Dynamics of point quadrupole vortices and of elliptical vortices in a plane

S. G. Chefranov

Atmospheric Physics Institute, Academy of Sciences of the USSR
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We obtain a finite-dimensional set of Hamiltonian equations for the dynamics of point quadrupole vortices—infinitesimally small elliptical vortices with an inhomogeneous nonstationary vorticity distribution corresponding to an exact weak nonstationary solution of the two-dimensional Helmholtz equation for the vortex field. We show that merging (convergence) to a single point after a finite time is possible for two identical point quadrupole vortices when the initial distance between them is less than some critical value $l_{\rm cr} = 2(3\rho/\varkappa)^{1/2}$, independent of the size of the corresponding elliptical vortices and determined solely by the ratio of the intensity of the quadrupole component ρ and the magnitude $\varkappa$ of the circulation of the quadrupole vortex. We find stable stationary regimes in a system consisting of a single point quadrupole vortex and a single point vortex with $\rho = 0$ interacting with it. We obtain a generalization of the equations for the dynamics of quadrupole vortices taking into account the effect of molecular viscosity and Ekman friction and we estimate the degree to which these dissipative factors affect the dynamics and the collapse of these vortex objects.

The phenomenon of the collapse of individual localized vortices, observed in experiments[1–7] and in Nature[8], which leads to a vorticity cascade in two-dimensional and quasi-two-dimensional turbulence, has also stimulated a corresponding theoretical study of strong vortex interactions.[9–14] In particular, it was shown in Refs. 10 to 13 that two identical finite-size elliptical vortices with uniform vorticity in a plane can merge if the initial distance between their centers is less than a critical value $l_{\rm cr} \approx 3.4(A/\pi)^{1/2}$ connected only with the fact that the vortices have a finite size and with their area A. At first sight the merging effect itself is in general caused only by the fact that the elliptical vortices have a finite size. Indeed, it is even impossible in principle for two point vortices in a plane to merge into a single point because of the well known invariance of their separation distance when there are no dissipative factors.[15] However, in contrast, it was noted in a recent paper[14] that in the plane two vortex objects can merge, even though they are point objects, namely, point vortex dipoles each of which may be in the form of two point vortices of different signs and strengths, separated by an arbitrarily small distance from one another.[1)] Therefore the decisive factor for the enabling localized vortices to merge may be not so much the size of these vortices, as the nature of the symmetry of the vorticity distribution in them.

We formulate in the present paper a Hamiltonian dynamical description and we obtain the conditions for collapse for arbitrarily small elliptical vortices with an inhomogeneous vorticity distribution inside the vortex core: point quadrupole vortices (PQV), for which, in contrast to elliptical Kirchhoff vortices with uniformly distributed vorticity (EKV)[17] or finite area vortex regions (FAVORs),[10–13] the kind of symmetry of the external stream function is scale invariant, i.e., independent of the size of the vortices, however small they are. In fact, when a Kirchhoff vortex with uniform vorticity goes over into a vortex of arbitrarily small size, it degenerates simply into a point vortex, whereas a PQV can also occur in the form of a vortex cluster, consisting of two point dipole vortices and one point vortex which are positioned arbitrarily close to one another in the plane. The new localized multipole-type[16] vortex objects (PQV) introduced in the present paper may facilitate the widening of possibilities for studying strong nonlinear interactions in two-dimensional turbulence, the simulation of which is necessary in many problems of physical, geophysical, and magneto-hydrodynamics, and also in plasma physics.

1. The vortex field distribution in a plane for a single point quadrupole vortex (i.e., a point vortex second-rank multipole[16]), positioned at the origin, has the form

$$\omega(\mathbf{x}) = \varkappa\delta(\mathbf{x}) + a_{ij}\frac{\partial^2\delta(\mathbf{x})}{\partial x_i\,\partial x_j}, \tag{1}$$

where there is a summation from 1 to 2 over repeated indices ($x_1 \equiv x$, $x_2 \equiv y$), δ is a delta-function, a_{ij} is the symmetric tensor of the quadrupole component of the strength of the PQV, and $\varkappa$ is the scalar strength of its monopole component. For $a_{ij} \equiv 0$, Eq. (1) describes the vorticity of a point vortex with strength $\varkappa$.[15] The stream function[15]

$$\Psi = -\frac{1}{2\pi}\int d^2x'\,\omega(\mathbf{x}')\ln|\mathbf{x}-\mathbf{x}'|,$$

corresponding to (1) has the form (in a reference frame for which the matrix a_{ij} is diagonal, i.e., where we have $a_{12} = 0$)

$$\Psi(\mathbf{x}) = -a_{22}\delta(\mathbf{x}) - \frac{\varkappa}{2\pi}\ln r - \frac{\rho\cos 2\varphi}{2\pi r^2}, \tag{2}$$

where we have put $\rho = a_{22} - a_{11}$ and where we use polar coordinates: $|\mathbf{x}| = r$, $x = r\cos\varphi$, $y = r\sin\varphi$.

We show in Fig. 1 the iso-level lines of the function (2) in a frame of reference rotated over an angle $\pi/4$, when $\cos 2\varphi$ in (2) is replaced by $-\sin 2\varphi$. The quantity Ψ is here expressed in units $\varkappa/2\pi$ and x and y in units of the length scale $l_* = (\rho/\varkappa)^{1/2}$. We note that neither the first singular term, nor the remaining terms in (2) correspond to a self-induced shift of the PQV from the origin, in contrast to point dipole vortices for which a singular directed self-induced velocity can be eliminated only as the result of the appropriate regularization.[14,16] In that respect the PQV are more convenient and are similar to point dipoles in a plane for which there is also no a priori directed self-induced motion.

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The term with a_{22} in (2) therefore affects neither the dynamics of the fluid particles surrounding the PQV, nor the PQV itself.

We note that for a solid elliptical cylinder (uniform and infinite along the z-axis and with an ellipse with semiaxes a, b with $a>b$ as transverse cross-section) rotating in a fluid with a frequency n when there is a circulation of the liquid around the cylinder which is independent of n and has a cyclic constant $\varkappa$ the flow outside the cylinder is described by a stream function which differs from (2) only by the sign [when we have $a_{22}=0$ in (2)], provided in (2) we have

$$\rho=\frac{\pi n(a+b)^2(a^2-b^2)}{8} \tag{3}$$

for $a+b\gg c\equiv(a^2-b^2)^{1/2}$ (see Ref. 17, Ch. IV and Ref. 18). We have the same stream function in the exterior region (relative to the localization boundary of the vortex field) and for an elliptical Kirchhoff vortex with uniform vorticity ω, rotating rigidly with a frequency n which is now rigorously connected with ω and $\varkappa=\pi ab\omega$ through the relation $\pi n(a+b)^2=\varkappa$ (Ref. 17, Ch. VII) provided we have the same relation (3) between ρ and n.

The relation given above between n and $\varkappa$ for a Kirchhoff EKV means that if we take the limit as $a\to 0$, $b\to 0$ for constant circulation $\varkappa$ the quantity $\rho\approx\varkappa(a^2-b^2)$ in (2) tends to 0 and, in fact, in that limit the elliptical Kirchhoff vortex degenerates simply into a point vortex with a strength $\varkappa$ and a stream function $\Psi=-(\varkappa/2\pi)\ln r$. Therefore the PQV (1) with $\rho\neq 0$ does not correspond to an infinitesimally small Kirchhoff EKV since for a PQV ρ and $\varkappa$ are, generally speaking, independent.

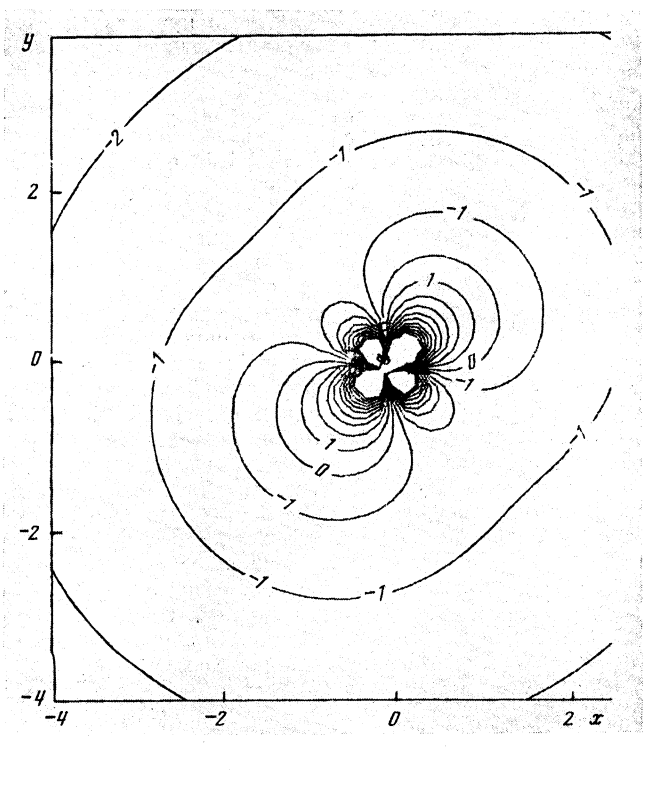

FIG. 1.

However, one can show easily that the PQV (1) with $\rho\neq 0$ can still correspond to an infinitesimally small elliptical vortex, but with a nonuniform vorticity distribution in its core, in contrast to a Kirchhoff vortex with uniform vorticity. Such a nonuniform elliptical vortex can exist, using the distribution (1), if we replace in (1) the δ-function by its regularized "smeared-out" form[14] $\tilde\delta$ inside a finite elliptical region with semiaxes a and b $(a>b)$. For instance, $\tilde\delta$ may have the form

$$\tilde\delta(\xi)=a_0\,\mathrm{sh}^2\,\xi\cdot\left(\frac{a+b}{c}-e^{\xi}\right)^2\theta\left(\frac{a+b}{c}-e^{\xi}\right), \tag{4}$$

where θ is the Heaviside function [$\theta(x)=1$ for $x\geqslant 0$ and $\theta(x)=0$ for $x<0$], we have $c^2=a^2-b^2$, and the necessary conditions $\tilde\delta(\xi)=d\tilde\delta(\xi)/d\xi=0$ on the boundary of the elliptical region $\xi=\ln[(a+b)/c]$ and for $\xi=0$ are satisfied. We use elliptical coordinates in (4) for which we have $x=c\cosh\xi\cos\varphi$, $y=c\sinh\xi\sin\varphi$ (on the boundary of the elliptical region we have $x=a\cos\varphi$, $y=b\sin\varphi$) and the quantity a_0 is determined from the normalization condition

$$1=\int d^2x\tilde\delta(\mathbf{x})=a_0c^2\int_0^{2\pi}d\varphi\int_0^{\ln p}d\xi\,(\mathrm{sh}^2\,\xi+\sin^2\varphi)\,\mathrm{sh}^2\,\xi\,(p-e^{\xi})^2,$$

where we have $p\equiv(a+b)/c$.

In the limit of weak ellipticity, $p\gg 1$, we construct in the Appendix an exact stationary solution for the vorticity distribution (1), replacing δ by $\tilde\delta$ with $\tilde\delta$ from (4), satisfying the rigidity condition for the stationary rotation of the elliptical boundary (and only the boundary) with a frequency n with unchanged a and b, and also the matching conditions on that boundary for the stream function and its first derivatives in the interior and exterior regions for (2) corresponding to the stream function in the exterior region. The quantities ρ and n are then related through the same Eq. (3) as for a Kirchhoff EKV, but for a nonuniform elliptical vortex the exterior stream function is now, in contrast to the EKV, scale invariant since there is no relation whatever between $\varkappa$ and n or ρ; this enables us to consider the PQV (1) as an infinitesimally small nonuniform elliptical vortex for $a\to 0$, $b\to 0$, $n\to\infty$ and finite, independent magnitudes of ρ and $\varkappa$.

We note that a rigid rotation of the boundary with frequency n does, in fact, not make a finite contribution to the total circulation of a nonuniform elliptical vortex which is equal to $\varkappa$. One can verify this directly by integrating the vorticity distribution [after replacing in (1) the δ-function by its smeared-out form $\tilde\delta$ from (4)] over the plane, bearing in mind that for the $\tilde\delta$ from (4) we have $\tilde\delta=d\tilde\delta/d\xi=0$ on the boundary of the ellipse and for $\xi=0$.

2. For a system consisting of N PQV the vortex field distribution has the form

$$\omega(\mathbf{x},t)=\sum_{\alpha=1}^{N}\left[\varkappa_\alpha\delta(\mathbf{x}-\mathbf{x}_\alpha)+a_{ij}{}^{\alpha}(t)\,\frac{\partial^2\delta(\mathbf{x}-\mathbf{x}_\alpha)}{\partial x_i\,\partial x_j}\right], \tag{5}$$

where the $\mathbf{x}_\alpha(t)$ are the coordinates in the plane of the PQV of number α for which $\varkappa_\alpha$ and a_{ij}^α correspond to the monopole and quadrupole components of its strength, in accord with the definition (1). The stream function of the fluid particles, corresponding to the vortex field (5), has the form

$$\Psi=-\frac{1}{2\pi}\sum_{\beta=1}^{N}\Big\{\varkappa_\beta\ln|\mathbf{x}-\mathbf{x}_\beta|+\frac{\tilde z_\beta[(x-x_\beta)^2-(y-y_\beta)^2]-2\tilde b_\beta(x-x_\beta)(y-y_\beta)}{|\mathbf{x}-\mathbf{x}_\beta|^4}\Big\},\qquad(6)$$

where $\tilde z_\beta=a_{22}^\beta-a_{11}^\beta$, $\tilde b_\beta=a_{12}^\beta+a_{21}^\beta=2a_{12}^\beta$. In (6) we have dropped the singular terms $a_{22}^\beta\delta(\mathbf{x}-\mathbf{x}^\beta)$ because, as mentioned earlier, it does not affect the dynamics of the fluid particles or the PQV themselves (see Appendix I). Expression (6) is the same as the stream function of the far field for a system of N FAVORs from Ref. 12, if the relations

$$\varkappa_\beta=\omega_\beta J^{(0,0)},\qquad \tilde z_\beta=\frac{\omega_\beta(J_\beta^{\,0,2)}-J_\beta^{(2,0)})}{2},$$

$$\tilde b_\beta=\omega_\beta J_\beta^{(1,1)},\qquad a_{22}{}^\beta=\frac{\omega_\beta J_\beta^{(0,2)}}{2},$$

$$a_{11}{}^\beta=\frac{\omega_\beta J_\beta^{(2,0)}}{2},\qquad J_\beta^{(m,n)}=\iint_{D_\beta}d\xi\,d\eta\,\xi^m\eta^n,$$

are satisfied, where D_β is the elliptical region of number β, $J_\beta^{(0,0)}$ is its area, and ω_β is the constant vorticity at each of its points. Bearing in mind what we said in the preceding subsection, the parameters $\varkappa_\beta$ and $\tilde z_\beta$, $\tilde b_\beta$ for the PQV in (6) are completely independent, which produces a difference between the PQV and the FAVORs of Refs. 12 and 13 and the Kirchhoff vortices.

The parameters $\varkappa_\alpha$, $\mathbf{x}_\alpha(t)$, and a_{ij}^α in (5) and (6) are determined in terms of the exact weak solution of the two-dimensional Helmholtz equation (A7) obtained by substituting in that equation the vorticity distribution (5) and integrating the expression obtained, multiplied by a quadratically nonlinear weight function (see Appendix II), over the plane. In the general case, taking into account the effect of the Ekman (or Stokes) friction and the molecular viscosity, the evolution of these parameters of (5) and (6) is then described by the finite-dimensional set of dynamical equations (A9) to (A11).

3. For an ideal fluid the dynamical set (A9) to (A11) can be written in Hamiltonian form

$$\varkappa_\alpha\dot x_i{}^\alpha=\varepsilon_{ij}\frac{\partial H_\alpha}{\partial x_j{}^\alpha},\qquad(7)$$

$$\dot q_i{}^\alpha=\varepsilon_{ij}\frac{\partial H_\alpha}{\partial q_j{}^\alpha},\qquad(8)$$

where the dot indicates differentiation with respect to the time. Here we have $\alpha=1,2,...,N$, $\varkappa_\alpha=\text{const}$, ε_{ij} is the antisymmetric second rank unit pseudotensor ($\varepsilon_{12}=1$, $\varepsilon_{21}=-1$, $\varepsilon_{11}=\varepsilon_{22}=0$), summation proceeds from 1 to 2 over repeated indices, and we have $q_1^\alpha\equiv\Psi_\alpha=\arctan(\tilde b_\alpha/\tilde z_\alpha)$, $q_2^\alpha=(\rho_\alpha^2+\widetilde C_\alpha^2)^{1/2}$, $\rho_\alpha^2=\tilde z_\alpha^2+\tilde b_\alpha^2$, $\widetilde C_\alpha^2=\tilde a_\alpha^2-\rho_\alpha^2$, $\tilde a_\alpha=a_{11}^\alpha+a_{22}^\alpha$. For (7), (8) we have $\widetilde C_\alpha=\text{const}$ and

$$H_\alpha=\varkappa_\alpha\Psi(\mathbf{x}^\alpha)+a_{ij}{}^\alpha\frac{\partial^2\Psi(\mathbf{x}^\alpha)}{\partial x_i{}^\alpha\,\partial x_j{}^\alpha}=\text{const}$$

for Ψ from (6). Since in (5) we have $\omega=\Sigma_{\alpha=1}^N\omega_\alpha$, the quantity $\tilde a_\alpha$ is related to the magnitude of the intrinsic angular momentum

$$M_s{}^\alpha=\int d^2x\,\omega_\alpha(\mathbf{x})\mathbf{x}^2$$

for the PQV of number α, shifted to the origin (i.e., we have $\mathbf{x}_\alpha=0$) since we find $M_s^\alpha=2\tilde a_\alpha$. For the finite nonuniform elliptical vortex corresponding to this PQV we have now

$$M_s{}^\alpha\simeq2\tilde a_\alpha+\frac{5\varkappa}{84}c_\alpha{}^2p_\alpha{}^3$$

for $p_\alpha\gg1$ and $c_\alpha\to0$, while we have $\rho_\alpha=(\pi n_\alpha/8)(a_\alpha+b_\alpha)^2c_\alpha^2$ with $c_\alpha^2=a_\alpha^2-b_\alpha^2$, $p_\alpha=(a_\alpha+b_\alpha)/c_\alpha$; a_α and b_α are the semiaxes of the vortex and n_α is the rotational frequency of the boundary of the vortex of number α [see (3)]. The quantity H_α in (7), (8) is related to the invariant interaction energy of the system of PQV defined in the form[15]

$$T'=\frac{-\tilde\rho_0}{8\pi}\int d^2x\int d^2x'\omega(\mathbf{x})\omega(\mathbf{x}')\ln|\mathbf{x}-\mathbf{x}'|=\frac{\tilde\rho_0}{2}\sum_{\alpha=1}^{N}H_\alpha,\qquad(9)$$

where $\tilde\rho_0$ is the density of the incompressible fluid, while the prime on T means that we have dropped the singular interaction energy of the PQV from the total energy of the set of N PQV of (5). The other hydrodynamical invariants,[15] the total momentum

$$P_i=\tilde\rho_0\int d^2x\omega(\mathbf{x})x_i$$

and the total angular momentum

$$M=\tilde\rho_0\int d^2x\omega(\mathbf{x})\mathbf{x}^2,$$

are also conserved for the dynamical system (7), (8), as one can check directly. For ω from (5) we have

$$P_i/\tilde\rho_0=\sum_{\alpha=1}^{N}\varkappa_\alpha x_i{}^\alpha(t)=\text{const},\qquad(10)$$

$$M/\tilde\rho_0=\sum_{\alpha=1}^{N}[2a_\alpha(t)+\varkappa_\alpha(\mathbf{x}^\alpha(t))^2]=\text{const}.\qquad(11)$$

Using the interpretation of the parameters of the function (6), noted in the preceding subsection, in the terms of Ref. 12 the quantity $\widetilde C_\alpha$ is given by $\widetilde C_\alpha=(\omega_\alpha/4\pi)(J_\alpha^{(0,0)})^2$ and Eqs. (7) are exactly the same as Eqs. (3.9) from that paper, while (8) differs from Eqs. (3.10) to (3.12) given in Ref. 12 only through the absence of the interaction terms describing the rigid self-induced rotation of the FAVORs with uniform vorticity. We note, however, that making the transition to infinitesimally small sizes of the FAVORs Eqs. (3.9) to (3.12) in Ref. 12 degenerate simply into the set of equations for N point vortices, which even qualitatively differs from the set (7) and (8) for the PQV corresponding to infinitesimally small nonuniform elliptical vortices (see Appendix I).

4. The case of two interacting EKV was considered in quite some detail in Refs. 10 to 13. We shall therefore restrict ourselves in what follows to a study of the interaction of two PQV when we put $N=2$ in (7), (8) and (A9) to (A11). To describe the relative dynamics of the PQV for $N=2$ we introduce new variables $x^1-x^2=l\cos\varphi$, $y^1-y^2=l\sin\varphi$, where l is the distance between the two PQV with numbers

$\alpha = 1$ and $\alpha = 2$, while φ is the polar angle which characterizes the orientation of the straight line passing through these PQV relative to some fixed frame of reference. The absolute dynamics of the PQV can if necessary easily be established from the given functions $l(t)$ and $\varphi(t)$ taking into account the invariance of the two momentum components in (10). It is also convenient to use for the internal variables $\tilde{z}_\alpha$ and $\tilde{b}_\alpha$ the representation introduced earlier, $\tilde{b}_\alpha = \rho_\alpha \sin \Psi_\alpha$, $\tilde{z}_\alpha = \rho_\alpha \cos \Psi_\alpha$ [see (8)], i.e., the interaction of the PQV leads to the rotation of the stream lines of each PQV (see Fig. 1) as a whole with a frequency $\dot{\Psi}_\alpha$. If we take into account the action of the dissipative factors, the general set (A9) to (A11) then has the form (A12) to (A19) (see Apendix II).

For an ideal fluid $\varkappa_\alpha = \text{const}$ and $\widetilde{C}_\alpha = \text{const}$. Let us have $\widetilde{C}_\alpha = 0$; we first consider the case of two identical PQV, the more because in Refs. 10 and 12, 13 the main attention was also focused on just the symmetric case (two identical EKV). In that case we have $\varkappa_1 = \varkappa_2 = \varkappa$, $\Psi_1 = \Psi_2 = \widetilde{\Psi}$, $\rho_1 = \rho_2 = \rho$ and the set (A12) to (A19) has the form

$$\dot{l} = \frac{4\rho \sin \tilde{y}}{\pi l^3}\left(1 - \frac{12\rho}{\varkappa l^2} \cos \tilde{y}\right), \tag{12}$$

$$\dot{\rho} = -\frac{\varkappa\rho \sin \tilde{y}}{\pi l^2}\left(1 - \frac{12\rho}{\varkappa l^2} \cos \tilde{y}\right), \tag{13}$$

$$\dot{\tilde{y}} = \frac{\varkappa}{\pi l^2}\left[2 - \left(1 + \frac{8\rho}{\varkappa l^2}\right)\left(\cos \tilde{y} - \frac{6\rho}{\varkappa l^2} \cos 2\tilde{y}\right)\right], \tag{14}$$

with $\tilde{y} = \widetilde{\Psi} + 2\varphi$, while the corresponding equations for $\widetilde{\Psi}$ and φ have the form

$$\dot{\widetilde{\Psi}} = \frac{\varkappa}{\pi l^2}\left(-\cos \tilde{y} + \frac{6\rho}{\varkappa l^2} \cos 2\tilde{y}\right), \tag{15}$$

$$\dot{\varphi} = \frac{\varkappa}{\pi l^2} - \frac{4\rho}{\pi l^4}\left(\cos \tilde{y} - \frac{6\rho}{\varkappa l^2} \cos 2\tilde{y}\right). \tag{16}$$

The closed dynamical set (12) to (14) corresponds to two invariants: the interaction energy,

$$T' = -\frac{\tilde{\rho}_0 \varkappa^2}{\pi}\left[\ln l + \frac{2\rho}{\varkappa l^2} \cos \tilde{y} - \frac{6\rho^2}{\varkappa^2 l^4} \cos 2\tilde{y}\right] \tag{17}$$

and the angular momentum

$$M = \frac{\tilde{\rho}_0 \varkappa l^2}{2}\left(1 + \frac{8\rho}{\varkappa l^2}\right). \tag{18}$$

For all the identical FAVORs or EKV in Ref. 13 in the limit of EKV sizes small compared to the distance l between them we have $\rho/\varkappa l^2 \ll 1$ for the quantity $\rho/\varkappa l^2$ and the terms of order $O(\rho/\varkappa l^2)$ inside the brackets in (12) and (13) are not present in the corresponding equations for l and ρ. At the same time, for the PQV corresponding to infinitesimally small nonuniform elliptical vortices (but not EKV) these terms are no longer small and may substantially determine the dynamics of the PQV according to (12) to (14).

For an analysis of the set (12) to (14) it is convenient to use its phase portrait in the plane of the dimensionless variables

$$\bar{x} = \frac{l}{l_p} \cos \tilde{y}, \quad \bar{y} = \frac{l}{l_p} \sin \tilde{y}$$

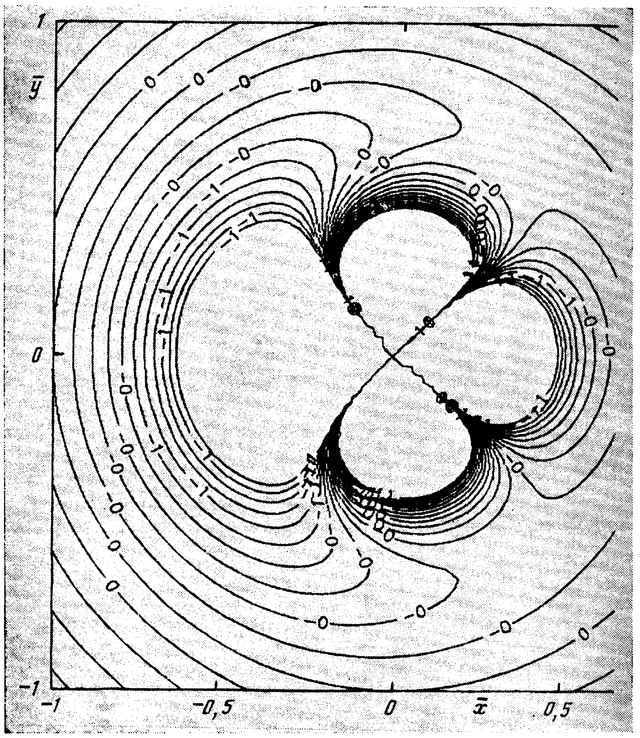

FIG. 2.

(Fig. 2) where we have $l_p = (2|M/\tilde{\rho}_0 \varkappa|)^{1/2}$ and where we have used the invariance of M and (18) to eliminate ρ from (12) and (14). We show in Fig. 2 the isolevel lines of the invariant energy $h = -T'\pi/\tilde{\rho}_0 \varkappa^2$ from (17) for the case $M/\varkappa > 0$. The set (12), (14) has two unstable stationary saddle-point type regimes which are symmetric with respect to the $\bar{x}$-axis (or for $\bar{y} = 0$) for

$$y = y_c = \pm \arccos \frac{2l_c^2/l_p^2}{3(\mu - l_c^2/l_p^2)}, \qquad l = l_c = l_p(105^{1/2} - 3\mu)^{1/2}/4,$$

where we have $\mu = 1$ for $M/\varkappa > 0$ and $\mu = -1$ for $M/\varkappa < 0$. The phase portrait of the system for $\mu = -1$ is qualitatively the same as Fig. 2. It is clear from Fig. 2 that for sufficiently small initial values of

$$l(t=0) \equiv l_0 < l_c \approx 0{,}67 l_p = 0{,}67 l_0\left(\frac{8\rho_0}{\varkappa l_0^2} + 1\right)^{1/2},$$

i.e., for $(\rho_0 \equiv \rho(t = 0))$

$$l_0 < l_{1\,\mathrm{cr}} \approx 2{,}57(\rho_0/\varkappa)^{1/2}, \tag{19}$$

the two identical PQV merge $(l(t) \to 0)$ into a single point after a finite time while for $l_0 > l_{1\,\mathrm{cr}}$ they rotate around one another without significantly changing the distance between them, like two point vortices of the same sign.

Moreover, for the set (12) to (14) there exist for $\sin \tilde{y} = 0$ and $\tilde{y} = \pi m$, $m = 0, \pm 1, \pm 2, \ldots$ turning points in which $\dot{l} = \dot{\rho} = 0$, while $\dot{\tilde{y}} y \neq 0$. Thus, if for $t = 0$ we take $\tilde{y} = \pi m$ and $l = l_0$, $\rho = \rho_0$ we have from (14)

$$y \simeq \pi m + \frac{\varkappa t}{\pi l_0^2}\left\{2 - (-1)^m + \frac{2\rho_0}{\varkappa l_0^2}[3 - 4\cdot(-1)^m] + \frac{48\rho_0^2}{\varkappa^2 l_0^4}\right\},$$

and we get from (12) and (13) $\dot{\rho} = -\dot{l}\varkappa l_0/4$ for $m = 0$ where

$$l=\frac{4t\rho_0\varkappa}{\pi^2 l_0^5}\left(1-\frac{12\rho_0}{\varkappa l_0^2}\right)\left(1-\frac{2\rho_0}{\varkappa l_0^2}+\frac{48\rho_0^2}{l_0^4\varkappa^2}\right). \tag{20}$$

Thus, for $\rho_0\varkappa>0$ the change in sign of $\dot l$ occurs for $l_0=l_{cr}=2(3\rho/\varkappa)^{1/2}$ and for

$$l_0<l_{cr}\approx 3{,}4(\rho_0/\varkappa)^{1/2} \tag{21}$$

the PQV tend to approach since we have $\dot l<0$, while for $l_0>l_{cr}$, on the other hand, there is a tendency for them to separate since we have $\dot l>0$. For $\rho_0\varkappa<0$ the tendency for an approach ($\dot l<0$) of the PQV is observed for all l_0.

We note that for passive fluid particles there exists in Fig. 1, in accordance of the shape of the stream function of a single PQV, a separatrix which separates the regions of "free" and "trapped" particles; it lies at a distance $l=l_{2\,cr}\approx 2.1(\rho_0/\varkappa)^{1/2}$ from the origin. For $l_0<l_{2\,cr}$ the fluid particles fall into the vortex center, i.e., they are trapped by the PQV. It is therefore not surprising that the critical scales $l_{1\,cr}$ and l_{cr} in (19) and (21) are of approximately the same magnitude as $l_{2\,cr}$. Moreover, one obtains better quantitative agreement of the criterion (21) and the condition for merging for two identical circular vortices of radius R_0 with uniform vorticity obtained in Ref. 10 and having the form $l_0<3.4R_0$, if in (21) we have $l_{0*}=(\rho_0/\varkappa)^{1/2}\approx R_0$ for the quantity l_{0*}. We emphasize, however, that in contrast to Ref. 10 the merger under the conditions (21) or (19) occurs just for infinitesimally small nonuniform elliptical vortices, i.e., for point vortex objects—the PQV.

Using the invariants (17) and (18) we can integrate the set (12) to (14) in quadratures since using (17) and (18) to eliminate the variables ρ and $\tilde y$ we get for l the equation

$$\dot l=\mp\frac{\varkappa L}{3\pi l|L|}\left[2B\left(9L^2-1+3\left(h-\ln\frac{l}{l_p}\right)\mp B^{1/2}\right]^{1/2}, \tag{22}$$

where

$$L=\frac{1}{4}\left(\frac{2M}{\bar\rho_0\varkappa l^2}-1\right)=\frac{2\rho}{\varkappa l^2},\quad h=-\frac{\pi T'}{\bar\rho_0\varkappa^2},$$

$$B=1+18L^2-12\left(h-\ln\frac{l}{l_p}\right).$$

In that case

$$\cos\tilde y=\frac{1\pm B^{1/2}}{6L}, \tag{23}$$

and the "plus" sign in (23) corresponds to the "minus" sign in (22) and vice versa. It follows from (22) that for $M=0$ and $L=-\frac{1}{4}$ the system performs periodic oscillations between $l=l_{min}$ and $l=l_{max}$ where the values of l_{min} and l_{max} are determined from the condition that the expression under the radical sign in (22) vanishes. For $M\neq 0$ we have from (22) and (23) in the limit as $l\to 0$

$$l=l_0\left(1\mp\frac{9|M|Mt}{\pi|\varkappa|\bar\rho_0^2 l_0^6}\right)^{1/4}, \tag{24}$$

where the "minus" sign, corresponding to merger of two identical PQV after a finite time $t_0=\pi|\varkappa|\tilde\rho_0^2 l_0^2/9M^2$ (for $M>0$) corresponds to the value $\cos\tilde y=2^{-1/2}$; For $\cos\tilde y=-2^{-1/2}$ we have the plus sign and the PQV have a tendency to separate. For sufficiently small values of the distances l_0 between the PQV, when we have $\delta=|\varkappa|l_0^2/|M/\tilde\rho_0|\ll 1$, the collapse time is $t_0=\pi l_0^2\delta^2/\varkappa\ll\tau_b$, where $\tau_b=\pi l_0^2/\varkappa$ is the rotational period of two identical point vortices. We note also that by virtue of the invariance of M in the merger ($l\to 0$) the magnitude of ρ must increase, which may correspond to an amplification of the deformation of the elliptical vortices which is, in fact, observed in numerical experiments[10] in the collapse of two circular vortices.

5. We now consider the case of two different PQV when the invariant interaction energy (9) has the form

$$T'=2\bar\rho_0 H_1$$
$$=-\frac{\tilde\rho_0}{\pi}\left[\varkappa_1\varkappa_2\ln l+\frac{\varkappa_1\rho_2\cos(2\varphi+\Psi_2)+\varkappa_2\rho_1\cos(2\varphi+\Psi_1)}{l^2}\right.$$
$$\left.-\frac{6\rho_1\rho_2\cos(4\varphi+\Psi_1+\Psi_2)}{l^4}\right], \tag{25}$$

while the angular momentum (11) [for $P_1=P_2=0$ in (10)] is for $\varkappa_1+\varkappa_2\neq 0$ equal to

$$M=\bar\rho_0\left[2(\rho_1+\rho_2)+\frac{\varkappa_1\varkappa_2 l^2}{\varkappa_1+\varkappa_2}\right]. \tag{26}$$

For the sake of simplicity, instead of PQV number 2 let us take a point vortex of strength $\varkappa_2$ with $\rho_2=\Psi_2=0$. We then have from (A12) to (A14) and (A16) for the relative motion of the point vortex and the PQV

$$\dot l=\frac{(\varkappa_1+\varkappa_2)\rho_1\sin\tilde y_1}{\pi\varkappa_1 l^3}, \tag{27}$$

$$\dot\rho_1=-\frac{\varkappa_2\rho_1\sin\tilde y_1}{\pi l^2}, \tag{28}$$

$$\dot{\tilde y}_1=\frac{\varkappa_2}{\pi l^2}\left\{\frac{\varkappa_1+\varkappa_2}{\varkappa_2}-\cos\tilde y_1\left[1+\frac{2(\varkappa_1+\varkappa_2)\rho_1}{\varkappa_1\varkappa_2 l^2}\right]\right\}, \tag{29}$$

where $\tilde y_1=2\varphi+\Psi_1$ while the equations for φ and Ψ_1 have the form

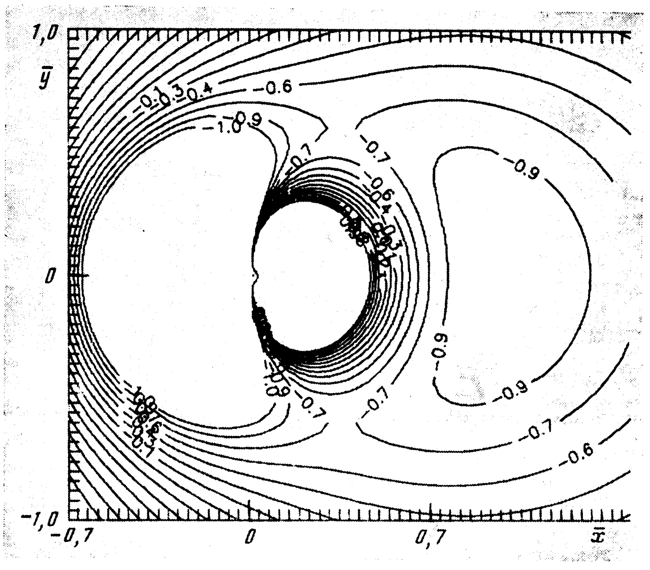

FIG. 3.

$$\dot{\varphi}=\frac{(\varkappa_1+\varkappa_2)}{2\pi l^2}\left(1-\frac{2\rho_1}{\varkappa_1 l^2}\cos\tilde{y}_1\right),\quad \dot{\Psi}_1=-\frac{\varkappa_2}{\pi l^2}\cos\tilde{y}_1. \tag{30}$$

The main difference[2)] (see Fig. 3) between the closed set (27) to (29) and the case of two identical PQV in (12) to (14) is the presence in (27) to (29) of stable, relatively small perturbations of stationary center-type regimes under the condition [in (27) and (29) we eliminate the variable ρ_1 by using the invariance of M in (26)]

$$\tilde{y}_1=\pi k,\quad l=\tilde{l}_c=[(M/\tilde{\rho}_0\varkappa_1)(-1)^k]^{1/2}, \tag{31}$$

where k is an even number for $M/\varkappa_1>0$ and k is odd for $M/\varkappa_1<0$.

The state (31) is stable for even k for $\varkappa_2/(\varkappa_1+\varkappa_2)>1$ and $\rho_1/\varkappa_1=\tilde{l}_c^2\varkappa_1/2(\varkappa_1+\varkappa_2)<0$ and for odd k for $\varkappa_2/(\varkappa_1+\varkappa_2)<-1$ and $\rho_1/\varkappa_1=-\tilde{l}_c^2(\varkappa_1+2\varkappa_2)/2(\varkappa_1+\varkappa_2)>0$. The first case with even k and $M/\varkappa_1>0$ is realized either for $\varkappa_1<0$ and $\varkappa_2>|\varkappa_1|$, or for $\varkappa_2<0$ and $0<\varkappa_1<2|\varkappa_2|$, and the second case for $\varkappa_2<0$ and $0<\varkappa_1<2|\varkappa_2|$ or for $\varkappa_1<0$ and $|\varkappa_1|<2\varkappa_2$. We note that a system consisting of different kinds of vortex objects—a system of point vortices and nonuniform finite length vortex sections (with a nonuniform vorticity distribution along the length) which simulates the interaction between localized vortices and vortex sheets, or of atmospheric fronts—was also considered in Refs. 19 and 20. In that case the possibility was noted in Ref. 20 for the existence of a stationary regime for a system of one nonuniform vortex section (which is an elliptical vortex degenerated in one of the two directions) and only two point vortices. At the same time for one PQV just one point vortex is sufficient for the realization of the stable stationary state (31), which definitely broadens the possibility for simulating real natural vortex interactions since two-particle interactions are significantly more probable and wide-spread than three-particle ones.

We note that the vortex sections in Refs. 19 and 20 can be the limiting case of nonuniform elliptical vortices which is the opposite to the one considered in the present paper (see Appendix I) since here we study the limit of weak ellipticity with $p=(a+b)/c\gg 1$, whereas in Refs. 19 and 20, in contrast, a finite vortex section with a nonuniform vorticity distribution along its length is obtained in the $p\to 1$ limit.

The motion in the field of the PQV proceeds for passive fluid particles with zero vortex charge, $\varkappa_2=0$, along the isolines of the stream function (2) (here we have $l\equiv r$, $\varkappa=\varkappa_1$, $\rho\equiv\rho_1=\text{const}$) and only two unstable stationary saddlepoint-type regimes (see Fig. 1) are admissible which are at a distance from the origin (i.e., from the PQV) equal to $l=(2|\rho_1/\varkappa_1|)^{1/2}\equiv 2^{1/2}l_*$, in accordance with (31).

The solution of the set (27) to (29), if we use the invariants (25) and (26) (for $\rho_2=0$), can be written in the form of quadratures

$$\int dl l\{Q^2-[h-\ln(l/\tilde{l}_c)]^2\}^{-1/2}=\frac{t(\varkappa_1+\varkappa_2)}{\pi}\frac{Q}{|Q|}+C_1,$$

$$Q=\frac{M}{2\tilde{\rho}_0\varkappa_1 l^2}-\frac{\varkappa_2}{2(\varkappa_1+\varkappa_2)},$$

while C_1 is an integration constant. Hence we get, in particular, for $l\to 0$ and $M\neq 0$

$$l=l_0\left[1+\frac{2t(\varkappa_1+\varkappa_2)M}{\pi\varkappa_1\tilde{\rho}_0 l_0^4}\right]^{1/4}. \tag{32}$$

Thus, after a finite time

$$\tilde{t}_0=\frac{\pi l_0^4}{2}\left|\frac{\tilde{\rho}_0\varkappa_1}{M(\varkappa_1+\varkappa_2)}\right|$$

merging of the point vortex and the PQV can occur for $(\varkappa_1+\varkappa_2)M/\varkappa_1<0$. In particular, for $\varkappa_1=\varkappa_2=\varkappa$ this merger occurs later than for the case of identical PQV since in that case we have

$$\frac{\tilde{t}_0}{t_0}=\frac{9}{4}\frac{|M|}{\tilde{\rho}_0|\varkappa|l_0^2}\gg 1$$

in the limit considered of small initial distances l_0 between the vortex objects.

As in the case of identical PQV, $l(t)$ can change for $M=0$ only within a finite range between $l_{\min}>0$ and $l_{\max}$ which is determined from the condition

$$Q^2-\left(h-\ln\frac{l}{\tilde{l}_c}\right)^2\geq 0$$

in the given quadrature. For $\varkappa_2\to 0$ the oscillation period has then the form [see (A20)]

$$T_0\approx\frac{\tilde{l}_c^2}{|\varkappa_1|}\exp\left[-\frac{2\pi T'}{\tilde{\rho}_0\varkappa_1\varkappa_2}\right],$$

where T' is the energy of the interaction for $\rho_2=0$ between the point vortex and the PQV from (25).

6. We now estimate the effect of dissipative factors on the PQV interaction process. Of course, when molecular viscosity is present even the consideration of point vortex objects itself is limited to the characteristic times over which they smear out to finite dimensions, $l_\nu\approx(\nu t)^{1/2}$. However, as long as the distance l between separate isolated PQV satisfies the condition $l\gg l_\nu$ the use of a vorticity distribution of the form (5) and the corresponding exact weak solution (A9) to (A11) [or (A12) to (A19)] of the Helmholtz equation (A7) for the vortex field in the presence of viscous dissipation is completely admissible. We note that the molecular viscosity coefficient ν occurs only in the equation for $\tilde{C}_\alpha$. In Ref. 13 the effect of the molecular viscosity was taken into account in fact in the same way as for our system (A9) to (A11).

In addition to taking $\nu\neq 0$ in (A7) we consider, in contrast to Ref. 13, also the case of Ekman (or Stokes) friction with a parameter γ which occurs not only in Eq. (A18) for $\tilde{C}_\alpha$, but also in (A14) and (A15) for ρ_α. To obtain solutions of (A9) to (A11) and (A12) to (A19), corresponding to the $\gamma\neq 0$ case (characteristic for real conditions of laboratory experiments and also for vortex processes occurring in the limits of the atmospheric boundary layer) it is sufficient to replace t in the solutions with $\gamma=0$ by $\tau=(1-e^{-\gamma t})/\gamma$ and add a factor $e^{-\gamma t}$ to $\tilde{C}_\alpha$, $\varkappa_\alpha$, and ρ_α. For instance, for the solution (32) this leads to the fact that the point vortex and the PQV for $(\varkappa_1+\varkappa_2)M/\varkappa_1<0$ do not collapse into a single point with $l=0$ even as $t\to\infty$, but only approach to a finite distance

$$L_{min}=l_0\left(1-\frac{2|\varkappa_1+\varkappa_2||M|}{\pi|\varkappa_1|\tilde{\rho}_0 l_0^4\gamma}\right)^{1/4}>0$$

for sufficiently large γ, i.e., for

$$\gamma > \frac{2|\varkappa_1+\varkappa_2||M|}{\pi|\varkappa_1|\tilde{\rho}_0 l_0^4}.$$

The action of the molecular viscosity can then be neglected, provided $L_{\min} \gg l_\nu \approx (\nu/\gamma)^{1/2}$. Thus, for sufficiently large Reynolds numbers,

$$\mathrm{Re} = \frac{L_{min}^2\gamma}{\nu} \gg 1$$

the viscous dissipation can only have a small effect on the approach process of PQV type localized vortices.

In conclusion we stress once again that the system (A9) to (A11) studied above is a regular finite-dimensional dynamical system which describes the interaction between singular vortices corresponding to infinitesimal small nonuniform vortices and the exact weak solutions of the hydrodynamical equations corresponding to them. A further study of this system for $N \geqslant 2$ and a more detailed analysis of the effect of molecular viscosity is of interest in connection with the well known problems of the spectral transformation of the energy and the entrophy in the theory of two-dimensional geophysical and MHD turbulence. Moreover, in three-dimensional hydrodynamics where there are definite difficulties in correctly introducing the dynamics of monopole-type point vortices[21] one can also consider the dynamics of quadrupole point vortices, but now with a monopole component of zero strength. Like the dynamical systems introduced in Refs. 14 and 16 for dipole point vortices the system (A9) to (A11) is adapted for a development of gridless algorithms for a numerical simulation of complex turbulent processes in hydro- and aerodynamics.

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APPENDIX I

The vorticity distribution $\tilde{\omega}(\mathbf{x})$ in a plane corresponds to a stream function Ψ satisfying the differential equation[15] $\Delta\Psi = -\tilde{\omega}$ where Δ is the two-dimensional Laplace operator. We shall consider as $\tilde{\omega}$ the distribution (1) in which we have $\tilde{\delta}(\xi)$ from (4), i.e., the function describing a special "smear" (the regularization of Ref. 14) of the δ-function in a finite elliptical region with semiaxes a and b $(a>b)$, instead of the δ-function. Let us have $\Psi = \Psi_{\text{in}}$ inside the ellipse and $\Psi = \Psi_{\text{ex}}$ in the outside region, i.e.,

$$\Delta\Psi_{ex}=0, \qquad \Delta\Psi_{in}=-\varkappa\tilde{\delta}-a_{22}\Delta\tilde{\delta}+\rho\frac{\partial^2\tilde{\delta}}{\partial x^2}.$$

In (4) we have $\tilde{\delta}(\xi) = \delta_0(\xi)\theta(p-e^\xi)$ where

$$p=\frac{a+b}{c}, \qquad c^2=a^2-b^2, \qquad \delta_0(\xi)=a_0\,\mathrm{sh}^2\,\xi\,(p-e^\xi)^2,$$

$$a_0=\frac{8}{\pi c^2}\left[\frac{p^6}{60}-\frac{p^4}{6}+p^2(2\ln p-3)\right.$$
$$\left.+\frac{32}{5}p-2\ln p-\frac{19}{6}-\frac{1}{p^2}\right].$$

Since $\delta_0(\xi) = d\delta_0(\xi)/d\xi = 0$ on the boundary of the ellipse for $\xi = \ln p$ we have for Ψ_{in} an equation in elliptical coordinates:

$$\Delta\Psi_{in}=\frac{1}{c^2(\mathrm{sh}^2\,\xi+\sin^2\varphi)}\left(\frac{\partial^2\Psi_{in}}{\partial\xi^2}+\frac{\partial^2\Psi_{in}}{\partial\varphi^2}\right)$$
$$=-\varkappa\delta_0(\xi)-a_{22}\Delta\delta_0+\rho\left\{\frac{d^2\delta_0}{d\xi^2}\frac{\cos^2\varphi\,\mathrm{sh}^2\,\xi}{c^2(\mathrm{sh}^2\,\xi+\sin^2\varphi)^2}\right.$$
$$\left.+\frac{d\delta_0}{d\xi}\frac{\mathrm{sh}\,\xi\,\mathrm{ch}\,\xi[(2+\cos 2\varphi)\sin^2\varphi-\mathrm{sh}^2\,\xi\cos 2\varphi]}{c^2(\mathrm{sh}^2\,\xi+\sin^2\varphi)^3}\right\}. \tag{A1}$$

In what follows we shall consider solely the case of a weakly elliptical localization boundary of the vorticity when we have $p \gg 1$, $a_0 \approx 480/\pi c^2 p^6$ and Eq. (A1) near the boundary of the elliptical region, i.e., for $\xi \gg 1$, takes the form

$$\frac{\partial^2\tilde{\Psi}_{in}}{\partial\xi^2}+\frac{\partial^2\tilde{\Psi}_{in}}{\partial\varphi^2}$$
$$=-\varkappa\frac{\delta_0(\xi)}{4}c^2e^{2\xi}+\frac{\rho}{2}\left(\frac{d^2\delta_0}{d\xi^2}-2\frac{d\delta_0}{d\xi}\right)\cos 2\varphi, \tag{A2}$$

where $\Psi_{\text{in}} = -a_{22}\delta_0 + \rho\delta_0/2 + \tilde{\Psi}_{\text{in}}$. For $p \gg 1$ the stream function (2) expressed in elliptical coordinates satisfies the equation $\Delta\Psi_{\text{ex}} = 0$ and has the form

$$\Psi_{ex}=-\frac{1}{2\pi}\left(\varkappa\xi+\frac{4e^{-2\xi}}{c^2}\rho\cos 2\varphi\right), \tag{A3}$$

which differs only in sign from the external stream function given in Ref. 17, Ch. IV of an elliptical cylinder rotating with a frequency n when there is a nonzero circulation, $\varkappa \neq 0$, and under condition (4).

We look for the solution of Eq. (A2) in the form $\tilde{\Psi}_{\text{in}} = \tilde{\Psi}_1(\xi) + \tilde{\Psi}_2(\xi)\cos 2\varphi$. Substitution of this expression into (A2) and integration of the equations

$$\frac{d^2\tilde{\Psi}_1}{d\xi^2}=-\frac{30\varkappa e^{4\xi}(p-e^\xi)^2}{\pi p^6}$$

and

$$\frac{d^2\tilde{\Psi}_2}{d\xi^2}-4\tilde{\Psi}_2=\frac{\rho}{2}\frac{d}{d\xi}\left(\frac{d\delta_0}{d\xi}-2\delta_0\right)=\frac{120\rho e^{3\xi}}{\pi c^2p^6}(-3p+4e^\xi)$$

for $\xi \gg 1$, $p \gg 1$ gives for Ψ_{in} the expression

$$\Psi_{in}=\left(\frac{\rho}{2}-a_{22}\right)\frac{e^{4\xi}}{4}(p-e^\xi)^2+C_2+\xi C_1$$
$$-\frac{30\varkappa e^{4\xi}}{\pi p^6}\left(\frac{p^2}{16}-\frac{2pe^\xi}{25}+\frac{e^{2\xi}}{36}\right)$$
$$+\left[C_3e^{-2\xi}+C_4e^{2\xi}+\frac{60\rho e^{3\xi}}{\pi c^2p^6}\left(\frac{e^\xi}{3}-\frac{3p}{5}\right)\right]\cos 2\varphi, \tag{A4}$$

where C_1 and C_2 are integration constants of the equation for $\tilde{\Psi}_1$ and C_3 and C_4 integration constants of the equation for $\tilde{\Psi}_2$. Therefore, (A4) describes the stream function inside the ellipse $\xi \leqslant \ln p$ only near its boundary where we have $\xi \gg 1$ (in the limit of weak ellipticity, $p \gg 1$) while (A2) describes the stream function outside the ellipse, $\xi \geqslant \ln p$. The matching condition at the boundary of the ellipse F (when we have

$\xi = \ln p$) of the functions Ψ_{in} and Ψ_{out} from (A4) and (A2) (i.e., $\Psi_{in}|_F = \Psi_{out}|_F$) and their derivatives

$$\frac{\partial\Psi_{ex}}{\partial\varphi}\bigg|_F = \frac{\partial\Psi_{in}}{\partial\varphi}\bigg|_F, \quad \frac{\partial\Psi_{ex}}{\partial\xi}\bigg|_F = \frac{\partial\Psi_{in}}{\partial\xi}\bigg|_F$$

determine the integration constants C_1, C_2, C_3, and C_4 uniquely; they have the form

$$C_1=0, \quad C_2=\frac{\varkappa}{2\pi}\left(\frac{37}{60}-\ln p\right),$$

$$C_3=-\frac{\rho}{\pi c^2}, \quad C_4=\frac{15\rho}{\pi p^2(a+b)^2}. \tag{A5}$$

From the condition for the time-independence of the solution (A4) for $\xi\to\ln p$ we have at the boundary the condition $\Delta\Psi_{in}|_F = -\tilde{k}\Psi_{in}|_F$ from which it follows that

$$\tilde{k}=\frac{240}{(a+b)^2}, \text{ a } a_{22}=\frac{\rho}{2}-\frac{\varkappa}{8}(a+b)^2\ln p.$$

Moreover, from the condition for the time-independence of the rigid rotation of the boundary of the elliptical region,

$$F=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

with frequency n, i.e., from the equation

$$\frac{\dot{x}x}{a^2}+\frac{\dot{y}y}{b^2}\bigg|_F=-\frac{nxy}{a^2}+\frac{nxy}{b^2}\bigg|_F$$

on the boundary F it follows that we also must satisfy the equation

$$\frac{\partial\Psi_{ex}}{\partial\varphi}\bigg|_F=\frac{nc^2}{2}\sin 2\varphi. \tag{A6}$$

In turn, it follows necessarily from (A6) that we have the interrelation between ρ and n in the form (3), mentioned earlier. In contrast to a Kirchhoff vortex with uniform vorticity now neither n nor ρ depends on the magnitude of the circulation $\varkappa$. We note also that the regularization of the term $-a_{22}\delta(\mathbf{x})\to\Psi_c$ in (2) in the form of the substitution $\delta\to\tilde{\delta}$ with $\tilde{\delta}$ from (4) gives only a contribution to the azimuthal velocity

$$\dot{\varphi}=-\frac{1}{c^2(\operatorname{sh}^2\xi+\sin^2\varphi)}\frac{\partial\Psi_c}{\partial\xi}$$

which at the boundary of the ellipse, $\xi=\ln p$, vanishes anyway while

$$\dot{\xi}=\frac{1}{c^2(\operatorname{sh}^2\xi+\sin^2\varphi)}\frac{\partial\Psi_c}{\partial\varphi}=0$$

for $\tilde{\delta}$ from (4). The self-induced rotation of the interior structure of a nonuniform elliptical vortex therefore does not affect the interaction of several vortices in the small size limit when they correspond to PQV.

APPENDIX II

1. The Helmholtz equation for the two-dimensional vortex field ω of an incompressible fluid has in the presence of multiple dissipative factors the form

$$\frac{\partial\omega}{\partial t}+u_l\frac{\partial\omega}{\partial x_l}=\nu\frac{\partial^2\omega}{\partial x_l^2}-\gamma\omega, \tag{A7}$$

where ν is the molecular viscosity coefficient, γ the Ekman or Stokes friction coefficient, and $\mathbf{u}$ the velocity field which corresponds to the vorticity distribution ω, i.e., we have $u_i=\varepsilon_{ij}(\partial\Psi/\partial x_j)$ for Ψ from (6), if ω satisfies (5).

We substitute the distribution ω from (5) into (A7), multiply the left- and right-hand sides of (A7) by an arbitrary, sufficiently smooth weight function φ, and integrate over the whole of the plane using the properties of the δ-function. We then have

$$\sum_{\alpha=1}^{N}\int d^2x\,\delta(\mathbf{x}-\mathbf{x}_\alpha)\Big\{\varphi(\dot{\varkappa}_\alpha+\gamma\varkappa_\alpha)$$

$$+\frac{\partial\varphi}{\partial x_l}\left[\varkappa_\alpha\dot{x}_l^\alpha-\varkappa_\alpha u_l(\mathbf{x}^\alpha)-a_{ij}^\alpha\frac{\partial^2 u_l}{\partial x_i\partial x_j}\right]$$

$$+\frac{\partial^2\varphi}{\partial x_i\partial x_j}\left(\dot{a}_{ij}^\alpha-a_{lj}^\alpha\frac{\partial u_i}{\partial x_l}-a_{il}^\alpha\frac{\partial u_j}{\partial x_l}+\gamma a_{ij}^\alpha-\varkappa_\alpha\nu\delta_{ij}\right)$$

$$+\frac{\partial^3\varphi}{\partial x_i\partial x_j\partial x_l}[a_{ij}^\alpha\dot{x}_l^\alpha$$

$$-a_{ij}^\alpha u_l(\mathbf{x}^\alpha)]+\frac{\partial^4\varphi}{\partial x_i\partial x_j\partial x_l\partial x_p}(-\nu a_{ij}\delta_{lp})\Big\}=0. \tag{A8}$$

It follows from (A8) that only for a function φ which is quadratic in $\mathbf{x}$ can one self-consistently and non-trivially satisfy this equation with the coefficients of φ, $\partial\varphi/\partial x_l$, and $\partial^2\varphi/\partial x_i\partial x_j$ vanishing; this leads to the equations

$$\dot{\varkappa}_\alpha=-\gamma\varkappa_\alpha, \tag{A9}$$

$$\varkappa_\alpha\dot{x}_l^\alpha=\varkappa_\alpha u_l(\mathbf{x}^\alpha)+a_{ij}^\alpha\frac{\partial^2 u_l(\mathbf{x}^\alpha)}{\partial x_i^\alpha\partial x_j^\alpha}, \tag{A10}$$

$$\dot{a}_{ij}^\alpha=a_{lj}^\alpha\frac{\partial u_i(\mathbf{x}^\alpha)}{\partial x_l^\alpha}+a_{il}^\alpha\frac{\partial u_j(\mathbf{x}^\alpha)}{\partial x_l^\alpha}-\gamma a_{ij}^\alpha+\varkappa_\alpha\nu\delta_{ij}, \tag{A11}$$

where in the expression for $u_i(\mathbf{x}^\alpha)$ in terms of Ψ from (6) in the sum over β we must drop the term with $\alpha=\beta$, i.e., exclude the singular self-interaction.

The system (A9) to (A11) is an exact weak solution of (A7) for the distribution ω from (5)—a solution in the narrow sense, in contrast to the weak solutions given in Ref. 14 for point dipole vortices obtained for arbitrary, and not just quadratic, weight functions.

2. For the $N=2$ case we have from (A9) to (A11), using the notation for the variables introduced in the main text,

$$\dot{l}=\frac{(\varkappa_1+\varkappa_2)}{\pi\varkappa_1\varkappa_2 l^3}\Big[\varkappa_1\rho_2\sin(2\varphi+\Psi_2)+\varkappa_2\rho_1\sin(2\varphi+\Psi_1)$$

$$-\frac{12\rho_1\rho_2}{l^2}\sin(4\varphi+\Psi_1+\Psi_2)\Big], \tag{A12}$$

$$\dot{\varphi}=\frac{(\varkappa_1+\varkappa_2)}{2\pi l^2}\Big\{1+\frac{2}{l^2}\Big[-\frac{\rho_2}{\varkappa_2}\cos(2\varphi+\Psi_2)-\frac{\rho_1}{\varkappa_1}\cos(2\varphi+\Psi_1)$$

$$+\frac{12\rho_1\rho_2}{\varkappa_1\varkappa_2 l^2}\cos(4\varphi+\Psi_1+\Psi_2)\Big]\Big\}, \tag{A13}$$

$$\dot\rho_1=-\gamma\rho_1+\frac{\varkappa_2}{\pi l^2}(\bar C_1^2+\rho_1^2)^{1/2}\Big[-\sin(2\varphi+\Psi_1)$$

$$+\frac{6\rho_2}{\varkappa_2 l^2}\sin(4\varphi+\Psi_1+\Psi_2)\Big], \tag{A14}$$

$$\dot\rho_2=-\gamma\rho_2+\frac{\varkappa_1}{\pi l^2}(\bar C_2^2+\rho_2^2)^{1/2}\Big[-\sin(2\varphi+\Psi_2)$$

$$+\frac{6\rho_1}{\varkappa_1 l^2}\sin(4\varphi+\Psi_1+\Psi_2)\Big], \tag{A15}$$

$$\rho_1\dot\Psi_1=\frac{\varkappa_2}{\pi l^2}(\bar C_1^2+\rho_1^2)^{1/2}\Big[-\cos(2\varphi+\Psi_1)$$

$$+\frac{6\rho_2}{\varkappa_2 l^2}\cos(4\varphi+\Psi_1+\Psi_2)\Big]. \tag{A16}$$

$$\rho_2\dot\Psi_2=\frac{\varkappa_1}{\pi l^2}(\bar C_2^2+\rho_2^2)^{1/2}\Big[-\cos(2\varphi+\Psi_2)$$

$$+\frac{6\rho_1}{\varkappa_1 l^2}\cos(4\varphi+\Psi_1+\Psi_2)\Big], \tag{A17}$$

$$\bar C_\alpha\dot{\bar C}_\alpha=-\gamma\bar C_\alpha^2+2\nu(\bar C_\alpha^2+\rho_\alpha^2)^{1/2}\varkappa_\alpha,\quad \alpha=1,\ 2, \tag{A18}$$

$$\dot\varkappa_\alpha=-\gamma\varkappa_\alpha,\quad \alpha=1,\ 2. \tag{A19}$$

3. To determine the oscillation period T_0 in the point vortex-PQV system for

$$h\equiv-\frac{\pi T'}{\bar\rho_0\varkappa_1\varkappa_2}\quad\text{and}\quad\frac{1}{2}\left|\frac{\varkappa_2}{\varkappa_1+\varkappa_2}\right|>h>0$$

we have for $M=0$ ($Q=-\varkappa_2/2(\varkappa_1+\varkappa_2)$) for the limits of the range of values of l:

$$l_{max}=\bar l_c\exp\left(h+\frac{1}{2}\left|\frac{\varkappa_2}{\varkappa_1+\varkappa_2}\right|\right),$$

$$l_{min}=\bar l_c\exp\left(h-\frac{1}{2}\left|\frac{\varkappa_2}{\varkappa_1+\varkappa_2}\right|\right).$$

For $l=l_{\min}$ and $l=l_{\max}$ the expression $Q^2-[h-\ln(l/\tilde l_c)]^2$ under the radical sign which occurs in the quadrature vanishes. Integration over l between these limits determines T_0 in the form

$$T_0=\frac{\pi Q}{|Q|(\varkappa_1+\varkappa_2)}\int_{l_{min}}^{l_{max}}dl\frac{l}{\{Q^2-[h-\ln(l/\bar l_c)]^2\}^{1/2}}$$

$$=\frac{\pi^2\bar l_c^2|\varkappa_2|e^{2h}}{\varkappa_2|\varkappa_1+\varkappa_2|}I_0\left(\left|\frac{\varkappa_2}{\varkappa_1+\varkappa_2}\right|\right), \tag{A20}$$

where I_0 is a zeroth-order modified Bessel function.

[1)] The possibility of two identical point dipole vortices merging was noted in Ref. 16 also for the three-dimensional case, where each of them may correspond to an infinitesimally small vortex ring or spherical Hill vortex.

[2)] In Fig. 3 for $\bar x=(l/\tilde l_c)\cos y_1$, $\bar y=(l/\tilde l_c)\sin y_1$, $M/\varkappa_1>0$, and $\varkappa_2(\varkappa_1+\varkappa_2)=2$ we have drawn the isolevel lines of the energy invariant, $h\equiv-\pi T'/\rho_0\varkappa_1\varkappa_2$, for the system (27) to (29) and $\tilde l_c$ from (31).

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