The topological classification of defects at a phase interface

T. Sh. Misirpashaev

L. D. Landau Institute for Theoretical Physics of the USSR Academy of Sciences

(Submitted 14 January 1991)
Zh. Eksp. Teor. Fiz. **99**, 1741–1757 (June 1991)

The problem of topological classification of point and line defects on the boundary surface between two phases is posed and solved; formulas are derived for the classification groups, which generalize the previously known relative homotopy groups, $\pi_2(R,\widetilde{R})$, $\pi_1(R,\widetilde{R})$. The separation boundary between the A- and B-phases of superfluid ^{3}He are discussed in detail; for two boundary conditions a complete classification of the defects was obtained, among them boojums on the surface of separation and an analog of the Dirac monopole. Interior defects in the boundary layer which separates ^{3}He-B from the wall of the vessel are considered. The appearance of these defects is caused by the lowering of the symmetry of this layer.

1. INTRODUCTION

The separation boundary between the A- and B-phases of superfluid ^{3}He, which is in many respects a unique example of a two-dimensional object in condensed-matter physics, is currently the subject of intensive experimental and theoretical investigations.[1–6] The large number of components of the order parameter (18 real numbers, forming a complex 3×3 matrix, as well as the high dimension of the phase degeneracy space (5 for the A-phase, 4 for the B-phase, for a maximal possible dimension of 7; see Ref. 7) necessitates the use of an appropriate mathematical apparatus for the classification of the singularities of the phases of ^{3}He. This apparatus is homotopy theory (Refs. 8, 9); the singularities are classified by indices which are topological invariants (see, for instance, Ref. 7).

In the presence of a boundary separating the phases, in addition to purely bulk singularities, there occur surface singularities of the following types: 1) isolated singular points and singular lines situated on the separation surface, including boojums (see, e.g., Ref. 10); defects in the bulk of one of the phases (singular lines and domain walls) having a termination point or line, respectively on the separation surface; to this group belongs also an analog of the Dirac monopole (Refs. 11–13); 3) singular lines and domain walls which intersect the separation surface.

Each of the bulk phases, which make contact along the separation surface, is characterized by its degeneracy space, the symmetries of which are, in general, completely different. For the A–B boundary they are $R_A = (S^3\times SO(3))/Z_2$ and $R_B = U(1)\times SO(3)$, respectively. The values of the order parameter on both sides of the boundary are not arbitrary and are related by some mutual boundary conditions which are not necessarily unique. (Thus, a simultaneous rotation of spin and coordinate space around the normal to the separation surface does not change the order parameter of the B-phase and leaves the separation plane in place, but may change the order parameter of the A-phase. Consequently, for a given boundary condition, to one value of the order parameter of the B-phase on one side of the boundary there may correspond different values of the order parameter of the A-phase on the other side of the boundary (see Ref. 6). Moreover, in the boundary there also arises a distribution of the order parameter, belonging in general to the maximal space $R_{\max}$ of order parameters, i.e., to the space $M(3,C)$ of complex 3×3 matrices. Therefore one should expect that the classification of singularities of types 1), 2), 3) will be a nontrivial problem. In the present paper we propose a topological method of classification of such defects.

For this one needs, first, to write the boundary conditions in a mathematically convenient way. We consider a distribution $A^{0}_{\alpha i}(x)$ of the order parameter in space, depending only on one coordinate x and having definite asymptotic values for $x\to\pm\infty$, of which one, $A^{0A}_{\alpha i}(x)$, belongs to the degeneracy space R_A of the A-phase, and the other, $A^{0B}_{\alpha i}(x)$, belongs to the degeneracy space R_B of the B-phase. Such distributions were obtained in Refs 1, 2, as solutions of the Ginzburg-Landau equations. The actual region of inhomogeneity of the solution turns out to be narrow (of the order of several coherence lengths) and forms properly the A–B boundary. Acting on the selected solution with the elements of the symmetry group of the Hamiltonian one can obtain other distributions of the order parameter, corresponding to the same value of the boundary energy. The set of asymptotic values of the new solutions obtained from $A^{0A}_{\alpha i}$ and $A^{0B}_{\alpha i}$ by the action of this group describes the selected boundary condition. Factoring the symmetry group of the Hamiltonian with respect to the subgroup which preserves this solution we obtain the degeneracy space of the boundary $\mathscr{G}$ for the given boundary condition.

Now we consider the distributions of the order parameter in the interior and on the boundary, depending on all three coordinates. Each such distribution represents a mapping which maps the volume of each of the phases into its own degeneracy space, and the boundary plane into the space $\mathscr{G}$. The matching conditions for these mappings consists in requiring that the values of the order parameter in the volume near a boundary point coincide with the asymptotic values of the distribution within the boundary of the order parameter at the same point (the thickness of the boundary is considered negligible). Thus, the classification of singularities of the types 1), 2), 3) reduces to a problem of homotopy classification of certain maps. These problems are posed and solved in Section 2.

Section 3 considers in detail the problem of classification of defects on the A–B boundary for two specific boundary conditions; the physical meaning of the classification indices is explained, and it is shown how all singularities on the given boundary are obtained by adding "basis" singularities.

 0038-5646/91/060973-10$03.00

In Section 4 the singularities at the boundary of the B-phase and the wall of the vessel are considered. Since the B-phase does not wet the surface of the vessel, an intermediate phase should arise on the surface, in distinction from Ref. 14. The experimental data (Refs. 15–18) force us to assume that in addition to the maximally symmetric structure with a planar phase at the boundary, there can appear additional structures with reduced symmetry, among them some which contain the A-phase at the boundary. Various solutions for the order parameter have been obtained in Refs. 19–23. In section 4 we study the singularities of such boundaries on the basis of the classification introduced in Ref. 23 of the presently known numerical and analytic solutions. Section 5 points out some problems which are beyond the scope of the present paper, but to which the present method is applicable.

2. TOPOLOGICAL CLASSIFICATION OF THE DEFECTS ON THE PHASE INTERFACE: GENERAL CASE

We choose the geometry of the structure in the following manner: the phase interface coincides with the $x = 0$, the phase a occupies the half-space $x < 0$ and the phase b occupies the half-space $x > 0$. The phase degeneracy spaces will be denoted by R_a and R_b, respectively. (The notations are chosen so as to remind us of the basic example—the phase separation surface of the A- and B-phases of superfluid ^{3}He, where the degeneracy spaces are denoted by R_A and R_B.) The existence of the boundary leads to a reduction of the symmetry group of the physical laws

$$G=O(2)^G\times O(3)^L\times SO(3)^S \tag{1}$$

to some subgroup G_i. For example, in the case of a flat boundary separating different phases in the absence of an external field the subgroup is

$$G_i=O(2)^G\times O(2)^L\times SO(3)^S. \tag{2}$$

Here $O(2)^G$ denotes the group containing the gauge transformations (which form a $U(1)$ subgroup) and their compositions with time reversal T; $O(2)^L$ denotes the group containing rotations of coordinate space around the x axis (the subgroup $SO(2)^L$) and the symmetries of coordinate space with respect to planes containing the x axis; $SO(3)^S$ is the rotation group of spin space. The group $O(3)^L$ contains all the rotations of coordinate space, as well as their compositions with the space reflection operator P; only its subgroup $O(2)^L$ leaves in place the boundary and the positions of the phases.

The values of the order parameter at points close to the boundary and symmetric with respect to it are not arbitrary and are related by boundary conditions which can be taken into account in the following manner. We prescribe some asymptotic forms of the order parameter at infinity:

$$A^0(x=-\infty)=A^{0a}\in R_a, \tag{3a}$$

$$A^0(x=+\infty)=A^{0b}\in R_b. \tag{3b}$$

After this one can determine the distribution of the order parameter $A^0(x)$ which has lowest energy, depends only on x, and satisfies the prescribed asymptotic behavior. In general, the order parameter A^0 takes values in the maximal space $R_{\max}$ of order parameters.

Let H be the symmetry group of the solution A^0, i.e., that subgroup of G_i consisting of elements $g\in G_i$ such that $gA^0(x) = A^0(x)$ for all x. In addition to H we consider the symmetry group H_{as} of the asymptotic solutions, i.e., the subgroup of all $g\in G_i$ such that $gA^{0a} = A^{0a}$ and $gA^{0b} = A^{0b}$. (The notation gA means the action of the group element g on the order parameter A.) Obviously, H is a subgroup of H_{as}. The degeneracy space of the states of the boundary is

$$\mathcal{G}=G_i/H. \tag{4}$$

In general, the homogeneous space $\mathcal{G}$ need not be a group; nevertheless the action of any element $g\in G_i/H$ on any point $A^0(x)\in R_{\max}$ is well-defined. Such an element $g\in\mathcal{G}$ is a coset $\tilde{g}H$, with $\tilde{g}\in G_i$. Since by the definition of H for each element $h\in H$ we have $hA^0(x) = A^0(x)$, the mapping $h_x:\mathcal{G}\to R_{\max}$, $h_x(g=\tilde{g}H) = \tilde{g}A^0(x)$ is well defined (does not depend on the choice of the representative $\tilde{g}$ in the coset $\tilde{g}H$). In particular, there are the mappings

$$h_a=h_{-\infty}:\mathcal{G}\to R_a, \tag{5a}$$

$$h_b=h_{+\infty}:\mathcal{G}\to R_b, \tag{5b}$$

which we will need in the sequel.

Thus, any element $g\in\mathcal{G}$ maps the initial solution $A^0(x)$ into some other distribution of the order parameter $gA^0(x)$ having, obviously, the same energy. The distribution of the order parameter at the boundary is described by a continuous map $F_0:R^2\to\mathcal{G}$. Here the plane R^2 is labeled by the coordinates (y,z) in the boundary plane and the value of $F_0(y,z)$ determines the element $g\in\mathcal{G}$ which has to act on $A^0(x)$ in order to yield the distribution of the order parameter over the thickness of the boundary at the given point (y,z). The topological classification of the point singularities of the boundary reduces to the homotopy classification of the mappings F_0 of the punctured plane R^2 into $\mathcal{G}$. Since a punctured plane contracts into the circle S^1 we reach the standard conclusion that if the fundamental group $\pi_1(\mathcal{G})$ is commutative the point singularities of the plane are classified by its elements, and the merging of point singularities corresponds to multiplication in $\pi_1(\mathcal{G})$. In all our examples the fundamental group will be abelian, and we therefore restrict our attention to that case. Further, line singularities of the boundary, i.e., boundaries of "islands" (see Ref. 10), are obviously classified by the set $\pi_0(\mathcal{G})$ of the connectivity components of the degeneracy space $\mathcal{G}$.

We are interested in the distributions of the order parameter throughout the volume surrounding a singularity on the boundary. We must figure out which singularities are isolated, which come from the interior inside one of the phases, which are endpoints of singular lines, or intersections of singular lines with the boundary. For this one has to find the topological classification of the distributions of order parameters in the four situations listed below. In all cases one searches for distributions of the order parameter within the interiors of the phases and on the boundary, with certain points of the volume and boundary removed.

Case 1. One point removed from the boundary (isolated singularity).

Case 2a. A line in the bulk (interior) of the a-phase is removed, with an endpoint on the boundary (endpoint of a singular line in the a-phase).

Case 2b differs from 2a in that the singular line comes from the b-phase.

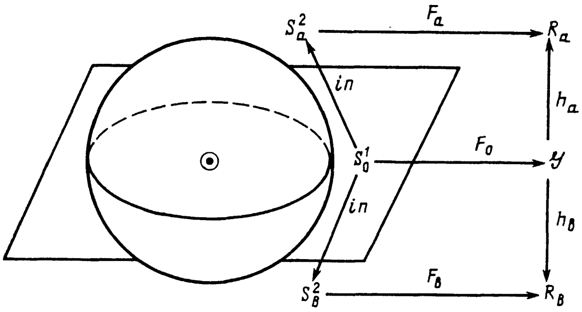

FIG. 1. The case of an isolated singularity. The volumes of the a- and b-phases can be contracted to hemispheres and the boundary to the equator of a two-sphere; each hemisphere and the equator are mapped into their own degeneracy spaces. The requirement of continuity of the order parameter implies the commutativity of the ensuing diagram of maps.

Case 3. A line which traverses the boundary is removed (singular line traversing the boundary).

Case 1 leads to problem 1 in the homotopy classification of triples of mappings,

$$F_0 : S_0^1 \to \mathcal{G}, \tag{6}$$
$$F_a : S_a^2 \to R_a, \tag{6a}$$
$$F_b : S_b^2 \to R_b, \tag{6b}$$

satisfying the matching conditions

$$h_a \circ F_0 \equiv F_a|_{S_0^1},$$
$$h_b \circ F_0 \equiv F_b|_{S_0^1}.$$

Here S_0^1, S_a^2, S_b^2 are respectively the equator and the upper and lower hemispheres of S^2, the two-sphere (Fig. 1). The mappings h_a, h_b have been defined above. The notation $h \circ F$ denotes the composition of the maps f and F.

The junction conditions which follow from the requirement that the order parameter should be continuous are tantamount to the commutative diagram

$$\begin{tikzcd} S_a^2 \arrow[r, "F_a"] & R_a \\ S_0^1 \arrow[u, "\text{injection}"] \arrow[d, "\text{injection}"'] \arrow[r, "F_0"] & \mathcal{G} \arrow[u, "h_a"'] \arrow[d, "h_b"] \\ S_b^2 \arrow[r, "F_b"] & R_b \end{tikzcd} \tag{7}$$

The difference in the way the problem is posed in cases 2a and 2b consists in the absence of the mappings F_a and F_b, respectively. Indeed, the volume from which a line is removed can be contracted to the circle S_0^1 situated on the boundary, and therefore the homotopy class of the distribution of order parameter in that volume is determined by the element $\pi_1(\mathcal{G})$ corresponding to the given point singularity. Replacing the hemisphere S_b^2 by the disk D_b homeomorphic to it, and considering the circle S_0^1 as its boundary ∂D_b, we can formulate problem 2a for case 2a in the following manner: find the homotopy classification of the pairs of mappings F_0, F_b:

$$F_0 : \partial D_b \to \mathcal{G},$$
$$F_b : D_b \to R_b,$$

such that $h_b \circ F_0 \equiv F_b|_{\partial D_b}$. The formulation of case 2b is similar. As regards case 3, the fact that the volumes of both phases can be contracted to the circle S_0^1 implies immediately that singular lines which intersect the separation boundary are classified by the homotopy group $\pi_1(\mathcal{G})$. Knowing the answer for cases 1 and 2a,b one can determine which of these lines are indeed topologically trivial.

Let us reformulate problem 1. Namely, the pair of mappings F_a, F_b can be considered as a single map $F = F_a \times F_b$,

$$F : D \to R_a \times R_b, \tag{8}$$

which maps the point q in the two-dimensional disk D bounded by the circle S_0^1 to the pair $(F_a(q_a) F_b(q_b))$ in the Cartesian product $R_a \times R_b$, where q_a is the preimage of the point q under orthogonal projection of the upper hemisphere S_a^2 onto the equatorial plane, and q_b is the preimage of q for a similar projection of the lower hemisphere S_b (see Fig. 2). We introduce the mapping

$$h = h_a \times h_b : \mathcal{G} \to R_a \times R_b, \tag{9}$$

which acts according to the formula $h(g) = (h_a(g), h_b(g))$. Now problem 1 can be reformulated as follows: find the homotopy classification of the pair of maps

$$F_0 : \partial D \to \mathcal{G}, \tag{10.1}$$
$$F : D \to R_a \times R_b \tag{10.2}$$

with the matching condition $h \circ F_0 \equiv F|_{\partial D}$, which is absolutely analogous to the way the problem is posed in case 2.

We now solve these problems, considering first the case $H = H_{as}$, and then we indicate the modifications necessary when this does not hold. In the case $H = H_{as}$ the mapping g: $\mathcal{G} \to R_a \times R_b$ will be an embedding. This means that each element $g \in \mathcal{G}$ shifts only one of the asymptotic values A^{0a}, A^{0b}, which obviously occurs since we have factored the total group of possible symmetries by the stabilizer $H = H_{as}$ of the asymptotic values. Therefore $\mathcal{G}$ can be identified with its image $h(\mathcal{G}) \subset R_a \times R_b$, and the set of pairs of mappings (10.1), (10.2) we are after turns out to be the relative homotopy group

$$\pi_2(R_a \times R_b, \mathcal{G})$$

according to its definition (see Refs. 8, 9). If the action of $\pi_1(\mathcal{G})$ on $\pi_2(R_a \times R_b)$ is trivial, the merging of singular points is described by multiplication in $\pi_2(R_a \times R_b, \mathcal{G})$.

By means of the exact sequence of the pair

$$\pi_2(Q) \to \pi_2(P) \to \pi_2(P, Q) \to \pi_1(Q) \to \pi_1(P) \tag{11}$$

the relative homotopy group $\pi_2(P,Q)$ can be decomposed into two components Π_1 and Π_2:

$$\Pi_1 = \mathrm{Ker}[\pi_2(P, Q) \to \pi_1(Q)] = \pi_2(P)/\mathrm{Im}[\pi_2(Q) \to \pi_2(P)], \tag{12}$$

$$\Pi_2 = \mathrm{Im}\,[\pi_2(P, Q) \to \pi_1(Q)] = \mathrm{Ker}\,[\pi_1(Q) \to \pi_1(P)], \tag{13}$$

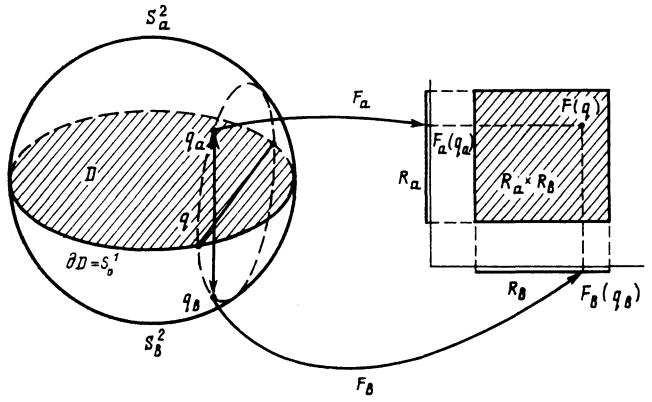

FIG. 2. The construction of the map $F = F_a \times F_b$. Each point q of the two-dimensional disk D with boundary $\partial D = S_0^1$ is mapped onto a pair of points $q_a \in S_a^2$ and $q_b \in S_b^2$ which is then mapped into the point $(F_a(q_a), F_b(q_b)) \in R_a \times R_b$.

which define $\pi_2(R_a \times R_b)$ up to an adjoint action, i.e., Π_1 is isomorphic to some subgroup in $\pi_2(P,Q)$ and Π_2 is isomorphic to the factor group of $\pi_2(P,Q)$ with respect to that subgroup. The intuitive meaning of the components Π_1 and Π_2 is the following: the component Π_2 describes those loops in Q which can be "spanned by films" in P, and the component Π_1 gives the difference between the different spannings for a fixed homotopy class of a loop in Π_2 (see Refs. 7, 14).

We introduce the homomorphisms of homotopy groups induced by the mappings h, h_a, h_b:

$$h_a^k : \pi_k(\mathcal{G}) \to \pi_k(R_a), \tag{14a}$$

$$h_b^k : \pi_k(\mathcal{G}) \to \pi_k(R_b), \tag{14b}$$

$$h^k = h_a^k \oplus h_b^k : \pi_k(\mathcal{G}) \to \pi_k(R_a \times R_b) = \pi_k(R_a) \oplus \pi_k(R_b). \tag{14c}$$

For the group

$$\pi_2(R_a \times R_b, \mathcal{G})$$

the components Π_1 and Π_2 are

$$\Pi_1 = \pi_2(R_a) \oplus \pi_2(R_b) / \mathrm{Im}\, h^2, \tag{15}$$

$$\Pi_2 = \mathrm{Ker}\, h^1 = \mathrm{Ker}\, h_a^1 \cap \mathrm{Ker}\, h_b^1. \tag{16}$$

We now come to problem 2. We introduce the images $\widetilde{R}_a$ and $\widetilde{R}_b$ under the maps h_a and h_b:

$$\widetilde{R}_a = h_a(\mathcal{G}) \subset R_a, \tag{17a}$$

$$\widetilde{R}_b = h_b(\mathcal{G}) \subset R_b. \tag{17b}$$

We consider, for example, the problem 2b. We need to find the set of homotopically distinct pairs of mappings $F_0 : \partial D \to \mathcal{G}$; $F_a : D \to R_a$, such that $h_a \circ F_0 \equiv F_b|_{\partial D}$. The mapping F_0 determines a loop in $\mathcal{G}$, is taken by h_a into R_a. The mapping F_a determines "stretching a film" across the image of that loop in R_a. The homotopy class $\Gamma \in \pi_1(\mathcal{G})$ of the map F_0 must be such that on the image $h_a(\gamma)$ of the loop $\gamma \in \Gamma$ one can "stretch a film" in R_a, i.e., one must have $\Gamma \in \mathrm{Ker}\, h_a^1$. Thus the set of homotopically distinct maps F_0 is $\mathrm{Ker}\, h_a^1$. Let γ be a loop of class $\Gamma \in \mathrm{Ker}\, h_a^1$ (Fig. 3). Then the distinguishing indices of different "film stretchings" in R_a of the loop $h_a(\gamma) \subset \widetilde{R}_a$ are described by the component Π_1 of the group $\pi_2(R_a, \widetilde{R}_a)$. Writing this component explicitly we find the answer to problem 2b for the desired pairs of mappings:

$$\pi_2(R_a)/\mathrm{Im}\,[\pi_2(\widetilde{R}_a) \to \pi_2(R_a)] \times \mathrm{Ker}\, h_a^1. \tag{18}$$

Similarly, the set for problem 2a is:

$$\pi_2(R_b)/\mathrm{Im}[\pi_2(\widetilde{R}_b) \to \pi_2(R_b)] \times \mathrm{Ker}\, h_b^1. \tag{19}$$

Here the symbols $\times$ denotes the Cartesian product of sets; generally speaking there is no natural group structure in the sets (18), (19), except in some special cases. For instance, if one of the components of the mapping vanishes, the remaining one is obviously a group, and multiplication in it will correspond to merging of singularities. Another special case is realized when $\mathcal{G} = \widetilde{R}_a$ (or $\widetilde{R}_b$). We then obtain the relative homotopy group $\pi_2(R_a, \widetilde{R}_a)$ [respectively, $\pi_2(R_b, \widetilde{R}_b)$], which corresponds to the situation considered in Refs. 7, 14, when a subspace $\widetilde{R}$ of possible values of the order parameter on the boundary was fixed, and no transition phase appears near the boundary.

Let now the symmetry H_{as} of the asymptotic solutions be higher than the symmetry H of the whole solution $A^0(x)$. As is easily seen, the solution of the problems 2a,b remains valid. Since the formulation of problem 1 is completely analogous to the formulation of problems 2a,b, it should be clear that the answer to problem 1 in the general case is given by the formula

$$\pi_2(R_a \times R_b)/\mathrm{Im}\,[\pi_2(\widetilde{R_a \times R_b}) \to \pi_2(R_a \times R_b)] \times \mathrm{Ker}\, h^1, \tag{20}$$

where we have introduced the set $\widetilde{R_a \times R_b} = h(\mathcal{G})$. For $H = H_{as}$ one can identify $\widetilde{R_a \times R_b}$ and $\mathcal{G}$ and we return to $\pi_2(R_a \times R_b, \mathcal{G})$.

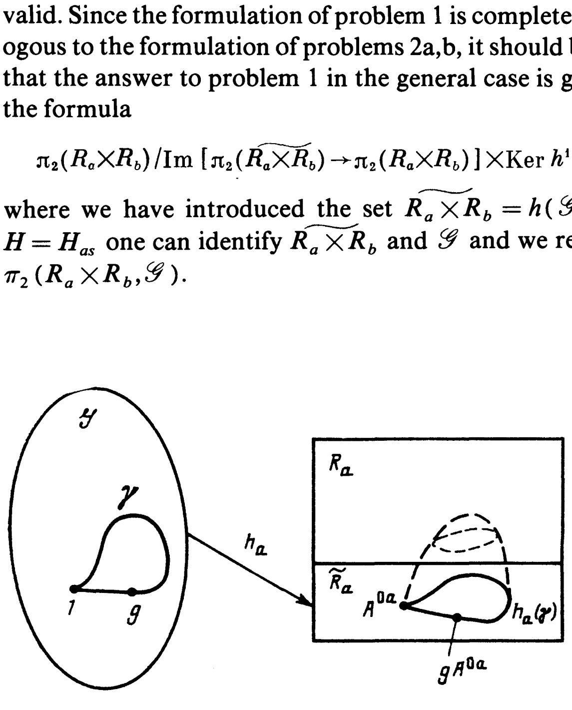

FIG. 3. The case 2. The map h_a maps the loop γ into $\widetilde{R}_a$; on its image $h_a(\gamma)$ one can "stretch" a film in R_a.

We still note that in the sets (18), (19), (20) for $H \neq H_{as}$ one can select the component responsible for the interior degrees of freedom of the boundary, i.e., the set of separators for the distributions of the order parameter, which differ only within the thickness of the boundary (but agree on the surface and in the bulk). Such a component will be

$$I=\mathrm{Ker}\,[\pi_1(G_i/H)\rightarrow\pi_1(G_i/H_{as})], \qquad (21.1)$$

where the mapping of the fundamental groups is induced by the projection of the bundle $G_i/H\rightarrow G_i/H_{as}$ with the fiber H_{as}/H. Using the exact sequence of this bundle one can also write

$$I=\mathrm{Im}\,[\pi_1(H_{as}/H)\rightarrow\pi_1(G_i/H)], \qquad (21.2)$$

from where we see, for instance, that in the case of discrete groups H, H_{as} the component I is trivial.

For the case of line singularities on the boundary and the case of domain walls which come from the interior, one can pose problems analogous to 1, 2a, 2b, 3; in the answers it is necessary to lower the dimensions of the homotopy groups by one unit. In particular, for the case $H = H_{as}$ we find that the isolated singularities on the boundary are described by the set

$$\pi_1(R_a\times R_b,\ \mathcal{G}).$$

3. THE TOPOLOGICAL CLASSIFICATION OF DEFECTS ON THE *A–B* BOUNDARY

We now consider directly the boundary separating the A- and B-phases of superfluid ^{3}He. We restrict our attention to the case of small distance scales, when the additional interactions can be neglected, and the degeneracy spaces of the phases are[7]

$$R_a=R_A=(S^2\times SO(3))/Z_2,$$
$$R_b=R_B=U(1)\times SO(3).$$

These spaces are subsets of the maximal space of order parameters, the space $R_{\max} = M(3,C)$ of complex 3×3 matrices. The order parameter in the A-phase has the form

$$A_{\alpha i}{}^{A}=\Delta_A(T)\,d_\alpha(e_{1i}+ie_{2i}), \qquad (22)$$

where $\mathbf{d}$ is a unit vector indicating the direction for which the projection of the spin of Cooper pairs is zero, and $\mathbf{e}_1$, $\mathbf{e}_2$ are a pair of orthogonal unit vectors; the third vector $\mathbf{l} = \mathbf{e}_1\times\mathbf{e}_2$, which together with the pair $(\mathbf{e}_1,\mathbf{e}_2)$ form a basis, points in the direction of the orbital momentum of the Cooper pairs.

The order parameter in the B-phase is

$$A_{\alpha i}{}^{B}=\Delta_B(T)\,\exp(i\Phi)R_{\alpha i}, \qquad (23)$$

where Φ is the phase of the Bose condensate; $R_{\alpha i}\in SO(3)$; the matrix $R_{\alpha i}$ determines the relation between the anisotropy axes of the orbital and spin properties of the phase; and $\Delta_A(T)$ and $\Delta_B(T)$ are the energy gaps in the two phases.

As was shown in Ref. 2, two maximal symmetry groups are possible for the distribution of the order parameter $A^0(x)\in M(3,C)$ within the boundary, namely

$$\mathcal{D}_2{}^{(1)}=\{1,\ C_{2x}T,\ C_{2y}P,\ C_{2z}PT\}, \qquad (24)$$
$$\mathcal{D}_2{}^{(2)}=\{1,\ C_{2x},\ C_{2y}PT,\ C_{2z}PT\}. \qquad (25)$$

Here C_{2l} ($l = x,y,z$) is a simultaneous rotation of the coordinate and spin spaces by π around the l axis (see Ref. 5). The most stable A–B boundary corresponds to the symmetry $H = \mathcal{D}_2^{(1)}$ and the following asymptotic forms of the solution (3) (see Refs. 1, 2):

$$A(x=-\infty)=A^{0A}=\Delta_A\hat{x}_\alpha(\hat{x}_i-i\hat{z}_i), \qquad (26a)$$
$$A(x=+\infty)=A^{0B}=\Delta_B\delta_{\alpha i}; \qquad (26b)$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are the unit vectors along the coordinate axes.

As is easy to see, $H_{as} = H$ and the value of the vector $\mathbf{l}$ for A^{0A} is $\mathbf{l} = \hat{\mathbf{y}}$. In the sequel we shall call such an A–B boundary a boundary with the boundary condition $\mathbf{l}\perp x$.

In addition to the boundary condition $\mathbf{l}\perp x$ we also consider a distribution of the order parameter with the symmetry $H = \mathcal{D}_2^{(2)}$ and the asymptotic forms

$$A(x=-\infty)=A^{0A}=\Delta_A\hat{y}_\alpha(\hat{y}_i+i\hat{z}_i), \qquad (27a)$$
$$A(x=+\infty)=A^{0B}=\Delta_B\delta_{\alpha i}, \qquad (27b)$$

for which also $H_{as} = H$ and $\mathbf{l}\|x$; this case will be called a boundary with the boundary condition $\mathbf{l}\|x$.

We consider the case $\mathbf{l}\perp x$. The degeneracy space of the states of the boundary is, according to Eq. (4),

$$\mathcal{G}=G_i/\mathcal{D}_2{}^{(1)}=U(1)\times SO(2)^L\times SO(3)^S. \qquad (28)$$

The point defects of the boundary are classified by the elements of the group $\pi(\mathcal{G}) = Z\oplus Z\oplus Z_2$, i.e., by three indices (n_1,n_2,n_3), where $n_1,n_2\in Z$; $n_3\in Z_2$. We have chosen the standard generators in $\pi_1(U(1))$, $\pi_1(SO(2)^L)$, $\pi_1(SO(3)^S)$, corresponding to loops which enclose the circles $U(1)$, $SO(2)^L$, $SO(2)^S$ once, respectively; the circle $SO(2)^S$ consists of rotations of the spin space around the x axis.

According to Eq. (5) there are maps $h_A:\mathcal{G}\rightarrow R_A$, $h_B:\mathcal{G}\rightarrow R_B$, which induce the homomorphisms (10):

$$h_A{}^1:\pi_1(\mathcal{G})\rightarrow\pi_1(R_A)=Z_4,$$
$$h_B{}^1:\pi_1(\mathcal{G})\rightarrow\pi_1(R_B)=Z\oplus Z_2.$$

We choose in $\pi_1(R_A) = Z_4$ the standard generator, represented by the loop whose projection onto S^2 joins diametrically opposed points of the sphere, and whose projection onto $SO(3)$ is a once-traversed circle $SO(3)\subset SO(3)$ (see Ref. 7); in $\pi_1(R_B) = Z\oplus Z_2$ the generators are selected as they were done in $\pi_1(\mathcal{G})$. In order to find the homomorphisms h_A^1, h_B^1, one must find the images of loops in $\mathcal{G}$ which are the standard generators in $\pi_1(\mathcal{G})$, obtained by the action of all elements of these loops on the asymptotic forms $A^{0A} = \Delta_A\hat{x}_\alpha(\hat{x}_i - i\hat{z}_i)$ and $A^{0B} = \Delta_B\delta_{\alpha i}$. We find thus

$$h_A{}^1:(n_1,\ n_2,\ n_3)\rightarrow(2(n_1+n_2)\ \mathrm{mod}\ 4), \qquad (29a)$$
$$h_B{}^1:(n_1,\ n_2,\ n_3)\rightarrow(n_1,\ n_3+n_2\ \mathrm{mod}\ 2). \qquad (29b)$$

(We represent the group Z_4 by the residues 0, 1, 2, 3, mod 4, and the group Z_2 by the residues 0, 1, mod 2. The addition $n_3 + n_2$ mod 2 denotes addition in Z_2. Equations (29) mean that under the homomorphisms h_A^1, h_B^1 the element (n_1,n_2,n_3) goes into the indicated elements of the groups Z_4 and $Z\oplus Z_2$, respectively.)

We now have

$$\operatorname{Ker} h_A{}^1=\{(n_1,\ n_2,\ n_3)\,|\,n.\ \ n_2\vdots 2\}. \tag{30}$$

$$\operatorname{Ker} h_B{}^1=\{(0,\ n_2,\ n_2 \bmod 2)\}=Z, \tag{31}$$

$$\operatorname{Ker} h^1=\operatorname{Ker} h_A{}^1\cap\operatorname{Ker} h_B{}^1=\{(0,\ n_2,\ 0)\,|\,n_2\vdots 2\}=2Z. \tag{32}$$

Here $a\vdots b$ indicates that b divides a. The notation $\{(n_1,n_2,n_3)|.....\}$ means the set of elements (n_1,n_2,n_3) such that the property to the right of the vertical bar is true. These formulas determine the values of the indices (n_1,n_2,n_3) of the singularity on the separation surface for which the loops surrounding the point can be "spanned by a film" respectively in R_A, R_B, $R_A\times R_B$ (the possibility of "stretching a film" in the latter case is, of course, equivalent to the possibility of doing this simultaneously in R_A and R_B).

We have thus defined the second components of the sets of answers (18), (19), (16) to the problems 2b, 2a, 1, respectively. We now calculate the first components, which determine the number of nontrivial "film stretchings" for a fixed homotopy class of the loop surrounding the singularity. We determine the first component Π_1 of the group $\pi_2(R_A\times R_B,\mathscr{G})$ according to Eq. (15):

$$\Pi_1=\pi_2(R_A)\oplus\pi_2(R_B)/\operatorname{Im}[\pi_2(\mathscr{G})\to\pi_2(R_A)\oplus\pi_2(R_B)]=Z, \tag{33}$$

where we have made use of the fact that $\pi_2(\mathscr{G})=0$ and $\pi_2(R_B)=0$. Since both components Π_1 and Π_2 of the group $\pi_2(R_A\times R_B,\mathscr{G})$ are infinite cyclic groups, $\pi_2(R_A\times R_B,\mathscr{G})$ equals their direct sum, $\pi_2(R_A\times R_B,\mathscr{G})=\mathscr{Z}\oplus 2\mathscr{Z}$; we denote the first index by p (the second index is equal to n_2 and is necessarily even). As regards the answers (19), (18) to the problems 2a and 2b, the first component vanishes and no additional indices appear. Indeed

$$\pi_2(R_B)/\operatorname{Im}[\pi_2(\tilde{R}_B)\to\pi_2(R_B)]=0,$$

since $\pi_2(R_B)=0$, and

$$\pi_2(R_A)/\operatorname{Im}[\pi_2(\tilde{R}_A)\to\pi_2(R_A)]=0,$$

since $\tilde{R}_A=(S^1\times S^1\times S^2)/Z_2$ and $\pi_2(\tilde{R}_A)\to\pi_2(R_A)$ turns out to be a homomorphism. The information obtained in this way is collected in Table I, which also contains an explanation of the physical meaning of the indices and singularities corresponding to certain choices of indices. This classification agrees with the results listed in Ref. 6.

In this classification isolated boojums appear naturally on the surface, as well as boojums which are end-points of vortices in the interiors of the phases. The latter are analogs of the Dirac monopole,[12,13] where the role of the vector potential is played by the superfluid velocity $\mathbf{v}_s$, and the role of the magnetic field by its vorticity curl $\mathbf{v}_s$. One should however, note the difference, that for the Dirac monopole the line singularity of the distribution of the vector potential is an artefact, and disappears when one considers the corresponding principal bundle, whereas vortices in ^{3}He are real and experimentally observed. This distinction is connected with the gauge invariance of electrodynamics which is absent in ^{3}He.

We note that since there are no additional distinguishing indices except n_1, n_2, n_3 in the cases 2a, 2b, and 3, the fusion and decay of the singularities we consider, with the exception of those coming from the bulk, is described by addition in $\pi_1(\mathscr{G})$, and all such singularities can be ob-

TABLE I. The boundary condition $\mathbf{l}\perp x$. Classification of singularities.

Type of singularity; classifying indices	Index or array of indices	Physical meaning of the index or of the singularity defined by an array of indices
1. Isolated singularity on the surface; $(\rho,0,n_2,0)$, $n_2\vdots 2$	ρ	Degree of the map of the hemisphere S^2_A onto the sphere of the vector $\mathbf{d}$
	n_2	The number of rotations of the vector $\mathbf{l}$ as the singularity is circumvented
	(1, 0, 0, 0)	Point singularity coming to the boundary from the bulk of the A-phase
	(0, 0, 2, 0)	Boojum on the surface
2a. Singular line from the A-phase ending at the surface of separation; $(0,n_2,$ mod 2)	n_2	See above
	(0, 1, 1)	One-quantum vortex in the A-phase with end on the boojum
2b. Singular line from the B-phase ending at the surface of separation; (n_1,n_2,n_3), $n_1+n_2\vdots 2$	n_1	Increase of the phase Φ (in units of 2π) in the A- and B-phases as the singularity is circumvented
	n_3+n_2 mod 2	The presence of a disclination in the field of the matrix $R_{\alpha i}$ in the B-phase
	(1, 1, 0)	One-quantum vortex in the B-phase with end on the boojum
	(0, 0, 1)	Disclination in the B-phase (ends without a boojum)
3. Singular line which intersects the boundary; (n_1,n_2,n_3)	(1, 0, 0)	One-quantum vortex intersecting the boundary
	(0, 1, 0)	One-quantum vortex in the A-phase, transforming into a disclination

tained as a result of the composition of "basis singularities," for example (1,0,0), (0,1,0), (0,0,1). It is easy to show that any singular line which intersects the boundary can be "torn up," i.e., decomposed into several singularities of the type 2a, b. This follows, e.g., from the fact that one can choose the basis (1,0,0), (0,1,0), (0,0,1). The supplementary index ρ corresponds to singular points coming from the bulk of the A-phase; we see that such points can merge with an isolated boojum on the surface (one thus obtains a singularity with the index $(\rho,0,n_2,0)$, and after merging with the end of a vortex the index ρ is forgotten. In fact, owing to the nontriviality of the action of $\pi_1(\mathcal{G})$ on $\pi_2(R_A)$, the index of the singular point arriving at the boundary from the interior is determined to within a sign (see Ref. 7), but the action of $\pi_1(\mathcal{G})$ on $\pi_2(R_A\times R_B)=\pi_2(R_B)$ is trivial, and therefore the index ρ is now uniquely defined. One more conclusion from Table I is that a half-quantum vortex in the A-phase cannot terminate on the interface with the B-phase.

We now consider the other boundary condition $\mathbf{l}\|x$. The degeneracy space of the states of the boundary, according to Eqs. (4) and (25), is

$$\mathcal{G}=G_i/\mathcal{D}_2^{(2)}=(O(2)^G\times(O(2)^L\times SO(3)^S)/Z_2)/Z_2. \qquad (34)$$

Here the factoring with respect to the internal Z_2 corresponds to an identification of elements which differ by a simultaneous spin–orbit rotation C_{2x}, and the factorization with respect to the external Z_2 identifies pairwise the connected components of G_i, reducing their number from four to two. The fundamental group is again

$$\pi_1(\mathcal{G})=Z\oplus Z\oplus Z_2,$$

but the generators have to be chosen differently. The generator of the first Z is represented by a loop $U(1)\subset O(2)^G$, the second generator Z is represented by a loop γ, where $\gamma(t)$ is a combined spin-orbit rotation by an angle πt around the x-axis. The generator of the group Z_2 is represented by the loop $SO(2)^s\subset SO(3)^s$. The point singularities of the boundary are described by three indices (n_1,n_2,n_3); $n_1,n_2\in Z$, $n_3\in Z_2$.

The homomorphisms (14) have the form

$$h_A{}^1:(n_1,\ n_2,\ n_3)\to((2n_1+n_2)\ \mathrm{mod}\ 4), \qquad (35a)$$

$$h_B{}^1:(n_1,\ n_2,\ n_3)\to(n_1,\ n_3). \qquad (35b)$$

Hence

$$\mathrm{Ker}\,h_A{}^1=\{(n_1,\ n_2,\ n_3)\,|\,2n_1+n_2\,\vdots\,4\},$$
$$\mathrm{Ker}\,h_B{}^1=\{(0,\ n_2,\ 0)\},$$
$$\mathrm{Ker}\,h^1=\{(0,\ n_2,\ 0)\,|\,n_2\,\vdots\,4\}.$$

The computation of additional separators leads, similar to the case $\mathbf{l}\bot x$, to the conclusion that $\pi_2(R_A\times R_B,\mathcal{G})=Z\oplus 4Z$ (there appears an additional index $\rho\in Z$ describing sinular points which come from the interior), and in the cases 2a,b no additional indices make their appearance. The results are summarized in Table II.

The assertion made above remains valid: that fusion of singularities corresponds to the addition of indices, and that

TABLE II. The boundary condition $\mathbf{l}\|x$. Classification of singularities.

Type of singularity; classifying indices	Index or array of indices	Physical meaning of the index or of the singularity defined by an array of indices
1. Isolated singularity on the surface; $(\rho,0,n_2,0)$, $n_2\vdots 4$	ρ	Degree of the map of the hemisphere S_A^2 onto the sphere of the vector $\mathbf{d}$
	n_2	Increase of the phase in the A-phase (in units of π) and the number of rotations of the vector $\mathbf{d}$ when circumventing the singularity
	(1, 0, 0, 0)	Point singularity coming to the boundary from the bulk of the A-phase
	(0, 0, 4, 0)	Boojum on the surface
2a. Singular line from the A-phase ending at the surface of separation; $(0,n_2,0)$	n_2	See above
	(0, 1, 0)	Half-quantum vortex in the A-phase with end on the boojum
	(0, 2, 0)	One-quantum vortex in the A-phase with end on the boojum
2b. Singular line from the B-phase ending at the surface of separation; (n_1,n_2,n_3), $2n_1+n_2\vdots 4$	n_1	Increase of the phase Φ (in units of 2π) in the A- and B-phases as the singularity is circumvented
	n_3	The presence of a disclination in the field of the matrix R_{ai} in the B-phase
	(1, 2, 0)	One-quantum vortex in the B-phase with end on the boojum
	(0, 0, 1)	Disclination in the B-phase (ends without a boojum)
3. Singular line which intersects the boundary; (n_1,n_2,n_3)	(1, 0, 0)	Half-quantum vortex in the A-phase, transforming into a one-quantum vortex in the B-phase

basis elements exist, with the observation that in this case the index ρ is defined only up to sign, even for a singularity which is already localized on the surface of separation, since in the image of the homomorphism h^1_A there appear odd elements which act nontrivially on $\pi_2(R_A)$. The essential distinction from the case $\mathbf{l}\|x$ is that here a half-quantum vortex of the A-phase may have its end on the surface of separation (cf. Refs. 12, 13).

As far as line-singularities on the boundary are concerned, it is easy to see that in the case $\mathbf{l}\perp x$ such singularities are exhausted by half-quantum vortices coming from the bulk of the A-phase, with both possible topological charges of such a vortex being equivalent:

$$\pi_1(R_A\times R_B, \mathscr{G})=\pi_1(R_A\times R_B)/\mathrm{Im}\, h^1=Z_2.$$

For the limiting case $\mathbf{l}\|x$ we have

$$\pi_1(R_A\times R_B, \mathscr{G})=\pi_0(\mathscr{G})=2,$$

and all line singularities reduce to the boundaries of "islands" which can appear on account of the non-connected character of the degeneracy space $\mathscr{G}$.

4. SINGULARITIES OF THE SURFACE LAYER IN SUPERFLUID ^{3}He

We now consider the boundary between ^{3}He–B and the wall of the vessel. The structure of the order parameter near the wall has been investigated in Refs. 19–23. Reference 23 lists eight structures of the order parameter; here we consider the possible singularities of these structures. The notations are according to Ref. 23, and the symmetry classes, according to Ref. 20.

The asymptotic behavior of the solution in the B-phase in all the structures is $A^{0B}_{\alpha i} = \Delta_B \delta_{\alpha i}$. The subgroup of the group (2) $G_i = O(2)^G\times O(2)^L\times SO(3)^S$ preserving the asymptotic behavior is $H_{as} = O(2)^J\times Z^T_2$, where $O(2)^J$ is the group consisting of combined spin-orbital rotations around the x axis, and their compositions with the element $C_{2y}P$ [Eqs. (24), (25)], and Z_2 is the group with generator T. The space $\widetilde{R}_B$ of the values of the order parameter which one can obtain by the action of all elements of G_i on $A^{0B}_{\alpha i}$ is $\widetilde{R}_B = U(1)\times SO(3)$ and coincides with the whole degeneracy space R_b of the B-phase. Therefore $\pi_2(R_B,\widetilde{R}_B) = 0$, holds, but the problem of boundary singularities remains nontrivial because the distribution of the order parameter in the bulk of the boundary may exhibit a lower symmetry H than H_{as}. We now consider the eight structures listed in Ref. 23 (see Fig. 4)

The structures P^1, $\widetilde{P}^1$. These are maximally symmetric distributions of the order parameter of the symmetry class 17. In the layer between the B-phase and the boundary there is a planar phase; the texture of the orbital vector $\mathbf{l}$ is homogeneous. The symmetry group of the solution is $H = O(2)^J\times Z^T_2 = H_{as}$. The degeneracy space of the states of the boundary is

$$\mathscr{G}=G_i/H=U(1)\times SO(3)=R_B.$$

There are no line defects on the boundary, all point-defects are exhausted by ends of singular lines coming from the bulk of the B-phase, and are classified by elements of the group $\pi_1(R_B) = Z\oplus Z_2$.

Before discussing the other structures corresponding to a lower symmetry we make a few remarks. In all the following structures we have $\pi_1(\mathscr{G}) = Z\oplus Z\oplus Z_2$, the point singularities of the boundary are classified by three indices (n_1,n_2,n_3); $n_1,n_2\in Z$, $n_3\in Z_2$, of which the indices n_1, n_2 will always correspond to ends of vortices and disclinations, coming from the bulk of the B-phase. The index n_2 describes "internal" isolated point singularities of the boundary (see the end of Sec. 2). Indeed, since $R_B = \widetilde{R}_B$, the map h_B: $\mathscr{G}\to R_B$ (5b) coincides with the projection of the bundle $G_i/H\to G_i/H_{as}$ with fiber H_{as}/H. Therefore the set $\mathrm{Ker}\, h^1_B$ which, according to Eq. (19) describes isolated point singularities of the boundary, coincides with the component I of "internal" degrees of freedom of the boundary, introduced in Eq. (21). The physical meaning of the index n_2 will be explained separately for each structure. In addition, for some structures, on account of the non-connectedness of the degeneracy space the appearance of singular lines is possible on the boundary, lines which are "internal," i.e., their presence influences only the distribution of the order parameter within the boundary, but has no influence on the distribution of the order parameter in the bulk of the B-phase.

The structures A^1, $\widetilde{A}^1$. In the layer between the B-phase and the wall of the vessel there is the A-phase, the texture of the vector $\mathbf{l}$ is homogeneous and $\mathbf{l}\|x$. The symmetry class is 11. The symmetry group of the solution is $H = \mathscr{D}^{(2)}_2 = \{1,C_{2x},C_{2y}PT,C_{2z}PT\}$. The degeneracy space

$$\mathscr{G}=(O(2)^G\times(O(2)^L\times SO(3)^S)/Z_2)/Z_2$$

has already been studied in Sec. 3, in connection with the A–B boundary with boundary condition $\mathbf{l}\|x$. The index n_2 can

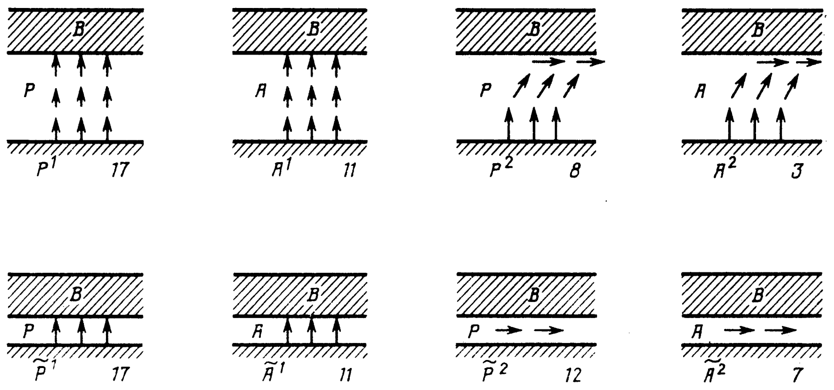

FIG. 4. Various surface structures in superfluid ^{3}He-B; the letter p denotes the planar phase, the letters A, B, the corresponding phases. The texture of the orbital vector l is schematically represented by arrows. Under the picture representing each structure is indicated its notation in Ref. 23 and the symmetry class of Ref. 20.

be interpreted as the accumulation of phase in units of π and the number of half-rotations of the vector $\mathbf{d}$ in the boundary layer of the A-phase when the singularity is circumvented. Since we have $\pi_0(\mathcal{G}) = 2$ the formation of internal singular lines is possible, separating the two islands with $\mathbf{l} = \pm\hat{\mathbf{x}}$.

The structure A^2. In the layer between the B-phase and the wall of the vessel there is the A-phase, the texture of the vector $\mathbf{l}$ is inhomogeneous: on the boundary of the vessel $\mathbf{l}$ satisfies the normal boundary condition $\mathbf{l}\|x$, and at the boundary with the B-phase $\mathbf{l}\|y$. The symmetry class is 3. The symmetry group of the solution is $H = \{1, C_{2z}PT\}$. The degeneracy space of the states of the boundary is

$$\mathcal{G}=((O(2)^G\times O(2)^L)/Z_2)\times SO(3)^S.$$

The index n_2 indicates the number of rotations of the vector $\mathbf{l}$ when the singularity is circumvented. Since $\pi_0(\mathcal{G}) = 2$ the formation of internal singular lines is possible, separating the two islands with the direction of the superfluid flow along the z axis or opposite to it (see Ref. 22).

The structure $\tilde{A}^2$. The difference from the structure A^2 is that owing to the narrowness of the surface layer of A-phase the texture of the vector $\mathbf{l}$ is absent, and everywhere $\mathbf{l}\|y$. An additional symmetry element arises and the symmetry group of the solution becomes $H = \mathcal{D}_2^{(1)} = \{1, C_{2x}T, C_{2y}P, C_{2z}PT\}$, corresponding to the symmetry class 7. The degeneracy space of the states of the boundary is

$$\mathcal{G}=U(1)\times SO(2)^L\times SO(3)^S,$$

and the only difference from the structure A^2 is the absence of singular lines, since now $\pi_0(\mathcal{G}) = 0$; this means that here the superfluid flow is forbidden by the symmetry.

The structure P^2. In the layer between the B-phase and the wall of the vessel there is a planar phase. The texture of the orbital vector $\mathbf{l}$ is inhomogeneous (as in the case A^2). The symmetry group of the solution is enlarged compared to the case A^2 to $H = \{1, T, C_{2z}P, C_{2z}PT\}$, corresponding to the class 8. The degeneracy space is

$$\mathcal{G}=U(1)\times SO(2)^L\times SO(3)^S,$$

and the distinction from the case A^2 is only in the absence of superfluid flow.

The structure $\tilde{P}^2$. Owing to the narrowness of the surface layer of A-phase the texture of the vector $\mathbf{l}$ is absent. A deformation of the order parameter of the planar phase occurs, leading to a conversion to the polar phase at the wall of the vessel (Ref. 23). (A similar phenomenon must also occur for the $\tilde{A}^2$ structure, for which so far neither an analytic nor a numerical solution has been found.) The symmetry group of the solution is $H = \{1, T, C_{2x}, C_{2y}P, C_{2z}P, C_{2x}T, C_{2y}PT, C_{2z}PT\}$, corresponding to the symmetry class 12. Compared to the symmetry group for P^2 there appears the new element C_{2x} which leads to a further factoring of the degeneracy space $\mathcal{G}\,U(1)\times(SO(2)^L\times SO(3)^S)/Z_2$. The index n_2 indicates the number of half-rotations of the vector $\mathbf{l}$ when the singularity is circumvented.

The internal singularities we have considered are vortex lines starting at the surface of the vessel and ending at the boundary with the B-phase, and therefore, on scales of the order of several coherence lengths ξ it is natural to consider them as line defects. At large scales they can be considered as peculiar point boojums of the boundary layer.

5. SEVERAL FURTHER EXAMPLES

The next problem to which it seems natural to apply the described method consists in the classification of the singularities at the A–B boundary with additional interactions taken into account (see Ref. 7). The presence of an external field leads to a lowering of the group G_i of symmetries of the physical laws in the presence of a boundary separating the phases. For example, suppose there is a magnetic field parallel to the A–B-interface, having the direction of the y-axis in the geometry we consider. Then the group G_i is $G_i = O(2)^{GS}\times O(2)^L\times SO(2)^S$; here the group $SO(3)^S$ reduced to the subgroup of spin rotations around the y axis, and time reversal T now enters G_i in a combination with a spin rotation by π around the x axis. The degeneracy spaces of the A- and B-phases are modified accordingly. The possibility of various boundary conditions was pointed out in Ref. 5. Thus, there are data for an analysis of the singularities of the A–B-boundary in the presence of a magnetic field.

In the same manner one can solve, for instance, the problem of classification of singularities of nontopological domain walls in the volumes of the A- and B-phases, which are analogs of the cosmic domain walls.[24,25] The spaces R_a and R_b are chosen identical and coincide with the degeneracy space of the phase under consideration.

We also note that a formal consideration of the problem of particle-like solitons containing the A–B boundary lead, for example in the case $H = H_{as}$ to the relative homotopy group $\pi_3(R_A\times R_B, \mathcal{G})$ (analogous to Ref. 7).

6. CONCLUSION

Thus, the procedure described in Section 2 yields a regular method for classifying singularities on the interfaces between condensed matter phases. One naturally distinguishes isolated singular points and singular lines, point and line singularities which are the endpoints of bulk defects in the phases, or intersections between bulk defects and the interface. Among the isolated singularities one distinguishes essential singularities of the boundary which cannot be taken into the interior of one of the phases (such point singularities are boojums), and singularities which come to the surface from the volume. Together with the classifying indices are defined laws which govern their addition when the singularities fuse. The formulas obtained here generalize the method of relative homotopy groups (Ref. 14). For example, in the case of maximal symmetry of the solution at the boundary, isolated point singularities are classified by the group $\pi_2(R_A\times R_B, \mathcal{G})$.

The classification obtained in Sec. 3 for the singularities on the A–B boundary contains completely the previously known information and allows one to draw some conclusions, in particular: 1) In the case of the boundary condition $\mathbf{l}\perp x$ a semiquantum vortex cannot end on the separation surface; 2) for any of the two boundary conditions we have considered the singular point which came from the bulk of the A-phase can merge with a boojum on the interface and when it merges with the end of a vortex (Dirac monopole) a loss of quantum number occurs. Thus, the singular points of the bulk of the A-phase can be annihilated at the end of a vortex. For various states near the boundary of ^{3}He-B with the wall of the vessel all possible internal degrees of freedom have been indicated, degrees which can play an essential role in the explanation of the experimental results.[15-18]

I am grateful to G. E. Volovik for posing the problem considered here and for numerous useful discussions and to S. P. Novikov, who made a series of valuable remarks.

[1] N. Schopol, Phys. Rev. Lett. **58**, 1664 (1987).
[2] M. M. Salomaa, J. Phys. C, **21**, 4425 (1988).
[3] D. S. Buchanan, G. W. Swift, and J. C. Wheatley, Phys. Rev. Lett. **57**, 341 (1986).
[4] S. T. Boyd and G. W. Swift, Preprint, Los Alamos Natl. Lab. 1989.
[5] G. E. Volovik, Zh. Eksp. Teor. Fiz. **97**, 1198 (1990); [Sov. Phys. JETP, **70**, 672 (1990)].
[6] G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **51**, 396 (1990); [JETP Lett. **51**, 449 (1990)].
[7] V. P. Mineev, Sov. Sci. Rev. **2**, 173 (1980).
[8] A. T. Fomenko and D. B. Fuks, *Gomotopicheskaya topologiya* (*Homotopic topology*), Akademiai Kiado, Budapest, 1986, Ch. 1.
[9] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, NY, 1966. Ch. 7.
[10] N. D. Mermin, in *Quantum Fluids and Solids*, Ed. by S. B. Trickey, E. D. Adams, and J. W. Dufty, Plenum Press, NY 1977, p. 3.
[11] S. Blaha, Phys. Rev. Lett. **36**, 874 (1976).
[12] M. M. Salomaa, Nature **326**, 367 (1987).
[13] M. M. Salomaa, Czech. J. Phys. **40**, 761 (1990).
[14] V. P. Mineev and G. E. Volovik, Phys. Rev. B **18**, 3197 (1978).
[15] J. P. Pekola and J. T. Simola, J. Low Temp. Phys. **58**, 555 (1985).
[16] P. J. Hakonen and K. K. Nummila, Phys. Rev. Lett. **59**, 1006 (1987).
[17] M. M. Salomaa and G. E. Volovik, J. Low Temp. Phys. **75**, 209 (1989).
[18] J. M. Kyynarainen, J. P. Pekola, A. J. Manninen, and K. Torizuka, Phys. Rev. Lett. **64**, 1027 (1990).
[19] E. V. Thuneberg, Phys. Rev. B **33**, 5124 (1986).
[20] W. Zhang, J. Kurkijärvi, and E. V. Thuneberg, Phys. Rev. B **36**, 1987 (1987).
[21] M. M. Salomaa and G. E. Volovik, J. Low Temp. Phys. **75**, 209 (1987).
[22] T. Sh. Misirpashaev and G. E. Volovik, J. Phys. Cond. Matt **2**, 7361.
[23] G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **52**, 976 (1990); [JETP Lett. **52**, 363 (1990)].
[24] M. M. Salomaa and G. E. Volovik, Phys. Rev. B **37**, 9298.
[25] M. M. Salomaa and G. E. Volovik, J. Low Temp. Phys. **74**, 319 (1989).

Translated by Meinhard E. Mayer