Application of an indefinite metric to go over to a Bose description of $SU(3)$ Hamiltonians: The excitation spectrum of spin nematics

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For the example of a spin nematic, a procedure for going over in a well defined manner to a Bose representation for quantum $SU(3)$ Hamiltonians is developed using a formalism with an indefinite metric and pseudo-Hubbard operators. An important aspect of the theory is that the finiteness of the number of physical states is taken into account. This is achieved by the introduction and systematic use of a metric operator, for which a simple form in terms of Bose operators is proposed. It is shown that in this approach the quantum Bose analog of the Hamiltonian is a Hermitian operator. This circumstance has removed a number of fundamental contradictions that have arisen previously when a Bose analog for the Hamiltonian has been obtained by essentially regarding the Hubbard and pseudo-Hubbard operators as identical and ignoring the fact that the number of physical states is bounded. For the spin-nematic state of a magnet with $S = 1$, integral equations determining the basic characteristics of the system are obtained with allowance for anharmonic effects and are solved in analytic form. This has made it possible, in the framework of a nonlinear theory, to write expressions for the two branches of the quantum excitation spectrum in explicit form. The character of the renormalization of the spectral parameters that is associated with the presence of zero-point quantum oscillations in the system is studied. The renormalization of the critical field in the transition of the system from the spin-nematic state is calculated analytically.

1. INTRODUCTION

In the development of the quantum theory of magnetically ordered systems several ways of going over to a quasiparticle description have been formulated.[1–4] At the basis of these methods lie the so-called representations of spin operators in terms of Bose operators creating and annihilating excitations at individual lattice sites. The best known representations are the Holstein-Primakoff[5] and Dyson-Maleev[6,7] representations. Recently, new transformations have appeared, among which we mention the Goldhirsch transformation,[8,9] the Baryakhtar-Krivoruchko-Yablonskiĭ representation,[10,11] which expresses the spin operators in terms of Bose and Fermi operators, and also the Belinicher-L'vov representation.[12] The current state of theory devoted to constructing Bose analogs of spin Hamiltonians and to taking systematic account of interaction in a magnon gas is reflected in the review in Ref. 13. The Bose-description methods mentioned turn out to be adequate if the spin Hamiltonian contains no interactions of a tensor character or the energy of such interactions is relatively small.

At the same time, there exist magnets[14–17] for which tensor interactions play a decisive role and the above schemes for constructing a Bose representation are consequently certainly inapplicable. Above all, this applies to systems described by those Hamiltonians (henceforth, $SU(3)$ Hamiltonians) for which a correct analysis requires the use of the $SU(3)$ algebra. In this case, as is well known, the basis of the generators is increased from the three operators of the $SU(2)$ algebra to eight. Physically, this is because a consistent description of the dynamics of such magnets requires that we take into account not only the dipole degrees of freedom but also the degrees of freedom associated with the presence of a quadrupole moment.

These features are manifested most sharply in those spin nematics (the terminology introduced in Ref. 16 is used) in which the spin-nematic state is induced by strong interaction of quadrupole moments. The ground state of a spin nematic possesses two distinctive features. First, as noted in Ref. 16, there is spontaneous breaking of the symmetry under spin-space rotations, while the invariance under time reversal is preserved. Second, the lowest single-ion level is the nonmagnetic state $|0\rangle$, whereas the two "magnetic" states $|+1\rangle$ and $|-1\rangle$ appear on an equal footing ($H = 0$) as the upper levels. This hierarchy of single-ion states leads to profound consequences in the construction of a Bose representation of the Hamiltonian of a spin nematic, since an adequate description of the system requires the introduction of two types of bosons.[18] This constitutes the principal difference between the Bose description of $SU(3)$ Hamiltonians and the Bose description of Hamiltonians with weak tensor interactions, for which it is possible to confine oneself to introducing just one boson field.[13] From a physical point of view, the necessity of introducing two boson fields is dictated by the participation of both the dipole and the quadrupole moment in the dynamics.

It is appropriate to note two important aspects of the program for going over to a Bose description of $SU(3)$ Hamiltonians. The first is the need to develop a bosonization procedure that will preserve the correct commutation relations for the initial (spin and quadrupole) operators. This problem is solved by constructing pseudo-Hubbard operators (see Ref. 19 and Sec. 3 of the present article) whose Lie algebra coincides with the Lie algebra of the true Hubbard operators.

The second aspect of the problem under discussion has a more subtle nature and is connected with the presence, in

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the Hilbert space in which the Bose analog of the Hamiltonian acts, of an infinite set of nonphysical states. When observable characteristics are calculated, these states, if their occupancy is nonzero, will make an additional contribution that does not exist in reality. Therefore, a correct procedure for obtaining a Bose analog of the Hamiltonian should also incorporate a way of cutting out the contributions from the nonphysical states.

The latter circumstance has special significance for systems in which so-called zero-point quantum oscillations (ZQO) are present. Since the exact wave function of the ground state of such systems is unknown, an approximate function is used as the bare vacuum. In these conditions, the Hamiltonian contains operator terms that lead to the creation of several excitations on one site, which gives rise to nonzero occupancy of the nonphysical states. The essential point is that this occupancy for $T\to 0$ is determined not by thermal excitation (as is the case in systems without ZQO, when the exact ground-state wave function is used), but by the intensity of the ZQO. Therefore, the contribution of the nonphysical states for systems with ZQO is algebraically small (in our case, proportional to $(\xi/I)^2$, where ξ/I is the parameter determining the intensity of the ZQO), rather than exponentially, proportional to $\exp(-T_c/T)$. For an exchange-anisotropic ferromagnet with spin $S=\frac{1}{2}$, these features were demonstrated in Ref. 20. The power-law character of the contribution under discussion implies that in the development of a nonlinear theory the cutting out of the contribution from the nonphysical states becomes obligatory, since the first anharmonic corrections have the same algebraic smallness (see below).

In view of what has been said, it appears appropriate to develop for the construction of an exact Bose analog of an $SU(3)$ Hamiltonian a regular procedure that preserves the algebra of the original operators and takes the presence of nonphysical states into account. In the paper it is shown that the use of a formalism involving the introduction of an indefinite metric and pseudo-Hubbard operators makes it possible to solve this problem. A nonlinear theory of the excitation spectrum of spin nematics is constructed by means of the technique developed. Integral equations determining the characteristics of the spectrum are obtained and solved in analytic form. This has made it possible to find the renormalized collective-excitation spectrum of a spin nematic with $S=1$ and biquadratic exchange with allowance for anharmonic effects. The renormalization (due to the presence of zero-point quantum oscillations in the system) of the critical value of the magnetic field is determined.

2. HAMILTONIAN OF A SPIN NEMATIC IN THE HUBBARD-OPERATOR REPRESENTATION

We demonstrate the development of a procedure for constructing a Bose representation for $SU(3)$ Hamiltonians, for the example of the Hamiltonian of a spin nematic,[16] which, for $S=1$, is characterized by the presence of tensor interactions of both a single-ion and a two-ion nature in the system:

$$\mathcal{H}=-\tfrac{1}{2}\sum_{fg}[I_{fg}(\mathbf{S}_f\mathbf{S}_g)+K_{fg}(\mathbf{S}_f\mathbf{S}_g)^2]-\sum_f[B_2^0O_{2f}^0+B_2^2O_{2f}^2+HS_f^z]. \tag{1}$$

In the absence of an external magnetic field, the spin-nematic state, in which the components of the tensor order parameter (in the molecular-field approximation) take the values

$$q_2^0=\langle O_2^0\rangle=-2,\quad q_2^2=\langle O_2^2\rangle=0,\quad \sigma=\langle S^z\rangle=0, \tag{2}$$

is realized if[16,21]

$$|B_2^2|<-3B_2^0-2(I_0-K_0),\quad I_0>K_0,\qquad |B_2^2|<-3B_2^0,\quad I_0<K_0. \tag{3}$$

Bearing this in mind, we go over to the representation using Hubbard operators,[22] the action of which on the single-ion states is specified by the rule

$$X_f^{nm}|\Psi_p(f)\rangle=\delta_{mp}|\Psi_n(f)\rangle,\quad n,\ m,\ p=0,\ 1,\ 2. \tag{4}$$

In our specific case, the single-ion states are eigenstates of the operator S_f^z; we use the following system of notation:

$$S^z|\Psi_0\rangle=0|\Psi_0\rangle,\quad S^z|\Psi_1\rangle=+1|\Psi_1\rangle,\quad S^z|\Psi_2\rangle=-1|\Psi_2\rangle, \tag{5}$$

where, for brevity, the lattice-site index has been omitted. The operators X_f^{nm} satisfy the following commutation relations:

$$[X_f^{nm},\ X_g^{pq}]_-=\delta_{fg}(\delta_{mp}X_f^{nq}-\delta_{nq}X_f^{pm}). \tag{6}$$

In the mathematical literature the set of operators X_f^{nm} is called a Weyl basis,[23] and can be used to expand elements of the $SU(3)$ algebra. The corresponding representation of the spin and quadrupole operators in terms of Hubbard operators has the following appearance:

$$\begin{aligned}&S_f^+=2^{1/2}(X_f^{10}+X_f^{02}),\quad S_f^-=(S_f^+)^+,\\ &S_f^z=X_f^{11}-X_f^{22},\quad O_{2f}^2=X_f^{12}+X_f^{21},\\ &O_{2f}^0=3X_f^{11}+3X_f^{22}-2,\quad O_{2f}^{xy}=i(X_f^{21}-X_f^{12}),\\ &O_{2f}^{zx}=2^{-1/2}(X_f^{01}+X_f^{10}-X_f^{20}-X_f^{02}),\\ &O_{2f}^{yz}=-2^{-1/2}i(X_f^{10}-X_f^{01}+X_f^{20}-X_f^{02}).\end{aligned} \tag{7}$$

Application of these relations makes it possible to write the Hamiltonian of a non-Heisenberg magnet in the atomic representation,[24] which is convenient to use in developing a procedure for going over to a Bose description:

$$\begin{aligned}\mathcal{H}=&2NB_2^0-NK_0+\sum_{jL}[\varepsilon_aX_f^{11}+\varepsilon_bX_f^{22}+B_2^2(X_f^{12}+X_f^{21})]\\ &-\sum_{fg}\{I_{fg}(X_f^{10}X_g^{01}+X_f^{02}X_g^{20})+(I_{fg}-K_{fg})(X_f^{10}X_g^{20}+X_f^{02}X_g^{01})\\ &+K_{fg}X_f^{12}X_g^{21}+\tfrac{1}{2}(I_{fg}+K_{fg})(X_f^{11}X_g^{11}+X_f^{22}X_g^{22})\\ &+(2K_{fg}-I_{fg})X_f^{11}X_g^{22}\},\end{aligned} \tag{8}$$

where

$$\varepsilon_a=K_0+D-H,\quad \varepsilon_b=K_0+D+H,\quad D\equiv-3B_2^0.$$

It can be seen from the structure of the Hamiltonian (8) that for $K_{fg}=I_{fg}$ the exact ground-state function (it is as-

sumed that the conditions (3) are fulfilled) can be written in the form of a product:

$$\mathscr{H}\prod_f |\Psi_0(f)\rangle = E_0 \prod_f |\Psi_0(f)\rangle, \quad E_0 = 2NB_2^0 - NK_0. \tag{9}$$

The existence of an exact eigenfunction in multiplicative form is closely related to the symmetry properties of the Hamiltonian.[1)] To demonstrate this thesis more simply, we assume first that $H=0$ and $B_2^0 = B_2^2 = 0$. We note that in this isotropic case the existence of the exact eigenfunction (9) was noted previously in Ref. 18. If $K_{fg} = I_{fg}$, then $[\Sigma_f O_{2f}^0, \mathscr{H}]_- = 0$ (Ref. 25) and the symmetry group of the exchange Hamiltonian is the group $SU(3)$. Then the states of the system are characterized not only by the value of the total dipole moment but also by the value of the total zz component of the quadrupole moment. In particular, there should exist a state Ψ such that

$$\left(\sum_f O_{2f}^0\right)\Psi = (-2N)\Psi. \tag{10}$$

However, the smallest of the possible values of the zz component of the quadrupole moment can occur only in the case when, on each lattice site, the ion is in the state $|\Psi_0(f)\rangle$, since $O_{2f}^0|\Psi_0(f)\rangle = (-2)|\Psi_0(f)\rangle$ and $O_{2f}^0|\Psi_{1,2}(f)\rangle = |\Psi_{1,2}(f)\rangle$. It follows from this that a function Ψ satisfying Eq. (10) can have only the multiplicative structure (9). Furthermore, it is easy to see that the inclusion of single-site interactions in the form (1) does not change the structure of such an eigenfunction, since $O_{2f}^2|\Psi_0(f)\rangle = 0$. Here it should be borne in mind that in the presence of a rhombic component in the operator of the single-ion anisotropy energy the other eigenfunctions of the Hamiltonian are not eigenfunctions of the operator of the zz component of the quadrupole moment.

For $K_{fg} = I_{fg}$ it is not difficult to obtain also exact expressions for the two branches of the spectrum of elementary excitations:

$$\omega^{\mp}(\mathbf{q}) = D + I_0 - I_{\mathbf{q}} \mp [H^2 + (B_2^2)^2]^{1/2}. \tag{11}$$

It can be seen that the spin-nematic state under consideration will become unstable as soon as the field H reaches the value

$$H_c = [(D-B_2^2)(D+B_2^2)]^{1/2}, \quad D - B_2^2 > 0. \tag{12}$$

For $K_{fg} \neq I_{fg}$, the function $\Pi_f|\Psi_0(f)\rangle$ is not the exact ground-state function, and ZQO are present in the system. In this case the solution of the problem of the excitation spectrum becomes substantially more complicated. However, when the condition

$$|I-K| \ll I+K, \quad I, K>0 \tag{13}$$

is fulfilled (henceforth we shall confine ourselves to the nearest-neighbor approximation) the intensity of the ZQO is insignificant and the solution of the problem can be obtained by perturbation theory. For this we go over to the Bose description. We note first that the Hamiltonian (8) can be written in the form

$$\mathscr{H} = \sum_f \sum_{nm} \varepsilon_{nm} X_f^{nm} + {}^1/_2 \sum_{fg} \sum_{nmpq} V_{fg}^{nm,pq} X_f^{nm} X_g^{pq}, \tag{14}$$

which shows explicitly that we are dealing with a set of interacting three-level subsystems. In physics such systems are being studied intensively, and the formalism developed below is applicable in equal measure without restrictions on the specifics, since the precise nature of the single-site states $|\Psi_n(f)\rangle$ in the construction of the Bose representation is not important.

3. PSEUDO-HUBBARD OPERATORS

As in Refs. 18, 19, and 26, with each lattice site we associate two types of Bose operators, with standard commutation relations

$$[a_f, a_g^+]_- = \delta_{fg}, \quad [b_f, b_g^+]_- = \delta_{fg}, \quad [a_f, b_g^+]_- = 0, \ldots \tag{15}$$

Next we introduce two sets of eigenstates of the operators a^+a and b^+b (so long as all the discussions pertain to a fixed lattice site, the subscript f, both on the operators and on the basis functions, need not be written; it is easy to restore it later):

$$a^+a\varphi_n = n\varphi_n, \quad b^+b\chi_m = m\chi_m, \tag{16}$$

where the indices n and m run from 0 to ∞. The functions φ_n and χ_m can be taken as basis vectors of Hilbert spaces $\mathfrak{h}_a$ and $\mathfrak{h}_b$, respectively. Choosing the usual normalization of the basis vectors:

$$\varphi_n = \frac{(a^+)^n}{(n!)^{1/2}}\varphi_0, \quad \chi_m = \frac{(b^+)^m}{(m!)^{1/2}}\chi_0, \quad a\varphi_0 = 0, \quad b\chi_0 = 0, \tag{17}$$

in $\mathfrak{h}_a$ and $\mathfrak{h}_b$ we define metrics such that

$$(\varphi_n, \varphi_{n'}) = \delta_{nn'}, \quad (\chi_m, \chi_{m'}) = \delta_{mm'}. \tag{18}$$

Then, in $\mathfrak{h}_a$ and $\mathfrak{h}_b$ with the metric (18),

$$(a^+)^+ = a, \quad (b^+)^+ = b. \tag{19}$$

Let us construct the direct product of the Hilbert spaces $\mathfrak{h}_a$ and $\mathfrak{h}_b$, i.e., introduce the Hilbert space $\mathfrak{h}_{ab} = \mathfrak{h}_a \otimes \mathfrak{h}_b$. As the basis of $\mathfrak{h}_{ab}$ we choose the basis that is the direct product of the bases of the Hilbert spaces $\mathfrak{h}_a$ and $\mathfrak{h}_b$. Then the set of basis functions Φ_{nm} of the space $\mathfrak{h}_{ab}$ is written in the form

$$\Phi_{nm} \equiv \varphi_n \otimes \chi_m = \frac{(a^+)^n (b^+)^m}{(n!m!)^{1/2}} \varphi_0 \otimes \chi_0. \tag{20}$$

It can be seen that $a\Phi_{0m} = 0$ and $b\Phi_{n0} = 0$. The scalar product in $\mathfrak{h}_{ab}$ is defined as the scalar product in the space that is the direct product of the subspaces.[27] Then, for the basis vectors,

$$(\Phi_{nm}, \Phi_{pq}) = (\varphi_n \otimes \chi_m, \varphi_p \otimes \chi_q) = (\varphi_n, \varphi_p)(\chi_m, \chi_q) = \delta_{np}\delta_{mq}. \tag{21}$$

From the obvious relations

$$(a^+a)(b^+b)\Phi_{nm} = (b^+b)(a^+a)\Phi_{nm} = nm\Phi_{nm} \tag{22}$$

it follows that the function Φ_{nm} describes a state with n bosons of type "a" and m bosons of type "b." The function Φ_{00}

describes a state without particles.

We set in correspondence to the function $|\Psi_0\rangle$ the function Φ_{00} (the vacuum state) of the space $\mathfrak{h}_{ab}$, to the function $|\Psi_1\rangle$ the function Φ_{10} (the state with one boson of type "a"), and, finally, to the function $|\Psi_2\rangle$ the function Φ_{01} (the state with one boson of type "b"). In the Hilbert space $\mathfrak{h}_{ab}$ there are states $\Phi_{11}, \Phi_{20}, \Phi_{21}, \Phi_{02}, \ldots$, to which correspond no states from the physical space $\mathfrak{h}^{(3)}$ generated by the basis of single-site states $|\Psi_n\rangle$. Therefore, those states Φ_{nm} with $n+m \geqslant 2$ are called nonphysical states.

We introduce, by construction, operators $\tilde{X}^{pq}$ acting in $\mathfrak{h}_{ab}$:

$$\begin{aligned} &\tilde{X}^{01}=(1-a^+a-b^+b)a, \quad \tilde{X}^{10}=a^+, \\ &\tilde{X}^{02}=(1-a^+a-b^+b)b, \quad \tilde{X}^{20}=b^+, \\ &\tilde{X}^{12}=a^+b, \quad \tilde{X}^{21}=b^+a, \\ &\tilde{X}^{11}=a^+a, \quad \tilde{X}^{22}=b^+b, \quad \tilde{X}^{00}=1-a^+a-b^+b. \end{aligned} \tag{23}$$

It is not difficult to convince oneself of the validity of the relations

$$[\tilde{X}^{nm}, \tilde{X}^{pq}]_-=\delta_{mp}\tilde{X}^{nq}-\delta_{nq}\tilde{X}^{pm}. \tag{24}$$

It can be seen that the Lie algebra constructed on the basis of the operators $\tilde{X}^{nm}$, as in Ref. 19, coincides with the Lie algebra constructed on the basis of the operators X^{nm}. However, in contrast to Ref. 19, the operators $\tilde{X}^{nm}$ introduced by the formulas (23) (the pseudo-Hubbard operators, henceforth) cannot be identified with the Hubbard operators X^{nm}. In addition, the definitions of the four pseudo-Hubbard operators have been changed, since when the corresponding formulas from Ref. 19 are used it is not possible to introduce an indefinite metric, which plays an essential role in the construction of the Bose analog of the Hamiltonian.

It is not difficult to understand that the fact that the commutation rules (6) and (24) coincide is entirely insufficient for X^{pq} to be identified with $\tilde{X}^{pq}$. In principle this cannot be done, if only because the Hubbard and pseudo-Hubbard operators act in spaces of different dimensionalities: The dimensionality of the physical space $\mathfrak{h}^{(3)}$ is equal to three, while the Hilbert space $\mathfrak{h}_{ab}$ is infinite-dimensional.

In addition, the Hubbard operators X^{pq} and X^{qp} are mutually adjoint, while not all the pseudo-Hubbard operators possess this property. We note also that the algebra of ordinary multiplication of the Hubbard operators differs from the corresponding algebra of the pseudo-Hubbard operators.

Thus, strictly speaking, the Hubbard operators, and, consequently, the Hamiltonian (14) as well, cannot be expressed in terms of any combinations of Bose operators. At the same time, it is possible to construct an operator that acts in the infinite-dimensional Hilbert space (and hence can be expressed in terms of Bose operators), such that its matrix elements on the class of physical states are equal to the corresponding matrix elements of the initial Hamiltonian, while its matrix elements between states at least one of which is nonphysical are equal to zero. The operator constructed in this manner is called a Bose analog. Therefore, when we speak of the construction of a Bose representation of the Hamiltonian (14) we shall have in mind the search for a way of constructing a Bose analog of the Hamiltonian. As will be seen from the following, a key role in the solution of this problem is played by pseudo-Hubbard operators in combination with the indefinite-metric formalism.

4. INTRODUCTION OF THE INDEFINITE METRIC AND THE BOSE ANALOG OF THE HAMILTONIAN

Following ideas of Dyson[6] (a detailed account of the technique of using an indefinite metric is contained in Ref. 2, and our construction is based on this material), we introduce a new metric in the Hilbert space $\mathfrak{h}_{ab}$, i.e., we redefine the scalar product of vectors. This scalar product, in contrast to the old one defined by Eq. (21), will be denoted not by round brackets but by angular brackets. To the new metric we assign properties so that in it the operators $\tilde{X}^{pq}$ and $\tilde{X}^{qp}$ for $q \neq p$ are mutually adjoint, while the operators $\tilde{X}^{pp}$ are self-adjoint. In their strict mathematical formulation these requirements take the following form:

$$\begin{aligned} &\langle \tilde{X}^{pq}\Phi_{nm}, \Phi_{rs}\rangle=\langle \Phi_{nm}, (\tilde{X}^{pq})^+\Phi_{rs}\rangle=\langle \Phi_{nm}, \tilde{X}^{qp}\Phi_{rs}\rangle, \\ &\langle \tilde{X}^{pp}\Phi_{nm}, \Phi_{rs}\rangle=\langle \Phi_{nm}, (\tilde{X}^{pp})^+\Phi_{rs}\rangle=\langle \Phi_{nm}, \tilde{X}^{pp}\Phi_{rs}\rangle. \end{aligned} \tag{25}$$

To specify the way in which the new scalar product is calculated we introduce the metric operator F, defined by the relation

$$\langle \Phi_{nm}, \Phi_{rs}\rangle=(\Phi_{nm}, F\Phi_{rs}). \tag{26}$$

In fact, the properties of the operator F determine the properties of the new metric in $\mathfrak{h}_{ab}$. From the requirement that the relations (25) be valid, and also from (26), we obtain two chains of equalities:

$$\begin{aligned} &\langle \Phi_{nm}, \tilde{X}^{qp}\Phi_{rs}\rangle=(\Phi_{nm}, F\tilde{X}^{qp}\Phi_{rs}) \\ &=\langle \tilde{X}^{pq}\Phi_{nm}, \Phi_{rs}\rangle=(\tilde{X}^{pq}\Phi_{nm}, F\Phi_{rs})=(\Phi_{nm}, (\tilde{X}^{pq})^+F\Phi_{rs}), \end{aligned} \tag{27}$$

$$\begin{aligned} \langle \Phi_{nm}, \tilde{X}^{pp}\Phi_{rs}\rangle&=(\Phi_{nm}, F\tilde{X}^{pp}\Phi_{rs})=\langle \tilde{X}^{pp}\Phi_{nm}, \Phi_{rs}\rangle \\ &=(\tilde{X}^{pp}\Phi_{nm}, F\Phi_{rs})=(\Phi_{nm}, \tilde{X}^{pp}F\Phi_{rs}). \end{aligned} \tag{28}$$

It follows from (28) that the operator F commutes with the operators $\tilde{X}^{pp}$. Therefore, in the basis of the states Φ_{nm} with the metric (21) the operator F is diagonal:

$$F_{nm, rs}\equiv(\Phi_{nm}, F\Phi_{rs})=\delta_{nr}\delta_{ms}F_{nm}. \tag{29}$$

To find the diagonal elements F_{nm} we make use of the chain of equalities (27). It can be seen that

$$F_{nm}(\Phi_{nm}, \tilde{X}^{qp}\Phi_{rs})=(\Phi_{nm}, (\tilde{X}^{pq})^+\Phi_{rs})F_{rs}. \tag{30}$$

Setting $p=0$ and $q=1, 2$, and making use of the expressions (23) for the pseudo-Hubbard operators, we find from (30) a system of equations that is satisfied by the diagonal matrix elements of the metric operator:

$$F_{n+1, m}=(1-n-m)F_{nm}=F_{n, m+1}. \tag{31}$$

Solving (31) we find $F_{00}=F_{10}=F_{01}=1$, while for all n and m such that $n+m \geqslant 2$ we have $F_{nm}=0$. Since the states Φ_{00}, Φ_{10}, and Φ_{01} correspond to physical states of the system, it can be seen that the metric operator F coincides with

the projection operator from the Hilbert space $\mathfrak{h}_{ab}$ on to the subspace of physical states.

It follows from (26) and the above properties of the metric operator that the new scalar product of a vector corresponding to a nonphysical state with itself is equal to zero, and therefore the new metric is indefinite.

We go over to a one-index system of notation, setting

$$\Phi_0\equiv\Phi_{00},\quad \Phi_1\equiv\Phi_{10},\quad \Phi_2\equiv\Phi_{01},\quad \Phi_3=\Phi_{20},\ldots.$$

Then it is not difficult to convince oneself that the equalities

$$\langle\Psi_n|X^{pq}|\Psi_m\rangle=(\Phi_n,\ F\bar{X}^{pq}\Phi_m),\ n,\ m,\ p,\ q=0,\ 1,\ 2. \tag{32}$$

are valid. The formula (32) is fundamental in the construction of an exact Bose analog of the Hamiltonian. It can be seen that to "get out" of the indefinite metric and work with the ordinary scalar product it is necessary to have an explicit form of the operator F. Taking into account that in the basis of states Φ_n of the Hilbert space $\mathfrak{h}_{ab}$ with the metric (21) the operator F has diagonal form, we can write

$$F=(1-a^+b^+ba)F_aF_b, \tag{33}$$

where

$$F_a=1+\sum_{n=2}^{\infty}A_n(a^+)^n(a)^n,\quad F_b=1+\sum_{n=2}^{\infty}A_n(b^+)^n(b)^n.$$

For the expansion coefficients A_n it is not difficult to obtain the following system of equations:

$$1+\sum_{m=2}^{n}[n(n-1)\,(n-2)\ldots(n-m+1)]A_m=0,\quad n=2,3,\ldots,$$

solving which, we find

$$A_2=-{}^1/_2,\ A_3={}^1/_3,\ A_4=-{}^1/_8,\ A_5={}^1/_{30},\ldots. \tag{34}$$

Returning to the analysis of the full Hamiltonian (14), we restore the lattice-site index f. After this we construct the physical space of dimensionality 3^N, by taking the direct product of the original three-dimensional physical spaces. In this space the $SU(3)$ Hamiltonian (14) acts. In an analogous way, from the space $\mathfrak{h}_{ab}(f)$ we construct the Hilbert space in which the Bose analog of the $SU(3)$ Hamiltonian under consideration will act. Then, using the relations (32), one can show rigorously that the rule for obtaining the Bose analog of the Hamiltonian has the form

$$\mathcal{H}_B=F^{\otimes}\mathcal{H}\{\bar{X}^{pq}\},\quad F^{\otimes}=\prod_f{}^{\otimes}F_f, \tag{35}$$

where the operator $F^{\otimes}$ is the direct product of the metric operators for the crystal-lattice sites.

Thus, the prescription for writing the exact Bose analog of the quantum Hamiltonian (14) is as follows: In the Hamiltonian (14) the Hubbard operators are replaced by pseudo-Hubbard operators, for which the expressions (23) are used. The important point is that it is necessary to multiply the operator obtained in this way by $F^{\otimes}$. As will be shown below, for systems with ZQO the contribution from the metric operator becomes important even for $T=0$. Running ahead, we note that taking $F^{\otimes}$ into account leads to restoration of the self-adjointness of the Hamiltonian operator in the Bose representation.

5. HAMILTONIAN OF A SPIN NEMATIC IN THE BOSE REPRESENTATION

Applying the formulas (35) and (23) for the Hamiltonian (8), we obtain (henceforth, for simplification, we set $B_2^2=0$)

$$\mathcal{H}_B=E_0+F^{\otimes}\mathcal{H}'_{(2)}+F^{\otimes}\mathcal{H}'_{(4)}+F^{\otimes}\mathcal{H}'_{(6)}, \tag{36}$$

where

$$\mathcal{H}'_{(2)}=\sum_p\{(\varepsilon_a-I_p)a_p^+a_p+(\varepsilon_b-I_p)b_p^+b_p -\xi_p(a_p^+b_{-p}^++b_pa_{-p})\},\quad \xi_p=I_p-K_p. \tag{37}$$

It is not difficult to convince oneself that $\mathcal{H}'_{(4)}$ and $\mathcal{H}'_{(6)}$ are non-Hermitian operators. Therefore, if we neglect the effect of $F^{\otimes}$, i.e., set $F^{\otimes}=1$, the entire Bose analog $\mathcal{H}_B$ will also be non-Hermitian. This circumstance leads to a number of problems, some of which have been discussed in Ref. 20. In reality, however, allowance for $F^{\otimes}$ automatically ensures that $\mathcal{H}_B$ is Hermitian. Representing $\mathcal{H}_B$ in a form normal in the Bose operators, we obtain

$$\mathcal{H}_B=E_0+\mathcal{H}_{(2)}+\mathcal{H}_{(4)}+\mathcal{H}_{(6)}+\mathcal{H}_{(8)}+\ldots. \tag{38}$$

Here, $\mathcal{H}_{(2)}=\mathcal{H}'_{(2)}$, whereas $\mathcal{H}_{(4)}$ differs substantially from $\mathcal{H}'_{(4)}$ and is given by the expression

$$\mathcal{H}_{(4)}=\frac{1}{N}\sum_{1234}\Gamma_a(12;34)a_1^+a_2^+a_3a_4 +\frac{1}{N}\sum_{1234}\Gamma_b(12;34)b_1^+b_2^+b_3b_4 +\frac{1}{N}\sum_{1234}\Gamma_{ab}(12;34)a_1^+b_2^+b_3a_4 +\frac{1}{N}\sum_{1234}\{(\xi_3+\xi_4)(a_1^+a_2a_3b_4+b_1^+b_2b_3a_4)+\text{H.c.}\}\Delta(1-2-3-4), \tag{39}$$

where we have introduced the following notation:

$$\Gamma_a(12;\ 34)=[I_1+I_4-\varepsilon_a-{}^1/_2(I_{1-4}+K_{1-4})]\Delta(1+2-3-4),$$
$$\Gamma_b(12;\ 34)=[I_1+I_4-\varepsilon_b-{}^1/_2(I_{1-4}+K_{1-4})]\Delta(1+2-3-4),$$
$$\Gamma_{ab}(12;\ 34)=[I_1+I_2+I_3+I_4-2(K_0+D) -{}^1/_2(2K_{1-4}+K_{4-2}+K_{1-3})+\xi_{1-4}]\Delta(1+2-3-4), \tag{40}$$

in which $\Delta(1-2)$ is the Kronecker symbol. The Hermiticity of the operator $\mathcal{H}_{(4)}$ is ensured by the following obvious property of the bare scattering amplitudes involving four bosons:

$$\Gamma_\alpha(12;\ 34)=\Gamma_\alpha(43;\ 21],\quad \alpha=a,\ b,\ ab. \tag{41}$$

The operator term $\mathcal{H}_{(6)}$ in (38) can be written in the form

$$\mathscr{H}_{(6)}=L_a+L_b+G_1+G_1^+, \tag{42}$$

where

$$\begin{aligned}
L_a=&-\frac{1}{2N}\sum_{p1234}(\varepsilon_a-I_p)a_p^+a_1^+a_2^+a_3a_4a_p\Delta(1+2-3-4)\\
&+\frac{1}{N^2}\sum_{1-6}(\varepsilon_a-I_1-I_6+I_{1-4}+K_{1-4}-I_{1-5-6})\\
&\times a_1^+a_2^+a_3^+a_4a_5a_6\Delta(1+2+3-4-5-6)\\
&-\frac{1}{N}\sum_{p1234}(\varepsilon_a-I_p)a_p^+a_1^+b_2^+b_3a_4a_p\Delta(1+2-3-4)\\
&-\frac{1}{2N}\sum_{p1234}(\varepsilon_b-I_p)a_1^+a_2^+b_p^+b_pa_3a_4\Delta(1+2-3-4)\\
&+\frac{1}{N^2}\sum_{1-6}\{2\varepsilon_a+\varepsilon_b-2(I_1+I_6)-I_3-I_4+3K_{1-6}-I_{3-4}+K_{1-4}\\
&+K_{3-6}+2K_{3-4}-I_{1-5-6}-I_{1-4-6}-I_{1+2-6}\}\\
&\times a_1^+a_2^+b_3^+b_4a_5a_6\Delta(1+2+3-4-5-6).
\end{aligned} \tag{43}$$

The operator L_b is obtained from L_a by the replacements $\varepsilon_a\to\varepsilon_b$, $\varepsilon_b\to\varepsilon_a$, $a_q\to b_q$, $b_p\to a_p$, etc. For G_1 the following form is valid:

$$\begin{aligned}
G_1=&\frac{1}{N}\sum_{p1234}\xi_p\{a_p^+b_{-p}^+a_1^+b_2^+b_3a_4+{}^1/_2a_p^+b_{-p}^+a_1^+a_2^+a_3a_4\\
&+{}^1/_2a_p^+b_{-p}^+b_1^+b_2^+b_3b_4\}\Delta(1+2-3-4)\\
&-\frac{1}{N^2}\sum_{1-6}\{(\xi_1+\xi_4+\xi_{1+2-6})\\
&\times(b_1^+a_2^+a_3^+a_4^+a_5a_6+a_1^+b_2^+b_3^+b_4^+b_5b_6)\\
&+(2\xi_1+2\xi_4+\xi_{1+2-6}+\xi_{1+4-6})b_1^+b_2^+a_3^+a_4^+a_5b_6\}\\
&\times\Delta(1+2+3+4-5-6).
\end{aligned} \tag{44}$$

Thus, taking $F^{\otimes}$ into account leads not only to restoration of the Hermiticity but also to the presence in $\mathscr{H}_B$ of operator terms describing interaction processes involving eight, ten, etc., quasiparticles. Since these terms do not contain any additional small parameters, in each specific case it is necessary to keep track of their influence on the results of the theory. The fate of $\mathscr{H}_{(8)}$, $\mathscr{H}_{(10)}$, ... is most simply resolved when the density of bare quasiparticles is small. Then processes involving an ever larger number of quasiparticles lead to contributions proportional to ever higher powers of the small parameters that control the quasiparticle density (T/T_c determines the thermal excitation of quasiparticles, and ξ/I is the quantity determining the intensity of the ZQO). For $T=0$, when there is no thermal excitation, the number of bare quasiparticles is determined entirely by the intensity of the ZQO. For small ξ/I the quasiparticle density is small and the terms $\mathscr{H}_{(8)}$ give a contribution proportional to $(\xi/I)^3$. Therefore, in developing a theory exact to second order in ξ/I one can neglect the terms $\mathscr{H}_{(8)}$, $\mathscr{H}_{(10)}$, ... in the Hamiltonian.

6. EXCITATION SPECTRUM OF THE SPIN-NEMATIC STATE OF AN ANISOTROPIC MAGNET WITH BIQUADRATIC EXCHANGE

To find the spectrum of the elementary excitations we apply to the Hamiltonian (38) a unitary transformation

$$\mathscr{H}_B\to\tilde{\mathscr{H}}_B=U\mathscr{H}_BU^+, \tag{45}$$

where the unitary operator U is determined by the expression

$$U=\exp\left\{\sum_p\varphi_p(a_pb_{-p}-b_{-p}^+a_p^+)\right\}. \tag{46}$$

It is not difficult to establish the transformation laws for the Bose operators:

$$\begin{aligned}
&\tilde{a}_p\equiv Ua_pU^+=u_pa_p+v_pb_{-p}^+,\\
&\tilde{b}_p\equiv Ub_pU^+=u_pb_p+v_pa_{-p}^+,\\
&u_p=\operatorname{ch}\varphi_p,\quad v_p=\operatorname{sh}\varphi_p.
\end{aligned} \tag{47}$$

Thus, the transformation (45) corresponds to the well known Bogolyubov u-v transformation; this aspect has been noted previously, e.g., in Ref. 26.

It is obvious that the Hamiltonian $\mathscr{H}_B$ does not have normal-ordering form. If we implement the procedure of reducing the operator terms of $\tilde{\mathscr{H}}_B$ to normal form, then the quadratic operator acquires corrections from $\mathscr{H}_{(4)}$ and from $\mathscr{H}_{(6)}$ (Refs. 13, 29–31). Analogous renormalizations also arise in terms containing products of a larger number of operators. As a result, the Hamiltonian $\tilde{\mathscr{H}}_B$ can be written in the form

$$\tilde{\mathscr{H}}_B=\tilde{E}_c+\tilde{\mathscr{H}}_{(2)}+\tilde{\mathscr{H}}_{(4)}+\dots, \tag{48}$$

where the quadratic form is given by the expression

$$\tilde{\mathscr{H}}_{(2)}=\sum_p\{\tilde{\varepsilon}_{ap}\tilde{a}_p^+\tilde{a}_p+\tilde{\varepsilon}_{bp}\tilde{b}_p^+\tilde{b}_p-B_p(\tilde{a}_p^+b_{-p}^++\text{H.c.})\}. \tag{49}$$

Here,

$$\begin{aligned}
\tilde{\varepsilon}_{ap}=&\varepsilon_a-I_p+\frac{1}{N}\sum_q[\Gamma_a(\mathbf{qp};\mathbf{qp})]_{\text{sym}}v_q^2\\
&+\frac{1}{N}\sum_q\Gamma_{ab}(\mathbf{qp};\mathbf{pq})v_q^2+\frac{2}{N}\sum_q(\xi_p+3\xi_q)u_qv_q\\
&-\frac{1}{N}\sum_q(\varepsilon_a+2\varepsilon_b-3I_q)u_q^2v_q^2\\
&+[2\xi_pv_p^2-2(\varepsilon_a-I_p)u_pv_p]\frac{1}{N}\sum_q u_qv_q\\
&+[6\xi_pu_pv_p-6(\varepsilon_a-I_p)v_p^2]\frac{1}{N}\sum_q v_q^2+\frac{1}{N^2}\sum_{ql}[8\varepsilon_a+4\varepsilon_b-9I_p\\
&-11(I_q+I_l)+3K_0+10K_{q-p}+15K_{q+l}-4I_{q+l}-5I_{q+p+l}]u_qv_qu_lv_l,
\end{aligned} \tag{50}$$

$$B_{\mathbf{p}}=\xi_{\mathbf{p}}-\frac{1}{N}\sum_{\mathbf{q}}\Gamma_{ab}(\mathbf{q},-\mathbf{q};\mathbf{p},-\mathbf{p})u_{\mathbf{q}}v_{\mathbf{q}}-\frac{2}{N}\sum_{\mathbf{q}}(\xi_{\mathbf{q}}+3\xi_{\mathbf{p}})v_{\mathbf{q}}^2$$
$$-[2\xi_{\mathbf{p}}u_{\mathbf{p}}v_{\mathbf{p}}-(\varepsilon_a+\varepsilon_b-2I_{\mathbf{p}})v_{\mathbf{p}}^2]\frac{1}{N}\sum_{\mathbf{q}}u_{\mathbf{q}}v_{\mathbf{q}}$$
$$-[6\xi_{\mathbf{p}}v_{\mathbf{p}}^2-3(\varepsilon_a+\varepsilon_b-2I_{\mathbf{p}})u_{\mathbf{p}}v_{\mathbf{p}}]\frac{1}{N}\sum_{\mathbf{q}}v_{\mathbf{q}}^2. \quad (51)$$

The expression for $\tilde{\varepsilon}_{b\mathbf{p}}$ can be obtained from (50) by replacing the index a by the index b, and vice versa. In writing (50) and (51) we have taken into account all the terms arising from $\mathscr{H}_{(4)}$, and also some of the terms arising from $\mathscr{H}_{(6)}$. Here we have discarded terms that certainly are of higher order than $(\xi_0/I_0)^2$.

From the condition that $\tilde{\mathscr{H}}_{(2)}$ be diagonal, we obtain an integral equation for the parameter of the transformation:

$$A_{\mathbf{p}}\operatorname{sh}2\varphi_{\mathbf{p}}-B_{\mathbf{p}}\operatorname{ch}2\varphi_{\mathbf{p}}=0, \quad (52)$$

where

$$A_{\mathbf{p}}=(\tilde{\varepsilon}_{a\mathbf{p}}+\tilde{\varepsilon}_{b\mathbf{p}})/2. \quad (53)$$

When (52) is fulfilled the quadratic form of the transformed Hamiltonian becomes diagonal:

$$\tilde{\mathscr{H}}_{(2)}=\sum_{\mathbf{p}}\{\Omega_a(\mathbf{p})a_{\mathbf{p}}^{+}a_{\mathbf{p}}+\Omega_b(\mathbf{p})b_{\mathbf{p}}^{+}b_{\mathbf{p}}\}, \quad (54)$$

and the two branches of the spectrum of the elementary excitations are determined by the expressions

$$\Omega_a(\mathbf{p})=\Omega_{\mathbf{p}}-\gamma,\quad \Omega_b(\mathbf{p})=\Omega_{\mathbf{p}}+\gamma,$$
$$\Omega_{\mathbf{p}}=(A_{\mathbf{p}}^2-B_{\mathbf{p}}^2)^{1/2},\quad \gamma=(\tilde{\varepsilon}_{b\mathbf{p}}-\tilde{\varepsilon}_{a\mathbf{p}})/2. \quad (55)$$

Let us analyze the distinctive features of the energy spectrum in the case when, in addition to the inequality $|\xi_0|\ll I_0$, which ensures a relatively low intensity of the ZQO, the inequality

$$D\ll I_0 \quad (56)$$

is fulfilled. In this case the solution of the integral equations can be obtained in the form of an expansion in powers of D/I_0 and ξ_0/I_0.

Performing the calculations to second order in the above-mentioned small parameters, we find that the expression for $A_{\mathbf{p}}$ can be written in the form

$$A_{\mathbf{p}}=D-\xi_0+{}^5/_2(W-1)\frac{\xi_0^2}{I_0}+(1+\mu)(I_0-I_{\mathbf{p}})$$
$$+\frac{1}{N}\sum_{\mathbf{q}}u_{\mathbf{q}}v_{\mathbf{q}}L_{\mathbf{p}}+3RM_{\mathbf{p}}, \quad (57)$$

while for $B_{\mathbf{p}}$ the following formula is valid:

$$B_{\mathbf{p}}=\left\{1+(W-1)\left(\frac{2D-5\xi_0}{2I_0}\right)\right\}\xi_{\mathbf{p}}-3RL_{\mathbf{p}}-\frac{1}{N}\sum_{\mathbf{q}}u_{\mathbf{q}}v_{\mathbf{q}}M_{\mathbf{p}}. \quad (58)$$

In these relations we have used the following notation:

$$L_{\mathbf{p}}=2\xi_{\mathbf{p}}v_{\mathbf{p}}^2-2(D-\xi_0+I_0-I_{\mathbf{p}})u_{\mathbf{p}}v_{\mathbf{p}},$$
$$M_{\mathbf{p}}=2\xi_{\mathbf{p}}u_{\mathbf{p}}v_{\mathbf{p}}-2(D-\xi_0+I_0-I_{\mathbf{p}})v_{\mathbf{p}}^2,$$
$$\mu=\left[(W-1)^2-\frac{7}{4}(W-1)\right]\left(\frac{\xi_0}{I_0}\right)^2-3R,\quad R=\frac{1}{N}\sum_{\mathbf{q}}v_{\mathbf{q}}^2, \quad (59)$$

where W denotes the numerical value of the Watson integral.[13] It follows from (57) and (58) that, to the accuracy that we are considering, $\Omega_{\mathbf{p}}$ is given by the expression

$$\Omega_{\mathbf{p}}=\left\{\Delta^2+\left[2(D-\xi_0)+(5W-3)\frac{\xi_0^2}{I_0}\right](I_0-I_{\mathbf{p}})\right.$$
$$\left.+C(D,\xi_0)(I_0-I_{\mathbf{p}})^2-(W-1)\left(\frac{\xi_0}{I_0}\right)^2\frac{I_{\mathbf{p}}(I_0-I_{\mathbf{p}})^2}{\omega_{\mathbf{p}}}\right\}^{1/2}, \quad (60)$$

where Δ denotes the gap in the elementary-excitation spectrum for $H=0$:

$$\Delta=[D(D-D_c)]^{1/2}. \quad (61)$$

It can be seen that for $H=0$ and $D=0$ the excitation spectrum in the phase under consideration has, in accordance with the Goldstone theorem, an activationless character. The quantity

$$D_c=2\xi_0[1-{}^3/_2(W-1)\xi_0/I_0] \quad (62)$$

that appears in (61) is the critical value of the anisotropy constant, such that for $D<D_c$ the phase under consideration becomes unstable. We note that $D_c=2\xi_0$ holds in the harmonic approximation, when the ZQO are not taken into account. The appearance of the second term in the square brackets in Eq. (62) is due to anharmonic corrections. It can be seen that the renormalization due to the ZQO leads to a decrease ($\xi_0>0$) of the threshold value of the single-ion anisotropy constant, this decrease being larger the more strongly developed are the ZQO. For $\xi_0<0$ the phase under investigation is stable for all $D\geqslant 0$.

The quantity $\omega_{\mathbf{p}}$ appearing in (60) can be written in the form

$$\omega_{\mathbf{p}}=[\Delta^2+2(D-\xi_0)(I_0-I_{\mathbf{p}})+(I_0-I_{\mathbf{p}})^2]^{1/2}. \quad (63)$$

For $C(D,\xi_0)$ the following representation is valid:

$$C(D,\xi_0)=1-\Phi(D,\xi_0)+{}^1/_2(9-15W+4W^2)(\xi_0/I_0)^2, \quad (64)$$

where

$$\Phi(D,\xi_0)=\frac{3^{3/2}\cdot 2^{1/2}}{\pi^2}\left(\frac{D-\xi_0+G}{I_0}\right)^{1/2}$$
$$\times\left\{\frac{D-\xi_0}{I_0}E(m)-\frac{D-\xi_0-G}{I_0}K(m)\right\}. \quad (65)$$

In this expression,

$$G=[(D-\xi_0)^2-\Delta^2]^{1/2}=|\xi_0|[1-3(W-1)(D/I_0)]^{1/2}, \quad (66)$$

and $K(m)$ and $E(m)$ are complete elliptic integrals of the first and second kind, respectively.[32] The modulus of these elliptic integrals is given by the formula

$$m=[2G/(D-\xi_0+G)]^{1/2}. \quad (67)$$

In the limiting case $\Delta \ll D-\xi_0$ the function $\Phi(D,\xi_0)$ has a simpler form:

$$\Phi(D,\xi_0)=\frac{2\cdot 3^{3/2}}{\pi^2}\left(\frac{D-\xi_0}{I_0}\right)^{3/2}\left\{1-\frac{3}{8}\delta^2\ln\left(\frac{8}{\delta}\right)-\frac{3}{16}\delta^2\right\} \quad (68)$$

where $\delta=\Delta/(D-\xi_0)$. It can be seen that in the region of small values of the gap in the excitation spectrum nonanalyticity in δ arises. In another limiting case $I_0 \gg D \gg |\xi_0|$ we obtain

$$\Phi(D,\xi_0)=\frac{3^{5/2}\cdot 2^{1/2}}{4\pi}\left(\frac{D}{I_0}\right)^{1/2}\frac{|\xi_0|^2}{I_0 D}. \quad (69)$$

From the expression (60) for the spectrum it can be seen that, when anharmonic effects are taken into account, in addition to the renormalizations of the gap Δ and of the coefficients multiplying the dispersion terms a new term arises. It possesses two distinctive features. First, in the region of small values of the quasimomentum, if $\Delta=0$, the contribution of this term is proportional to $|\mathbf{p}|^3$. Second, as $\mathbf{p}$ changes there is a change of sign of the term under discussion, corresponding to the change of sign of $I_{\mathbf{p}}$. The latter can be explained if we take two facts into account. The first is that this term appears because we have taken into account the kinematic interaction and the finiteness of the number of physical states. The second fact is the result, obtained earlier in Ref. 33, that in Hubbard systems with strong correlation the kinematic interaction that arises leads to a scattering amplitude with opposite signs in the regions of small and large values of the quasimomentum.

For $\Delta=0$ and $pa \ll 1$ the excitation spectrum is linear in the quasimomentum. If $D=0$ holds, this situation is realized for $K_0 > I_0$. In this case the square of the velocity of propagation of an elementary excitation is given by the expression

$$v^2=[-2\xi_0+(5W-3)\xi_0^2/I_0]Ia^2. \quad (70)$$

To estimate the influence of anharmonic corrections on this quantity, we remark that

$$\frac{v^2-v_0^2}{v_0^2}=\frac{5}{2}(W-1)\left(1-\frac{|\xi_0|}{I_0}\right)\frac{|\xi_0|}{I_0},$$

where v_0^2 is the square of the velocity in the harmonic approximation. If we set $|\xi_0/I_0|=\frac{1}{3}$, we find that the relative contribution of the anharmonic effects amounts to about 30%.

In the region $K_0 < I_0$, when $\xi_0 > 0$, the quadrupolar ordering under consideration is stable only for $D > D_c$. At the point $D=D_c$ we obtain

$$v^2=[2\xi_0+(3-W)\xi_0^2/I_0]Ia^2. \quad (71)$$

Despite the considerable quantitative renormalizations of the parameters under consideration, the inclusion of anharmonic effects does not lead to the appearance of parts of the spectrum with negative dispersion, as did occur with increase of the ZQO in an exchange-anisotropic ferromagnet.[20] This weakening of the role of the ZQO is connected with the fact that, in the given case, as K_0 increases the increase of the parameter $|\xi_0|=K_0-I_0$ determining the intensity of the ZQO is accompanied by a simultaneous increase of the exchange quadrupole field, which is proportional to K_0 and which suppresses the ZQO. Therefore, an increase in K_0 does not cause the relative intensity of the ZQO to reach a level such that beyond it the excitation spectrum undergoes qualitative changes.

If, however, $K_0 < I_0$ holds, an increase in ξ_0 is accompanied by a decrease of the exchange quadrupole field, and this naturally facilitates growth of the ZQO. In the given case, however, we have $D \geqslant D_c$. But since D_c increases as a function of ξ_0, we are dealing in fact with a system in which there is an "external" field of a quadrupolar nature, stabilizing the quadrupolar phase and suppressing the ZQO. Here it is necessary to take this external quadrupolar field to increase as a function of ξ_0. Therefore, for positive values of ξ_0 as well, the intensity of the ZQO does not reach the threshold level.

To corroborate these arguments we give the results of a calculation of the quadrupole parameter of order q_2^0. Using the representation

$$O_2^0=3X^{11}+3X^{22}-2, \quad (72)$$

we use the above-described scheme to construct the Bose analog of the operator

$$\frac{1}{N}\sum_f O_{2f}^0=-2+\frac{3}{N}\sum_f (X_f^{11}+X_f^{22}). \quad (73)$$

Then

$$q_2^0=-2+\frac{3}{N}\sum_{\mathbf{p}}\langle a_{\mathbf{p}}^+ a_{\mathbf{p}}+b_{\mathbf{p}}^+ b_{\mathbf{p}}\rangle$$
$$-\frac{3}{N}\sum_{1234}\langle 2a_1^+ b_2^+ b_3 a_4+a_1^+ a_2^+ a_3 a_4+b_1^+ b_2^+ b_3 b_4\rangle$$
$$\times\,\Delta(1+2-3-4)+\ldots. \quad (74)$$

Here we have not written out terms containing products of six or more operators, since, to the accuracy that we are considering, there is no contribution from them. We note that, without allowance for the metric operator, the value of q_2^0 would be determined entirely by the first two terms. In fact, however, when we go over to the Bose method of description and of performing the calculations, extra terms arise that are due to the necessity to cut out the contributions from the nonphysical states.

After the necessary calculations we find that

$$q_2^0=-2+\Phi(D,\xi_0)-\tfrac{3}{2}(W-1)^2(\xi_0/I_0)^2, \quad (75)$$

where $\Phi(D,\xi_0)$ is defined by Eq. (65). It follows from (75) that even for $\xi_0/I_0=-\frac{1}{2}$ and $D=0$ we have $q_2^0=-1.7$, i.e., the change of the order parameter in comparison with its value in the zeroth approximation in r_0^{-3} is insignificant. This also implies that in the given case the ZQO are relatively weakly developed.

In a finite magnetic field, in accordance with the expression (55), splitting of the two branches of the spectrum occurs. In the limiting case under consideration ($|\xi_0|, D \ll I_0$)

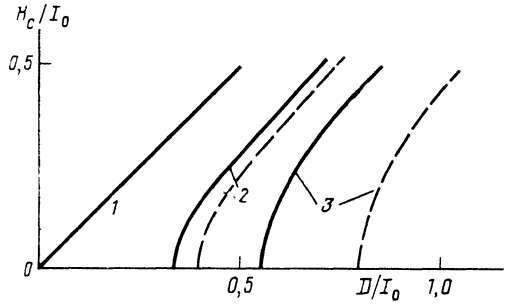

FIG. 1. Displacement of the characteristic curves with growth of the zero-point quantum oscillations.

it is also necessary to impose a restriction on the magnitude of the field H. It is obvious that it is also subject to the inequality $H \ll I_0$, and, to the accuracy under consideration, $\gamma = H$. The relation between the limiting value of H and D follows from the condition that the excitation spectrum be positive-definite. Therefore,

$$H_c = [D(D-D_c)]^{1/2}. \tag{76}$$

Thus, in an external magnetic field the phase under consideration can exist only for $D > 0$ and $H \leqslant H_c$.

In the figure the solid curves show how the critical value H_c of the magnetic field depends on the magnitude of the anisotropy for different values of ξ_0/I_0. The dashed curves correspond to the same dependences, but obtained in the harmonic approximation. The curve 1 corresponds to the exactly solvable case when $\xi_0 = 0$ and there are no anharmonic corrections. The curves 2 and 3 correspond to the values $\xi_0 = 0.2I_0$ and $\xi_0 = 0.4I_0$. It can be seen that allowance for anharmonic corrections leads to significant quantitative changes of the positions of the characteristic curves (in comparison with the corresponding results of the theory in the harmonic approximation) as soon as the ZQO become sufficiently developed.

7. CONCLUSION

We discuss a few fundamental points that have arisen as a result of the introduction of the indefinite-metric formalism into the bosonization procedure. In the present paper the method of construction of an exact Bose analog of the Hamiltonian of interacting three-level subsystems describable by the $SU(3)$ algebra has been demonstrated for the specific example of a spin nematic. Naturally, the formalism that we have developed for introducing the indefinite metric is also equally applicable for any other systems describable by an effective spin Hamiltonian with $S = 1$ and containing tensor interactions.

A general feature of a correct transition to a Bose method of description is that operator terms containing products of an arbitrary number of Bose operators arise in the analog Hamiltonian $\mathscr{H}_B$. An important aspect of the method developed here is that allowance for the metric operator leads to restoration of the Hermiticity of the Bose analog of the Hamiltonian. These two factors mean that the results of this approach and the results of the theory when the pseudo-Hubbard operators were identified with the Hubbard operators differ considerably. For example, the extra terms in the operators $\mathscr{H}_{(4)}$ and $\mathscr{H}_{(6)}$ leads to new contributions from anharmonic effects, not to speak of the fact that the non-Hermiticity of the Bose analog of the Hamiltonian (a property that arose when the above-mentioned identification was made) would induce fundamentally insuperable contradictions in the nonlinear theory.

One further distinctive feature, associated with the determination of average values in the use of the Bose method of description, should be noted. In the paper, for the example of the calculation of q_2^0, it is shown that to obtain in the Bose representation the form of an operator whose average gives a calculable characteristic it is not sufficient to make a simple replacement of the Hubbard operators X^{pq} by pseudo-Hubbard operators $\tilde{X}^{pq}$ with subsequent use of their relation to the Bose operators via the formulas (23). The correct approach requires that, in this case too, the metric operator be taken into account. Here, in the operator Bose analog, new terms, leading to additional contributions, are formed. Therefore, the expression for an average is given, generally speaking, by another formula, containing terms that cancel the contributions from the nonphysical states.

Summarizing the above account, we note that a correct procedure for going over to the Bose method of description of systems with ZQO requires that the contributions from the nonphysical states be accurately removed even for $T = 0$, as soon as the interaction of the quasiparticles is taken into account in the theory. The formalism developed here for using an indefinite metric, together with the proposed structure of the metric operator, solves this problem for a wide class of physical systems describable by $SU(3)$ Hamiltonians. In view of this, it appears promising to use the technique discussed to investigate quasi-two-dimensional magnets, since in such systems the ZQO are rather strongly developed. In particular, it is of special interest (in connection with the problem of the ground state of a high-temperature superconductor) to study the effect of the interaction of bare quasiparticles on the physical characteristics of a quasi-two-dimensional antiferromagnet with $S = \frac{1}{2}$ (Refs. 34, 35).

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[1)] The existence of this connection and the restoration of the $SU(3)$ symmetry of the exchange Hamiltonian for $K_{fg} = I_{fg}$ were noted by A. F. Andreev during a preliminary discussion of the work.

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