of elliptic functions. The function $\mathscr{P}(\alpha)$ is the elliptic Weierstrass function (of second order). Its periods are 2ω and $2\omega'$. They are related to the periods $4K(k)$ and $2iK'(k)$ of the function $sn\ u$ in Eq. (19) by the equations

$$K(k)=(e_1-e_3)^{1/2}\omega,\quad iK'(k)=(e_1-e_3)^{1/2}\omega'. \tag{21}$$

In turn, e_1 and e_3 are the values of the same Weierstrass function at the points ω and ω':

$$e_1=\wp(\omega),\quad e_3=\wp(\omega'). \tag{22}$$

The ratio of the numbers e_1 and e_3 is uniquely determined by the parameter k:

$$\frac{e_1}{e_3}=\frac{2-k^2}{-(1+k^2)}. \tag{23}$$

The numbers e_1 and e_3 can be multiplied by a common factor, on which the physical quantities do not depend.[1)] The function ζ in (20) is the well-known Weierstrass ζ-function. Consequently, within this latitude associated with the multiplication of e_1 and e_3 by an arbitrary factor, all the functions on the right side of Eqs. (20) are determined uniquely.

The two-band spectrum shown in Fig. 1 corresponds to Eqs. (20). In the lower allowed band we have $0 \leqslant \varepsilon \leqslant 1-k^2$, and the quasimomentum changes from zero to $p = \pi/2K$. In the band lying above the gap, $1-k^2 < \varepsilon < 1$, the quasimomentum is a complex quantity. The lower edge of the upper allowed band again corresponds to the value $p = \pi/2K$, and the entire upper allowed band corresponds to a variation of the quasimomentum from $\pi/2K$ to ∞. The details associated with the derivation of this spectrum are given in the Appendix.

We thus see that all the eigenvalues satisfy $\varepsilon(p) \geqslant 0$. Correspondingly, the second variation of the energy, $\delta^2 E$, is positive [Eq. (18)]. The system is therefore locally stable. We turn now to the topological reason for this stability.

In calculating the energy variation $\delta^2 E$, we did not vary the function at the ends of the integration interval. We set

$$\delta\Phi(\chi_0)=\delta\Phi(\chi_0+\Delta\chi)=0$$

[this approach made it possible to discard the term outside the integral in the course of the integration by parts in Eq. (18)]. These variations might have made the solution unstable. Actually, this solution found for the linear region of the range of the variable χ, $\chi_0 \leqslant \chi \leqslant \chi_0 + \Delta\chi$, is not stable if the ends are not fixed. Taken by itself, this solution starts to become smooth at its ends and ultimately spreads out. This behavior can be seen particularly clearly when we look at the particular case of a field variation corresponding to a change in the parameter k:

$$\delta\varphi=\left(\frac{d\varphi}{dk}\right)\delta k.$$

The energy density in (14) changes explicitly in the course of such variations ($d\varepsilon/dk \neq 0$), and it decreases with increasing k (with increasing period). At first glance, this fact would seem to be totally inconsistent with the assertion that our solution corresponds to an extremum of the energy. The resolution of this paradox is that the discarded "surface" term

$$\delta\varphi\left(\frac{d\varphi}{d\chi}\right)\Big|_{\chi_0}^{\chi_0+\Delta\chi}$$

which arises as a result of the variation of the ends turns out to be proportional to the "volume" in this case. In other words, it is proportional to the length of the interval, because we have $d\varphi/dk \sim u$ at large u. As a result, for the energy density $\varepsilon(k)$ the quantity $\delta\varepsilon(k)/\delta k$ turns out to be finite, as would follow from the explicit expression (14).

We thus see that the field variations at the ends of the range of integration, $\delta\Phi(\chi_0)$ and $\delta\Phi(\chi_0+\Delta\chi)$, are dangerous. Actually, however, we are dealing with a cyclic variable χ, so there are no "ends" for a closed universe. We can assume that χ varies over the interval $0 \leqslant \chi < 2\pi$ (in this case, of course, the range of φ must be shrunk: $0 \leqslant \varphi < \pi$). The values $\chi = 0$ and $\chi = 2\pi$ then correspond to the same physical point. Over the entire range of χ, the field φ, which is of an angular nature, must undergo a whole number of rotations:

$$\varphi(\chi=2\pi)-\varphi(\chi=0)=2\pi N.$$

For the solution in (11), which has a period

$$\Delta\chi=\frac{4K(k)k}{ma},$$

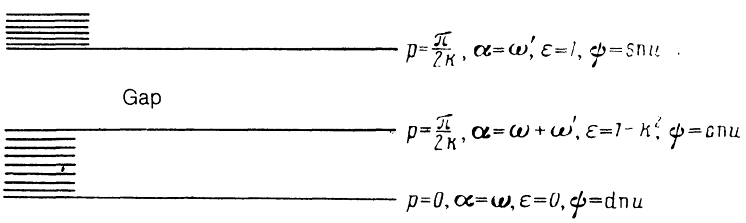

FIG. 1.

along the variable χ, the number N is found from the equation

$$N=\frac{2\pi}{\Delta\chi}=\frac{\pi}{2K(k)k}ma. \tag{24}$$

Equation (24) determines a discrete set of eigenvalues $k=k_N$, which is the same as the set (13) found from the condition that the solution not be singular at the poles. At the values $k=k_N$, the Goldstone field φ performs a mapping $\pi_1(S_1)=Z_N$. The quantity N is obviously the winding number, which makes our solution stable.

The rigorous stability of this model solution thus results from both the topology of the metric of the closed universe and the Goldstone nature of the field Φ. In particular, it is obvious that there is no such stability for the interaction $\lambda\Phi^4$, for which the field is not of an angular nature. This point is demonstrated clearly in the Appendix through an analysis of the spectrum of the corresponding differential equation.

For the solution which we have found, with the asymptotic behavior (11), the poles—i.e., the points $\rho=0$ and $\rho=\pi ma$—are special physical points: the "poles of the universe." Specifically, in a frame of reference with origin at the point $\rho=0$ (or $\rho=\pi ma$), the solution is spherically symmetric, while in other frames of reference, e.g., one moving with the earth, this is not the case. A simple geometric model corresponding to this situation would be the surface of a sphere on which lines of latitude are inscribed. The spherical surface corresponds to the closed universe, and the lines of latitude correspond to (for example) lines of a maximum energy density. For a marked globe of this sort, the poles would obviously have an objective meaning.

We can write explicit expressions for the distribution of the energy density in two frames of reference. In the frame with origin at the pole $\rho=0$ we have, from Eq. (14),

$$\varepsilon-\varepsilon_0=2m^2v^2\left[\frac{1+k^2}{k^2}-2\,\mathrm{sn}^2\left(\frac{ma}{k}\arcsin\frac{r}{a}-K(k)\right)\right], \tag{25}$$

where $r=a\sin\chi$ is the radial coordinate. In a frame of reference moving with the earth, on the other hand, the density distribution is found through a shift of $\mathbf{r}$ by an amount $-\mathbf{r}_0$, where $\mathbf{r}_0$ is the coordinate of the pole for observation on the earth:

$$\mathbf{r}\to\mathbf{r}-\mathbf{r}_0\left[\left(1-\frac{r^2}{a^2}\right)^{1/2}+\left(1-\left(1-\frac{r^2}{a^2}\right)^{1/2}\right)\left(\frac{\mathbf{r}_0\mathbf{r}}{r_0^2}\right)\right]. \tag{26}$$

Substitution of the magnitude of the right-hand side of this expression into Eq. (25) gives us the distribution of the energy density for an observer on the earth. For the immediate neighborhood ($r\ll r_0, a$), we find the simple formula

$$\varepsilon-\varepsilon_0=2m^2v^2\left[\frac{1+k^2}{k^2}-2\,\mathrm{sn}^2\left(\frac{m}{k}r\cos\theta+K(k)-\arcsin\frac{r_0}{a}\right)\right]. \tag{27}$$

Here θ is the angle between the ray along which the observation is made and the direction to the pole of the universe. The observation of an angular variation of the density oscillation period would be direct proof of the existence of a special point (or at least a special direction) in the universe.

4. IS A REAL COSMOLOGICAL SCENARIO POSSIBLE?

It appears that the classical state of a Goldstone field Φ described in the preceding sections of this paper might be capable in principle of giving rise to large-scale cosmological structure. However, two circumstances seem to rule out a direct extension of the solution (11) to the entire universe for a real cosmology.

In the first place, a simple estimate of the second derivative of the field Φ with respect to the time [more precisely, $a^{-3}\partial_t^2(a^3\Phi)$], which we discarded from Eq. (4), leads to the following for the case of the solution (11):

$$\frac{1}{a^3}\frac{\partial^2}{\partial t^2}(a^3\Phi)\approx\frac{\partial^2\Phi}{\partial t^2}\approx\rho^2\left(\frac{\dot a}{a}\right)^2\frac{\partial^2\Phi}{\partial\rho^2},\qquad \rho=ma\chi \tag{28}$$

(in this estimate we assumed $\chi\approx 1$, i.e., $\rho\gg 1$).

We see that $\partial^2\Phi/\partial t^2$ can be discarded in the field equation only for distances

$$d=a\chi<c\left(\frac{\dot a}{a}\right)^{-1},$$

i.e., distances smaller than the cosmological horizon.

Second, it is generally not clear how we are to reconcile a late phase transition to a highly ordered state throughout the universe, as described above, with causality.

However, there is still the possibility that a fragment of this solution will form in some region of the range of the variable χ—in a region whose linear dimension $d=a\chi$ is smaller than or on the order of the distance to the horizon. As we learned in the preceding section of this paper, such a solution would be unstable at the ends of the interval, but it might have a fairly long lifetime. In particular, it is obvious that this solution would at any rate be no smaller than d/c, where d is the length of the region under consideration. Actually, the lifetime could be even longer, because the "burning" of the periodic region of the field starts from the ends and propagates at a velocity $\approx c$. The ends themselves, however, move away from the observer by virtue of the cosmological expansion. The velocity at which they do so is comparable to c if the region has a length comparable to the dimensions of the horizon. It is for this reason that we cannot ignore the quantity $\partial^2\Phi/\partial t^2$ in the field equation at such large distances. It thus seems natural that for fragments with dimensions on the order of the cosmological horizon the expansion would increase the lifetime with respect to a decay coming from the ends. A quantitative description of the behavior of the field would of course require solving the equation in which the term $\partial^2\Phi/\partial t^2$ is retained.

The question now is whether, over the lifetime of the fragment, the gravitational coupling of the field Φ with matter would lead to the formation of a periodicity in the distribution of galaxies. This possibility would appear to be ruled out if the irregularities caused in the distribution of the total energy density by the presence of the field were sufficiently large: $\delta\rho/\rho\sim 1$ (i.e., if the energy density stored in the field were not small in comparison with the total energy density). An estimate in the Introduction to this paper showed that this case is quite possible. After periodicity was established in the distribution of galaxies, the classical field itself might dissipate.

I wish to thank Z. G. Berezhiani, V. K. Dubrovich, O. V. Kancheli, and N. G. Ural'tsev for useful discussions. The decision to write this paper arose from a conversation with J. Bjorken, to whom I also extend my sincere gratitude.

APPENDIX

In this Appendix we analyze the differential equation (19). Although some of the material below can be found in the corresponding mathematical literature (e.g., Ref. 9), we will point out some details for readers who are not intimately familiar with the theory of elliptic functions.

Equation (19) is a Lamé equation in so-called Jacob form. In general, this equation is

$$\frac{d^2\psi}{du^2}+[h-m(m+1)k^2\,\mathrm{sn}^2\,u]\psi=0, \tag{A1}$$

where m is an integer. We are therefore dealing with the particular case $m=1$.

The theory of periodic solutions of the Lamé equation is well developed.[9] Let us assume that a solution is periodic with an interval $2pK$, where p is an integer, and $K=K(k)$ is the complete elliptic integral ($4K$ is the period of the function sn u). Within one period, the solution can have pq zeros, where q is some other integer (or $q=0$). Finally, this solution can be an even or odd function of the variable $u-K$. We denote by $Ec_m^q(u)$ solutions which are even with respect to $u-K$ and which have pq zeros over one period, and we denote by $Es_m^q(u)$ the corresponding odd solutions. These are the so-called Lamé polynomials. It follows from the theory that there are in general $2m+1$ Lamé polynomials, so that for the case $m=1$ we have three such polynomials:

$$\begin{aligned}\psi_0&=Ec_1^0(u)=\mathrm{dn}\,u, \quad \varepsilon_0=0,\\ \psi_1&=Es_1^1(u)=\mathrm{cn}\,u, \quad \varepsilon_1=1-k^2,\\ \psi_2&=Ec_1^1(u)=\mathrm{sn}\,u, \quad \varepsilon_2=1.\end{aligned} \tag{A2}$$

The other solutions cannot be represented by finite polynomials of sn, cn, and dn.

The existence of a solution ψ_0 with a zero energy is obvious from general considerations. This solution is a Goldstone degree of freedom, which corresponds to the possibility of a general translation of the classical field. It is proportional to the derivative of the classical field with respect to the coordinate u:

$$\psi_0=\frac{d\varphi}{du}=\frac{d}{du}(\mathrm{arcsinsn}\,u)=\mathrm{dn}\,u. \tag{A3}$$

Physically, Eq. (19) may be thought of as a Schrödinger equation with a periodic potential, with a period of $2K$ in the variable u. The first of solutions (A2), ψ_0, has the same period of $2K$, while the other two solutions, ψ_1 and ψ_2, change sign when u is shifted by the "lattice constant" $2K$. The meaning here is that these solutions correspond to the value

$$p=\frac{\pi}{2K} \tag{A4}$$

of the quasimomentum.

We will see below that ψ_1 corresponds to the upper edge of the lower allowed band, and ψ_2 to the lower edge of the upper allowed band (Fig. 1).

For a comprehensive analysis of the spectrum, we need to find the Bloch functions for Eq. (19) for arbitrary values of the quasimomentum. For this purpose it is convenient to rewrite Eq. (19) in so-called Weierstrass form.

The elliptic sine in Eq. (19) can be expressed in terms of the Weierstrass function $\mathcal{P}$ (Ref. 9):

$$\mathrm{sn}(u,k)=\frac{(e_1-e_3)^{1/2}}{[\wp(y)-e_3]^{1/2}}, \quad y=(e_1-e_3)^{-1/2}u. \tag{A5}$$

The function $\mathcal{P}(y)$ is a second-order elliptic function with periods of 2ω and $2\omega'$. Here we have the parameters

$$e_\alpha=\wp(\omega_\alpha),$$

where $\omega_1=\omega$, $\omega_3=\omega'$, $\omega_2=-\omega-\omega'$. Since $\mathcal{P}'(\omega_\alpha)=0$, the function $[\mathcal{P}(y)-e_\alpha]^{1/2}$ has no branch point at $y=\omega_\alpha$. The ratios of the numbers e_α are determined by the modulus k of the elliptic sine:

$$e_1:e_2:e_3=2-k^2:2k^2-1:-(1+k^2). \tag{A6}$$

Here

$$e_1+e_2+e_3=0,$$

and the common factor in e_α is not determined. However, physical quantities do not depend on it [the right side of (A5) is invariant under the substitution $e_\alpha\to\lambda e_\alpha$].

If we want the function $\mathcal{P}$ to appear in the numerator, rather than the denominator, of the resulting differential equation, we should introduce yet another shift of the variable, by an amount equal to the second half-period of the elliptic sine, iK' [$K'=K(k'^2)$, $k'^2=1-k^2$]. We assume

$$u=(e_1-e_3)^{1/2}z+iK'. \tag{A7}$$

Then

$$k\,\mathrm{sn}(u,k)=\frac{1}{\mathrm{sn}[(e_1-e_3)^{1/2}z,k]}=\frac{[\wp(z)-e_3]^{1/2}}{(e_1-e_3)^{1/2}}, \tag{A8}$$

where

$$K=(e_1-e_2)^{1/2}\omega, \quad iK'=(e_1-e_3)^{1/2}\omega'.$$

According to (A8), we can rewrite Eq. (19) as

$$\frac{d^2\psi}{dz^2}+[H-2\wp(z)]\psi=0, \qquad \psi=\psi_n, \quad H=(e_1-e_3)(\varepsilon_n+k^2)+2e_3. \tag{A9}$$

Equation (A9) is a Lamé equation in Weierstrass form.[9] The general solution of this equation is[10]

$$\psi(z)=\frac{\sigma(z+\alpha)}{\sigma(z)}\exp\{-z\zeta(\alpha)\}. \tag{A10}$$

Here ζ and σ are the Weierstrass functions

$$\zeta'=-\wp, \quad \frac{\sigma'}{\sigma}=\zeta,$$

and (as we will see below) the parameter α specifies unambiguously the value of the quasimomentum. It is related to the "energy" H by

$$H=-\wp(\alpha). \qquad (A11)$$

To verify that (A10) is a solution of (A9), it is sufficient to substitute (A10) into (A9) and to use the known identities for Weierstrass functions:

$$\zeta(u+v)-\zeta(u)-\zeta(v)=\frac{1}{2}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)},$$
$$\wp(u+v)+\wp(u)+\wp(v)=\frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2. \qquad (A12)$$

The function $\sigma(z)$ is an entire function, which has zeros at the points $2m\omega+2n\omega'$, where m and n are integers which are not simultaneously zero. Consequently, ψ as a function of u has poles at

$$u=2mK+(2n+1)iK'.$$

These poles never reach the real axis; i.e., for real values of u, the field ψ is a finite quantity.

To find the quasimomentum corresponding to a certain value of the parameter α, it is sufficient to find the factor which the function $\psi(z)$ in (A10) acquires under a shift $z\rightarrow z+2\omega$, which corresponds to a shift $u\rightarrow u+2K$. Using known properties of the Weierstrass functions,[9] we easily find

$$\psi(z+2\omega)=\psi(z)\exp\{2[\alpha\zeta(\omega)-\omega\zeta(\alpha)]\}. \qquad (A13)$$

By virtue of the definition of the quasimomentum p, the exponential factor in (A13) corresponds to a factor of $\exp(2iKp)$. We thus have

$$p(\alpha)=\frac{1}{iK}[\alpha\zeta(\omega)-\omega\zeta(\alpha)]. \qquad (A14)$$

On the other hand, from Eq. (A11), along with (A9) and (A6), we find the following expression for the energy $\varepsilon(p)$ (the discrete parameter n is being replaced by the continuous parameter p):

$$\varepsilon(p)=\frac{1}{3}(2-k^2)-\frac{\wp(\alpha)}{e_1-e_3}. \qquad (A15)$$

Let us assume that α runs along the boundary of the first quarter of the fundamental parallelogram of periods [(1/4)*FPP*] (Fig. 2). As α varies from 0 to ω, the quantity $p(\alpha)$ remains purely imaginary, since $\zeta(\alpha)$ is a real function on the real axis. For $\alpha=\omega$, the quasimomentum vanishes: $p=0$. We see from (A15) that in this case we have $\varepsilon(0)=0$ [$\mathscr{P}(\omega)=e_1$]. The point $p=0$, $\varepsilon=0$ corresponds to the lower edge of the first allowed band in Fig. 1. Using the relations between Weierstrass functions and the relations between the latter and elliptic Jacobi functions, we can easily show that the corresponding wave function is proportional to $\psi_0(u)=dn(u,\ k)$, which is the same as (A2).

We now assume that α varies along the imaginary axis, from the point $\alpha=\omega$ to $\alpha=\omega+\omega'$. It is easy to verify that on this interval the quantity $p(\alpha)$ is real, with the value

$$p(\alpha)=\frac{1}{K}[\mathrm{Im}\,\alpha\zeta(\omega)-\omega\,\mathrm{Im}\,\zeta(\alpha)]. \qquad (A16)$$

Expression (A16) follows from (A14) if we note that, by virtue of the symmetry properties of the function $\zeta(\alpha)$,

$$\zeta(z+2\omega)=\zeta(z)+2\zeta(\omega),\qquad \zeta(-z)=-\zeta(z),$$

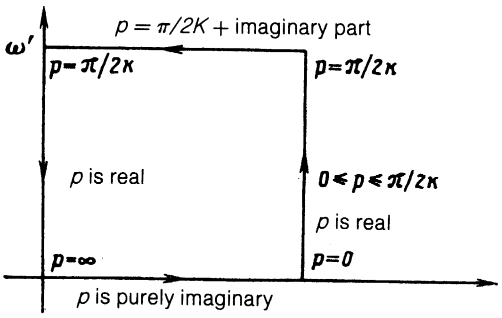

FIG. 2.

its real part remains constant as α varies from ω to $\omega+\omega'$. We easily find from (A16) that the value of the quasimomentum changes from $p(\omega)=0$ to $p(\omega+\omega')=\pi/2K$ on this interval. The latter conclusion follows from the symmetry properties of ζ under a shift of its argument by an amount $2\omega'$:

$$\zeta(\omega+\omega')=\zeta(\omega-\omega')+2\zeta(\omega').$$

Hence

$$\mathrm{Im}\,\zeta(\omega+\omega')=-i\zeta(\omega').$$

In addition, from the Legendre relation we have

$$\omega'\zeta(\omega)-\omega\zeta(\omega')=i\pi/2.$$

With regard to the value of the energy, we note that we have $\varepsilon(\omega)=0$ and

$$\varepsilon(\omega+\omega')=1-k^2.$$

It is easy to show that the wave function corresponding to the upper edge of the first allowed band is $\psi=\mathrm{cn}\,u$.

As we continue to move along α, from $\alpha=\omega+\omega'$ to $\alpha=\omega'$, the value of the real part of $p(\alpha)$ remains constant:

$$\mathrm{Re}\,p(\alpha)=\frac{\pi}{2K}.$$

However, $p(\alpha)$ has an imaginary part. This interval corresponds to the gap

$$1-k^2\leqslant\varepsilon(\alpha)\leqslant 1.$$

At $\alpha=\omega'$, the upper allowed band begins. At this point the quasimomentum again becomes real (and equal to $\pi/2K$), and the wave function is $\psi=\mathrm{sn}\,u$.

Finally, as we move from the point $\alpha=\omega'$ to $\alpha=0$, the energy increases from unity to infinity, while the quasimomentum changes from $\pi/2K$ to infinity (remaining real). Consequently, as we have already asserted, all the eigenvalues $\varepsilon(p)$ are nonnegative.

The situation is different in the $\lambda\Phi^4$ theory, in which, by virtue of the topological arguments presented above, we would not expect all the eigenvalues to be positive. Let us assume

$$V(\Phi)=-{}^1/_2 m^2\Phi^2+{}^1/_4\lambda\Phi^4, \qquad (A17)$$

so that the sign of the mass corresponds to spontaneous symmetry breaking ($\Phi\rightarrow-\Phi$). Instead of the sine-Gordon equation in this case we have the equation

$$-\frac{d^2\Phi}{dr^2}-m^2\Phi+\lambda\Phi^3=0, \quad r=a\chi. \tag{A18}$$

The general solution of this equation can be written in the form

$$\Phi=\Phi_0\,\mathrm{sn}\,u, \quad \Phi_0=\left(\frac{m^2}{\lambda}\right)^{1/2}\left(\frac{2k^2}{1+k^2}\right)^{1/2}, \quad u=\frac{m(r-r_0)}{(1+k^2)^{1/2}}. \tag{A19}$$

The differential equation for excitations, equivalent to (13), is

$$\frac{d^2\psi_n}{du^2}+(\varepsilon_n+1-k^2-6k^2\,\mathrm{sn}^2\,u)\psi_n=0. \tag{A20}$$

The zeroth mode, which corresponds to the eigenvalue $\varepsilon=0$, is

$$\psi_0=\mathrm{cn}\,u\,\mathrm{dn}\,u\sim\frac{d\Phi}{du}. \tag{A21}$$

It is obvious at the outset that this solution could not correspond to the smallest eigenvalue, since it has a zero on the periodicity interval of the potential; i.e., it is a function of $Es_2^1(u)$ (see the discussion above). By virtue of the Sturm oscillation theorem,[10] there must exist a solution with a smaller (i.e., negative) eigenvalue. This solution is written out explicitly in Ref. 11:

$$\psi(u)=1-[(1+k^2)-(1-k^2+k^4)^{1/2}]\mathrm{sn}^2\,u=Ec_2^0(u). \tag{A22}$$

The corresponding eigenvalue is

$$\varepsilon=1+k^2-2(1-k^2+k^4)^{1/2}<0. \tag{A23}$$

The $\lambda\Phi^4$ theory thus has no stable classical solution equivalent to that which we have been discussing in the present paper.

[1)] If $e_i\to\lambda e_i$, then ω, $\omega'\to\lambda^{-1/2}\omega$, $\lambda^{-1/2}\omega'$, and Eqs. (20) remain the same in form if we redefine a parameter: $\alpha\to\lambda^{-1/2}\alpha$.
Note added in proof, 30 May 1991. Unfortunately, recent work has shown that C in (2), (2′) is zero for arion models.

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Translated by D. Parsons