Nonlinear model of autophasing of classical oscillators

Yu. A. Kobelev, L. A. Ostrovskii, and I. A. Soustova

Institute of Applied Physics, USSR Academy of Sciences
(Submitted 22 August 1990)
Zh. Eksp. Teor. Fiz. **99**, 470–480 (February 1991)

We investigate analytically the process of autophasing of classical oscillators coupled by a radiation field in its nonlinear stage. The equations of motion (coupled Duffing equations) are first written in terms of amplitudes of oscillation for the individual oscillators; rewriting these equations in terms of collective variables (i.e., modes) that characterize the phase distribution of the oscillators allows us to decrease the number of these equations. When the nonlinear processes are ineffective compared to the radiative processes, we also can obtain the time dependence of the amplitude of the radiated wave. A model from acoustics is proposed that exhibits this autophasing effect.

An interesting example of a "transition from chaos to order" is the autophasing of nonlinear oscillators that interact with each other through a self-generated radiation field. An example of this process is "Dicke superradiance",[1] in which initially excited atoms come into phase with one another in the process of radiating and emit short and powerful electromagnetic wave trains. Another example is the radiation of spatially localized ensembles of electrons moving in a magnetic field, a phenomenon investigated in Refs. 2 and 3. Using a semiclassical description of the autophasing, in which the individual oscillators obey quantum (usually two-level) equations while their ensemble obeys a balance equation, it is possible to carry the analysis of this phenomenon quite far, up to the point where the nonlinear stages must be dealt with.[1,4]

The investigation of autophasing of classical oscillators is a topic of interest in its own right. Under this heading we note the "classical maser" effect[5] and its acoustic analogue discussed in Ref. 6. However, despite the almost 30-year history of this question, only the linear stage of the autophasing process has been successfully treated analytically, while the process of evolution of the radiation pulse and its parameters have been studied only numerically.[7,8]

In this paper we will attempt to describe the nonlinear stage of the autophasing of classical oscillators and propose a model which exhibits mechanical "superradiance".

1. THEORETICAL MODEL

Without specifying the physical nature of the system, let us consider an ensemble of N oscillators with cubic nonlinearities interacting with each other through a field $\mathscr{P}$ that is proportional to the sum of their velocities:

$$\ddot{x}_k+\delta\dot{x}_k+x_k(1+\alpha x_k^2)=-\mathscr{P}, \tag{1}$$

$$\mathscr{P}=\varkappa\sum_{k=0}^{N-1}\dot{x}_k; \tag{2}$$

here δ is the attenuation coefficient, α is the nonlinearity coefficient, and $\varkappa$ is the coupling coefficient of the oscillators. Multiplying (1) by $\dot{x}_k$, integrating with respect to time, and summing over k, we obtain an equation for the time variation of the total energy E of the oscillators:

$$\dot{E}=-\delta\sum_{k=0}^{N-1}\dot{x}_k^2-\mathscr{P}^2/\varkappa, \tag{3}$$

$$E=\sum_{k=0}^{N-1}\varepsilon_k,\qquad \varepsilon_k=(\dot{x}_k^2+x_k^2+\alpha x_k^4/2)/2. \tag{4}$$

It is clear from (3) that the field $\mathscr{P}$ (which we will refer to as "macroscopic") supplies an additional mechanism for the dissipation of energy.

Let us further clarify what we mean by the effect of oscillator autophasing. We first set $\alpha=0$, so that Eq. (1) becomes linear, and assume that at time $\tau=0$ all of the oscillators are excited in such a way that $\mathscr{P}(\tau=0)=0$; then by virtue of the linearity of Eq. (1) and the fact that all the oscillators are identical, we have $\mathscr{P}\equiv 0$ for all $\tau>0$ as well. In this case the energy of the system decreases according to an exponential law with damping coefficient δ. If at $\tau=0$ all the oscillators are excited with identical phases and amplitudes, it follows from (1) and (3) that the energy will again decrease exponentially now, however, the damping coefficient is $\delta+N\varkappa$. Usually the condition $N\varkappa\gg\delta$ is satisfied. For $\alpha\neq 0$ the phenomenon of autophasing becomes possible, in which oscillators initially excited with random phases at $\tau=0$ ($\mathscr{P}(\tau=0)=0$) come into phase with one another during the oscillation process, and then rapidly "burn out" their energy with a damping coefficient close to $N\varkappa$. In discussing processes of this kind we will assume the nonlinearity is small, i.e., $\alpha x_k^2\ll 1$. This allows us to write the variables x_k and $\mathscr{P}$ in the form

$$x_k=(Y_k e^{i\tau}+\text{c.c.})/2,\quad \mathscr{P}=i(Ze^{i\tau}+\text{c.c.})/2, \tag{5}$$

where Y_k is the complex amplitude of oscillation of the k th oscillator. By virtue of the additional condition

$$\dot{x}_k=i(Y_k e^{i\tau}-\text{c.c.})/2 \tag{6}$$

we obtain from (1) and (2) an equation for Y_k and Z:

$$\dot{Y}_k+(\delta/2)Y_k-ia|Y_k|^2Y_k=-Z/2, \tag{7}$$

$$Z=\varkappa\sum_{k=0}^{N-1}Y_k, \tag{8}$$

where we have introduced a new nonlinearity coefficient $a=3\alpha/8$. In what follows we will attempt to reduce the system (7) to a certain set of equations that in the simplest cases admits analytic solutions. To do this we first change from the variables Y_k to new variables (modes) according to the formula

 0038-5646/91/020262-06$03.00

$$Y_k=\sum_{l=0}^{N-1} A_l \exp(i2\pi kl/N). \tag{9}$$

Substituting (9) into (7), we then multiply the right and left sides of Eq. (7) by $\exp(-i2\pi lk/N)$ and sum over all k, after which we can make use of the equation[9]

$$\sum_{k=0}^{N-1}\exp(i2\pi km/N)=\begin{cases} N & \text{for } m=0;\ \pm N;\ \pm 2N\ldots, \\ 0 & \text{for } m\neq 0;\ \pm N;\ \pm 2N\ldots, \end{cases} \tag{10}$$

where m is an integer, to obtain the following equations for A_n:

$$A_n+\frac{1}{2}[\delta+N\varkappa\delta(n)]A_n$$
$$=ia\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}A_l A_m(A^*_{l+m-n}+A^*_{l+m-n\pm N}), \tag{11}$$

where $\delta(n)$ is the Kronecker symbol. The subscripts j attached to the quantities A_j on the right side of Eq. (11) lie between 0 and $N-1$; for j outside this range we have $A_j=0$. For the amplitude of the macroscopic field Z we have from Eqs. (8) and (9)

$$Z=\varkappa NA_0. \tag{12}$$

In an analogous fashion we obtain an expression for the total energy of the system:

$$E=\frac{1}{2}\sum_{k=0}^{N-1}|Y_k|^2=\frac{1}{2}N\sum_{l=0}^{N-1}|A_l|^2. \tag{13}$$

Let us perform still another change of variables that will allow us to sum a portion of the terms in (11). We transform from A_n to b_n according to the expression

$$A_n=b_n\exp\left[(4ia/N)\int_0^\tau E\,d\tau'\right]. \tag{14}$$

Then from (11) and (14) we obtain the following equation for b_n

$$\dot{b}_n+\frac{1}{2}[\delta+N\varkappa\delta(n)+ia|b_n|^2]b_n$$
$$=ia\Bigg[\sum_{l=0,l\neq n}^{N-1} b_l^2(b^*_{2l-n}+b^*_{2l-n\pm N})$$
$$+2\sum_{l=1}^{N-1}\sum_{m=0}^{l-1}b_l b_m(b^*_{l+m-n}+b^*_{l+m-n\pm N})\Bigg]. \tag{15}$$

Of course the transformation to Eqs. (15) in itself does not simplify the problem; however, an approximation is now feasible that will leave us with only three equations We will demonstrate this first for the linearized problem.

2. LINEAR THEORY

Let us first discuss the problem of the evolution of small perturbations superposed on a single given mode $b_{\bar{n}}$, $\bar{n}\neq 0$, with $Z(\tau=0)=0$. We will linearize Eqs. (15) with respect to all the variables b_j with $j\neq\bar{n}$, which we assume are small compared to b_n, and neglect the right-hand side in writing the equation for the "pump" $b_{\bar{n}}$. Then it is not difficult to see that for any fixed n we can separate out two equations

$$\dot{b}_n+\frac{1}{2}[\delta+N\varkappa\delta(n)]b_n=iab_{\bar{n}}^2 b_m^*,$$
$$\dot{b}_m^*+\frac{1}{2}[\delta+N\varkappa\delta(m)]b_m^*=-ia(b_{\bar{n}}^*)^2 b_n, \tag{16}$$

while b_n satisfies the equation

$$\dot{b}_{\bar{n}}+\left(\frac{1}{2}\delta+ia|b_{\bar{n}}|^2\right)b_{\bar{n}}=0. \tag{17}$$

In Eq. (16) m takes on three values: $m=2\bar{n}-n$ or $m=2\bar{n}-n\pm N$ (the sign is chosen to ensure the condition $m\geqslant 0$); recall that m lies in the range 0 to $N-1$. It is clear from (16) that in the linear approximation the modes are coupled in pairs by the given pump; in this case the behavior of all the pairs is the same, except for the cases $n=0$ and $m=0$, in which the damping coefficient becomes equal to $\delta+N\varkappa$. The autophasing process is in fact associated with this specific pair of modes: the additional damping is due to the presence of coupling in Eq. (12) between the macroscopic field Z and the amplitude of the mode b_0.

We first write the solution to Eq. (17) for the pump mode in the form of a nonlinearly decaying amplitude with the initial condition $b_{\bar{n}}(\tau=0)=b_{\bar{n}0}$:

$$b_{\bar{n}}=b_{\bar{n}0}\exp[-\delta\tau/2+i\eta(1-e^{-\delta\tau})/\delta], \tag{18}$$

where $\eta=a|b_{\bar{n}0}|^2$. We seek the solution to Eq. (16) in the form

$$b_n=b_{n0}\exp[(\lambda_n-2i\eta)\tau],\quad b_m^*=b_{m0}\exp(\lambda_n\ \tau), \tag{19}$$

where the real part λ_n determines whether the corresponding pair of modes grow ($\mathrm{Re}\lambda_n>0$) or decay ($\mathrm{Re}\lambda_n<0$). We will further neglect the damping of the pump for the moment, setting $\delta=0$ in (19). Then, when we substitute (19) into (16) we obtain the following equation for λ_n

$$(\lambda_n+\delta/2)\{\lambda_n-2i\eta+[\delta+\delta(n)N\varkappa]/2\}=\eta^2, \tag{20}$$

from which we find the quantities λ_n and λ_0:

$$\lambda_n=-\delta/2+i\eta, \tag{21}$$

$$\lambda_0=\frac{1}{2}\left\{-\delta-\frac{N\varkappa}{2}+\frac{N\varkappa}{2^{3/2}}\left[1+\left(1+\frac{64\eta^2}{N^2\varkappa^2}\right)^{1/2}\right]^{1/2}\right\}$$
$$+i\eta\left\{1-2^{1/2}\left[1+\left(1+\frac{64\eta^2}{N^2\varkappa^2}\right)^{1/2}\right]^{-1/2}\right\}. \tag{22}$$

Apparently the expression (22) for $\mathrm{Re}\lambda_0$ was obtained directly from Eqs. (7) and (8) for the first time in Ref. 7; see also the later Ref. 8. It follows from (21) that all pairs of modes with $n>0$ decay, while the only pair that can grow is the pair that contains the zero mode and a mode with index m equal to $2\bar{n}$ if $2\bar{n}\leqslant N-1$ or $2\bar{n}-N$ if $2\bar{n}>N-1$. This result is understandable in light of what was said above: an instability is possible only when a coherent component is excited that is coupled to the zero mode. Thus, for the case where the first mode is pumped ($\bar{n}=1$) the unstable pair will be a combination of the zero and second mode. As a function of $N\varkappa$ the magnitude of the growth rate $\mathrm{Re}\lambda_0$ has a maximum

$$(\mathrm{Re}\,\lambda_0)_{max}\approx 0{,}3|\eta|-\delta/2, \tag{23}$$

which occurs for

$$N\varkappa\approx 2|\eta|. \tag{24}$$

This reflects the participation of the macroscopic field in two processes: autophasing and the dissipation of energy. For small $N\varkappa$ the coupling between the oscillators decreases, while for large $N\varkappa$ the energy dissipation increases. Equation (23) defines a condition on the quantity η that ensures amplification: $|\eta| > 1.7\delta$. However, for the case of arbitrary $N\varkappa$, according to (22) the analogous condition has the form

$$|\eta|>{}^1/_2\{\delta(\delta+N\varkappa)[1+4\delta(\delta+N\varkappa)/(N\varkappa)^2]\}^{1/2}. \tag{25}$$

Let us note one specific limiting case. One possible choice among the various values the indices m and n can take in (16) is $m = n = 0$ (e.g., $N = 10$, while $\bar{n} = 5$); after substituting (19) into (16) we obtain for this choice

$$\lambda_0=-(\delta+N\varkappa)/2+i\eta, \tag{26}$$

i.e., without the participation of any other mode the zero mode does not grow. In the presence of instability the growth of a pair of modes must be limited either by the damping of the pump mode or by the nonlinearity. Let us first investigate the first factor within the framework of the linear system (16) and (17). It is possible to find an analytic solution for this case under the additional condition

$$|\dot{b}_0|\ll N\varkappa|b_0|; \tag{27}$$

then for the magnitude of b_0 we obtain from (16) the equation

$$|b_0|=|b_{00}|\exp[-3\delta\tau/2+\mu^2N\varkappa(1-e^{-2\delta\tau})/4\delta], \tag{28}$$

where $b_{00} = b_{2\bar{n}0}$ is the initial value of b_0 (within the approximation (27) it is determined only by the initial value of the index of the mode $2\bar{n}$); $\mu = 2\eta/N\varkappa$ is a coefficient that characterizes the degree of influence of the nonlinearity in comparison with the collective losses. In (28) we also assume δ small in comparison to $N\varkappa$. The following condition for the validity of (27) follows from (28):

$$\mu^2\ll 1, \tag{29}$$

i.e., we are discussing the case of a nonlinearity that is relatively small in comparison with the collective losses. According to (28), growth in $|b_0|$ is possible for $\mu^2N\varkappa > \delta$, whereas (22) gives the condition $\mu^2N\varkappa < \delta$ for small μ. From this we see that the damping of the pump in the limiting case $\mu^2 \ll 1$ increases the threshold for the instability with respect to intensity by a factor of three. Note that for

$$\tau=-(1/2\delta)\ln(3\delta/\mu^2N\varkappa)$$

the quantity $|b_0|$ is a maximum:

$$|b_0|_{max}=|b_{00}|(3\delta/N\varkappa\mu^2)^{3/4}\exp\left[\frac{3}{4}\left(\frac{N\varkappa\mu^2}{3\delta}-1\right)\right]. \tag{30}$$

This implies that when δ is small the quantity $|b_0|$ may be large enough to cause the linear theory to be inapplicable even when condition (29) is satisfied.

3. NONLINEAR THEORY

Thus, it follows from the linear theory that when the initial excitation involves only a single mode it is possible to separate all the modes into two groups: one consisting of a pair of modes with exponentially growing amplitudes and the other including all the remaining modes, which are exponentially decaying. If we normalize the amplitudes of the modes by the amplitude of the pump at $\tau = 0$, i.e., introduce new variables according to the formula

$$b_j/|b_{\bar{n}0}|=C_j, \tag{31}$$

it is not hard to see that the transition from the linear to the nonlinear approximation is signaled by the appearance of terms of the form $C_{\bar{n}}C_{2\bar{n}}C_0^*$ on the right sides of (16) and (17); however, these terms do not play a significant role until $C_{2\bar{n}}$ and C_0 have increased to values of order unity. This makes it possible to use the three-mode model in the nonlinear regime as well for times prior to this stage. In what follows we will complement these general considerations with an attempt to compare the results of our model with numerical calculations from Ref. 8.

Let us choose the mode with $\bar{n} = 1$ as the pump mode and pass to the variable $\bar{\tau} = N\varkappa\tau$; doing this, we obtain from (15) the following system of equations for C_j:

$$\begin{aligned}
&C_0'+\frac{1}{2}C_0+\frac{i\mu}{2}|C_0|^2C_0=i\frac{\mu}{2}C_1^2C_2^*,\\
&C_1'+\frac{\delta}{2N\varkappa}C_1+\frac{i\mu}{2}|C_1|^2C_1=i\mu C_1^*C_0C_2,\\
&C_2'-\frac{\delta}{2N\varkappa}C_2+\frac{i\mu}{2}|C_2|^2C_2=\frac{i\mu}{2}C_1^2C_0^*,
\end{aligned} \tag{32}$$

where $C_j' = dC_j/d\bar{\tau}$. Here we have also assumed $N > 3$, in order to exclude additional terms from the right-hand sides of the equations that are due to terms in (15) that contain modes with subscripts $2l–n \pm N$ and $l + m \pm N$. With an eye to further simplification of Eq. (32), let us pass from the complex variables C_j to real variables, i.e., the intensities J_j and phases ψ_j, according to the formula $C_j = J_j^{1/2}\exp(i\psi_j)$, after which we obtain

$$\psi'=\mu\left\{-J_1+\frac{1}{2}(J_2+J_0)+2\left[(J_0J_2)^{1/2}-\frac{1}{4}J_1\left(\left(\frac{J_2}{J_0}\right)^{1/2}+\left(\frac{J_0}{J_2}\right)^{1/2}\right)\cos\psi\right]\right\}, \tag{33}$$

$$\frac{d}{d\bar{\tau}}(J_0-J_2)+\frac{\delta}{N\varkappa}(J_0-J_2)=-J_0, \tag{34}$$

$$J_0'+J_0=-\mu J_1(J_0J_2)^{1/2}\sin\psi, \tag{35}$$

$$J_1=(1+2J_0)\exp(-\delta\bar{\tau}/N\varkappa)-2J_2, \tag{36}$$

where $\psi = 2\psi_1 - \psi_2 - \psi_0$, and $J_{20} = J_2(\bar{\tau} = 0)$.

Neglecting the intrinsic attenuation ($\delta = 0$) and the small "seed" J_{20} ($J_{20} \ll 1$), let us simplify Eq. (34) and expression (36) as follows:

$$\frac{d}{d\bar{\tau}}(J_0-J_2)=-J_0, \tag{37}$$

$$J_1=1-2J_2. \tag{38}$$

These equations should be considered together with Eqs. (33) and (35). For arbitrary μ this system cannot be integrated analytically. Therefore, we discuss first the range of variation and asymptotic behavior of the solutions to (37) and (38). To do this we we use Eqs. (7), (9), (12) and (13) to write an equation for the change in energy:

$$d\varepsilon/d\bar{\tau}=-J_0, \tag{39}$$

where

$$\varepsilon=E/E_0,\quad E=\frac{1}{2}N|b_{10}|^2\sum_{j=0}^{2}J_j,\quad E_0=\frac{1}{2}N|b_{10}|^2. \tag{40}$$

Since we have $J_0>0$, the energy can only decrease with time from $\varepsilon=1$ at $\bar{\tau}=0$ to a certain value $0\leqslant\varepsilon(\bar{\tau}\to\infty)<1$; therefore, $\varepsilon'(\bar{\tau}\to\infty)=0$. Then (39) implies $J_0(\bar{\tau}\to\infty)=0$. Furthermore, substituting (38) into (40), we obtain

$$\varepsilon=1+J_0-J_2. \tag{41}$$

Since $\varepsilon(\bar{\tau}\to\infty)<\varepsilon(\bar{\tau}\to 0)=1$ holds when a coherent field is generated, while $J_0(\bar{\tau}\to\infty)=0$, we have $J_2(\bar{\tau}\to\infty)\neq 0$. Let us write (35) in a somewhat different way:

$$2\frac{dJ_0^{1/2}}{d\bar{\tau}}+J_0^{1/2}=-\mu J_1 J_2^{1/2}\sin\psi.$$

Since $J_0^{1/2}\to 0$, $dJ_0^{1/2}/d\bar{\tau}\to 0$, while $J_2\neq 0$, we have $J_1\sin\psi\to 0$ for $\bar{\tau}\to\infty$. However, $J_1(\bar{\tau}\to\infty)\neq 0$ causes the factor with the cosine in the left-hand side of (33) to diverge, which in turn leads to $d\psi/d\bar{\tau}\to\infty$, and, of course, $J_0(\bar{\tau}\to\infty)\neq 0$ as well. Therefore, $J_1(\bar{\tau}\to\infty)\to 0$, while from (38) and (41), we have $J_2(\bar{\tau}\to\infty)=1/2$ and $\varepsilon(\bar{\tau}\to\infty)=1/2$. From this we see that this model gives a total loss of energy equal to half the initial value.

In order to write the analytic solution, let us consider as before the case $\mu^2\ll 1$. This condition allows us to assume $J_0'\ll J_0$ in (35); then Eq. (33) for the phase becomes

$$\frac{d}{d\bar{\tau}}\cos\psi+\frac{1}{2}\cos\psi=\mu\left(1-\frac{5}{2}J_2\right)\sin\psi. \tag{42}$$

In Eq. (42) we once again neglect the derivative of $\cos\psi$ in comparison with $\frac{1}{2}\cos\psi$, which gives

$$\cos^2\psi=4\mu^2\left(1-\frac{5}{2}J_2\right)\sin^2\psi\approx 0\qquad\text{or}\qquad\sin^2\psi=1.$$

Using this fact, we have from (35), (37) and (38) that

$$J_2'=\mu^2(1-2J_2)^2J_2,\quad J_2'=J_0. \tag{43}$$

Integrating Eq. (43) we finally obtain

$$\ln\frac{J_2(1-2J_{20})}{J_{20}(1-2J_2)}+\frac{1}{1-2J_2}-\frac{1}{1-2J_{20}}=\mu^2\bar{\tau}. \tag{44}$$

The expression (44) shows that the variation of J^2 has the form of a falloff from J_{20} at $\bar{\tau}=0$ to 1/2 for $\bar{\tau}\to\infty$, while $J_0(\bar{\tau})$ has the form of a pulse with a maximum equal to

$$J_{0\,max}=2\mu^2/27. \tag{45}$$

At the point where the value of J_0 is a maximum we have $J_2=1/6$, and Eq. (44) gives the following expression for the time $\bar{\tau}_{\max}$:

$$\bar{\tau}_{max}=(1/2-\ln 4J_{20})/\mu^2, \tag{46}$$

from which it follows that the time delay corresponding to the appearance of the radiation pulse is determined by the initial value of the mode J_{20}, and that for small values of this quantity the delay can be arbitrarily large. At the initial stage, (43) and (44) imply exponential growth of J_0, which for larger values of the time goes over to power-law decay. Let us use a level of $0.5J_{0\max}$ to define the length of the pulse;

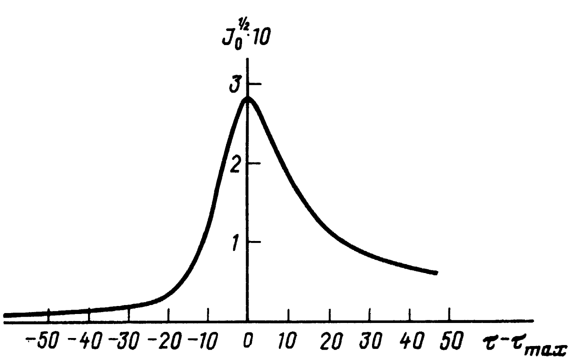

FIG. 1. Time dependence of the radiation intensity.

then from (44) we obtain

$$\Delta\bar{\tau}=\bar{\tau}_2-\bar{\tau}_1=\frac{1}{\mu^2}\left[\ln\frac{J_{22}(1-2J_{21})}{J_{21}(1-2J_{22})}+\frac{1}{1-2J_{22}}-\frac{1}{1-2J_{21}}\right], \tag{47}$$

where $\bar{\tau}_1$ is the time it takes J_0 to increase to a value $0.5J_{0\max}$ and $\bar{\tau}_2$ the time for it to decrease to a level $0.5J_{0\max}$; here $J_{21}=J_2(\bar{\tau}_1)$ and $J_{22}=J_2(\bar{\tau}_2)$. According to (43) and (45), the quantities $J_{21,2}$ are roots of the equation

$$J_2^3-J_2^2+\frac{1}{4}J_2-\frac{1}{108}=0. \tag{48}$$

The roots of interest to us have the values $J_{21}=0.045$ and $J_{22}=0.333$, and the length of the pulse, according to (47), is given by the expression

$$\Delta\bar{\tau}=5/\mu^2. \tag{49}$$

Figure 1 shows the radiation pulse $J_0(\tau)$ corresponding to the analytic formulas (43) and (44) for $\mu=1$ and $N\varkappa=0.3$. It is interesting to compare the values of the amplitude $J_{0\max}$ and duration $\Delta\tau$ ($\Delta\bar{\tau}=N\varkappa\Delta\tau$) given by Eqs. (45) and (49) with the values given in Ref. 8 obtained by numerical integration of Eqs. (7) and (8). The parameters of the problem used in Ref. 8 (in our notation) equal $\varkappa=2/N$, $\mu=a$ and $|b_{10}|^2=1$. For $\mu=1$ we have $J_{0\max}=0.07$ from (45), while Ref. 8 gives the value $J_{0\max}=0.1$, an agreement that is still rather good although our results are formally valid only for $\mu\ll 1$. For $\Delta\tau$ Eq. (49) gives the value $\Delta\tau\approx 2.5$, while from Ref. 8 we have $\Delta\tau\approx 5$, i.e., here the disagreement is large. Unfortunately, no calculations were given in Ref. 8 for $\mu<1$.

From this we see that we can use expressions (45) and (49) for estimates up to $\mu\sim 1$, while for $\mu>1$ there are limitations of a fundamental character. From (35) and (36) it follows that Eq. (45) gives the value of the absolute maximum, i.e., for increasing μ ($\mu>1$) the quantity $J_{0\max}$ can be only smaller than $2\mu^2/27$. Consequently, Eq. (45) always gives an overestimated value. Using (41) and (45) and the substitution $J_2=1/6$, we can write the variation of the energy $\Delta\varepsilon$ for times after J_0 has increased to its maximum value in the form

$$\Delta\varepsilon=1-\varepsilon=\frac{1}{3}\left(\frac{1}{2}-\frac{2}{9}\mu^2\right)>0,$$

from which we have for μ the condition $\mu<3/2$.

4. ACOUSTIC MODEL

In conclusion let us consider a specific physical example which, in principle at least, suggests that it is possible for

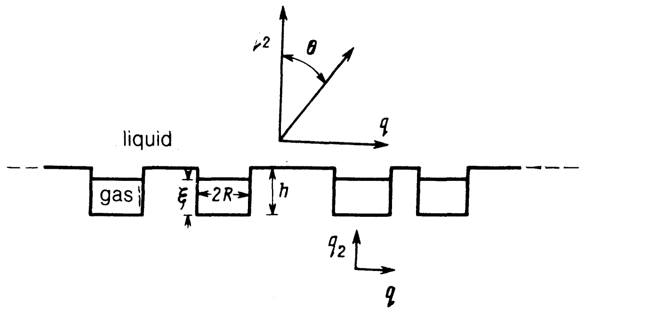

FIG. 2. Geometry of the problem under discussion: (q_2, q_1), where q_1 is parallel to the boundary, is a system of coordinates for describing the macroscopic field; the angle θ shows the direction of the radiated sound wave. $(q_2, \mathbf{q})$ is a system of coordinates for describing the individual wave, $(q_2, \mathbf{q})$ is a system of coordinates for describing the individual cavities; the dimensions (R, ξ, h) of the cavities are the same and the distances between them are random.

mechanical oscillators in acoustics to exhibit the effect of superradiance. Consider a set of cylindrical cavities with identical radii R and heights h filled with gas and embedded in an unbounded planar acoustically-rigid surface. The half-space $q_2 > 0$ (where $q_2 = 0$ is the position of the surface) is filled with a liquid that partially occupies the cavities so that the height of the air columns is $\xi \leqslant h$ (Fig. 2). These cavities are oscillators with a resonant frequency ω_0 determined by the elasticity of air and the effective mass of the neighboring liquid.[10] If the oscillators are excited at $t = 0$, they will vibrate at a frequency close to the intrinsic frequency ω_0. In this case it is possible to divide the pressure P and the velocity of liquid motion v into two components: an average component for variation on a scale including many cavities (i.e., for macroscopic fields) and a local component that varies rapidly in the component space.

Let us choose an area S on the surface with linear dimensions much smaller than the characteristic size of the nonuniformity of the macroscopic fields while still containing many cavities. For this area the condition of conservation of mass of the liquid gives an equation for $\langle v \rangle_\perp$, i.e., the component of the macroscopic velocity of liquid motion normal to the surface near the plane $q_2 = 0$:

$$S\langle v\rangle_\perp = \sigma \sum_{n=1}^{N_s} \frac{d\xi_n}{dt},$$

where $\sigma = \pi R^2$ is the area of the cavity and N_s the number of cavities in the area S. We assume that the cavities are distributed uniformly over the entire surface. Taking into account that the phase and amplitude of the oscillators can be repeated many times across S, we assume that at $t = 0$ there are only $N < N_s$ oscillators with differing phases and/or amplitudes; for clarity we will divide S into K small elements, each of which contains these N different oscillators, and let $s = S/K$. This allows us to reduce the summation to N terms, i.e.,

$$\langle v\rangle_\perp = \frac{\sigma}{s} \sum_{n=1}^{N} \frac{d\xi_n}{dt}. \tag{50}$$

Consider a planar sound wave propagating from the surface at an angle θ; then the average sound pressure is related to the velocity by the condition

$$\langle P\rangle = (\rho c/\cos\theta)\langle v\rangle_\perp, \tag{51}$$

where ρ is the density of the liquid and c the velocity of sound in the liquid. At this point we need an equation for the vibration of the individual cavities coupled by the overall field (51) [see Appendix, Eq. (A5)]. If we set the coefficients $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha$ in this equation, and pass to the dimensionless time variable $\tau = \omega_0 t$ and pressure $\mathscr{P} = \langle P\rangle/\gamma P_0$, then from (50) and (51) we obtain Eqs. (1) and (2), with the coefficient $\varkappa$ given by the expression

$$\varkappa = V_c\, \rho c \omega_0/\gamma s O_0 \cos\theta, \tag{52}$$

where V_c is the volume of the gas cavity, γ is the adiabatic exponent of the gas, and P_0 is the statistical pressure. If, however, α_1 and α_2 are not equal to zero, we obtain equations with a more complicated nonlinearity; however, the equations for the amplitude Y_k have the same form as (7) and (8), with a nonlinearity coefficient a equal to[11]

$$a = {}^1/_4(3\alpha_3/2 - 5\alpha_2^2/3 - \alpha_1^2/3). \tag{53}$$

In order to obtain an idea of the possible parameters of such a system, let us take ω_0 equal to $3\cdot 10^3$ sec^{-1} ($\sim$500 Hz), and the cavity radius $R = 3\cdot 10^{-3}$ m; then from the expression for ω_0 [see (A6)] under the condition $h = \xi_0$ we have $h = 0.6\cdot 10^{-2}$ m. The other parameters entering into (A6) are assumed to have following values: $P_0 = 10^5$ Pa (i.e., atmospheric pressure), $\gamma = 1.4$, $\rho = 10^3$ kg/m^3. Let us also set $|\eta| = 0.15$ [see Eq. (18)], in which case we have $\mathrm{Re}\,\lambda_0 = 0.045$, and $\mu = 1$, corresponding to $N\varkappa = 0.3$. The radiation pulse corresponding to the parameters of this model is shown in Fig. 1. Let us take $N = 20$; then from (52) we find at $\theta = 0$ the area $s = 0.018$ m^2. From expression (53) and (A7) we find the value of the nonlinearity coefficient $a = 0.6$, which gives $Y(t = 0) = 0.5$ for the initial value of the amplitude. The dimensional energy per unit area T_0 of the system at $t = 0$ can be written in the form

$$T_0 = \gamma P_0 V_c E_0/s, \tag{54}$$

while (36) implies the following expression for the flux of sonic energy Φ:

$$\Phi = \omega_0 \varkappa N T_0 J_0. \tag{55}$$

In the case under discussion here we have $T_0 = 6.5$ J/m^2, and the maximum value is $\Phi_{\max} = \omega_0 \varkappa N T_0 J_{0\max} = 1$ kW/m^2.

In conclusion let us note that this value of the energy T_0 for the phased linear oscillators would give a sonic flux Φ_L exponentially decaying with time, with a maximum value at $t = 0$ equal to $\Phi_L = \omega_0 \varkappa N T_0 J_{0\max} = 13.5$ kW/m^2. From this we see that the quantity $J_{0\max} = \Phi_{\max}/\Phi_L$ allows us to estimate the effectiveness of the autophasing of the oscillators.

APPENDIX

Let us choose the nonlinear equations for the oscillations in the gas cavities under the assumption that their dimensions are small compared with the wavelength of a sound wave of frequency ω_0 in the liquid and in the gas (i.e., the approximation of an incompressible liquid) and that the boundaries between liquid and gas are rigid during the oscillations (i.e., the piston approximation).

The change in momentum of the liquid column located in the cavity between the liquid-gas boundaries and $q_2 = 0$ can be written in the form

$$\sigma\rho \frac{d}{dt}[(h-\xi)\dot{\xi}] = \sigma P_G - \int P(\mathbf{q}, q_2=0)\, d\sigma, \quad \text{(A1)}$$

where P_G is the pressure in the gas, $P(\mathbf{q}, q_2 = 0)$ is the pressure at an arbitrary point with coordinates $\mathbf{q}$ on the cross section of the cavity at $q_2 = 0$ (see Fig. 2), and $\mathbf{q}$ is a vector that connects the center of the $q_2 = 0$ cross section of the cavity with an arbitrary point on this cross section. The integration is carried out over the cross section $q_2 = 0$. The pressure $P(\mathbf{q}, q_2 = 0)$ in our approximation is written in the form[12]

$$P(\mathbf{q}, q_2=0) = \langle P\rangle - \rho\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}v^2\right)$$
$$= \langle P\rangle + \frac{\rho\ddot{\xi}}{2\pi}\int\frac{d\sigma}{r} - \frac{\rho\dot{\xi}^2}{2}, \quad \text{(A2)}$$

where $\varphi = \varphi(\mathbf{q}, q_2 = 0)$ is the potential in the liquid, $v^2 = |\mathbf{v}(q_2 = 0)|^2$, and r is the distance between the point $\mathbf{q}$ and a point within the region of integration (which is carried out over these points). From (A1) and (A2) we make use of the equation[12]

$$\int\int\frac{d\sigma\, d\sigma'}{r} = 16\pi R^3/3,$$

to obtain

$$(8R/3\pi + h - \xi)\ddot{\xi} - \left(\frac{3}{2}\right)\dot{\xi}^2 = (P_G - \langle P\rangle)/\rho. \quad \text{(A3)}$$

For adiabatic vibrations of the gas in the cavity we have

$$(P_G + P_0)\xi^\gamma = P_0\xi_0^\gamma, \quad \text{(A4)}$$

where ξ_0 is the height of the cavity at rest. From (A3) and (A4) we obtain the following equation for $x = (\xi - \xi_0)/\xi_0$ up to cubic nonlinear terms:

$$\frac{d^2x}{dt^2} + \delta\omega_0\frac{dx}{dt} + \omega_0^2 x + \alpha_1\left[x\frac{d^2x}{dt^2} + \frac{3}{2}\left(\frac{dx}{dt}\right)^2\right]$$
$$+ \alpha_2\omega_0^2x^2 + \alpha_3\omega_0^2x^3 = -\mathscr{P}, \quad \text{(A5)}$$

$$\omega_0^2 = \gamma P_0/\rho\xi_0(8R/3\pi + h - \xi_0), \quad \text{(A6)}$$

$$\alpha_1 = -\xi_0/(8R/3\pi + h - \xi_0), \quad \alpha_2 = -(\gamma+1)/2,$$

$$\alpha_3 = (\gamma+1)(\gamma+2)/6.$$

In Eq. (A5) we have added a dissipative term with decay rate δ that includes various types of losses, i.e., thermal conductivity.

[1] A. V. Andreev, V. I. Emil'yanov, and Yu. A. Il'inskii, *Cooperative Phenomena in Optics*, Nauka, Moscow, 1988.
[2] N. S. Ginzburg, Pis'ma Zh. Eksp. Teor. Fiz. **13**, 173 (1988) [*sic*].
[3] N. S. Ginzburg and I. V. Zotova, Pis'ma Zh. Eksp. Teor. Fiz. **15**, 83 (1989) [*sic*].
[4] V. V. Zheleznyakov, V. V. Kocharovskii, and Vl. V. Kocharovskii, Izv. vuzov. Radiofizika **29**, 1095 (1986) [in Russian].
[5] A. V. Gaponov, M. I. Petelin, and V. G. Yulpatov, Izv. vuzov. Radiofizika **10**, 1414 (1967) [in Russian].
[6] Yu. A. Kobelev, L. A. Ostrovskii, and I. A. Soustova, Izv. vuzov. Radiofizika **29**, 1129 (1986) [in Russian].
[7] A. Yu. Stabinis, *Diss. for Candidacy in Physical and Mathematical Sciences*, Moscow State Univ., Moscow, 1972.
[8] Yu. A. Il'inskii and N. S. Maslova, Zh. Eksp. Teor. Fiz. **94**, 171 (1988) [Sov. Phys. JETP **67**, 307 (1988)].
[9] I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic, New York (1980).
[10] Yu. A. Kobelev, Akust. Zh. **33**, 903 (1987) [Sov. Phys. Acoust. 33, 523 (1987)].
[11] Yu. A. Mitropol'skii, *The Method of Averaging in Nonlinear Mechanics*, p. 176 [in Russian], Nauk. Dumka, Kiev, 1971.
[12] J. W. Strutt (Lord Rayleigh), *Theory of Sound, Vol. 2*, Macmillan, London (1922).

Translated by Frank J. Crowne