Quadratic algebras and dynamical symmetry of the Schrödinger equation

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(Submitted 11 May 1990)
Zh. Eksp. Teor. Fiz. **99**, 369–377 (February 1991)

The quadratic Jacobi algebra with three generators is the dynamical symmetry algebra of exactly solvable potentials, for which the Schrödinger equation reduces to the hypergeometric equation. The commutation relations provide information on the most important physical properties such as the energy spectrum and coefficients of the scattering matrix.

1. INTRODUCTION

As is well known, all exactly solvable problems in quantum mechanics admit a treatment in the language of Lie algebras or groups.

It was shown in Ref. 1 that the $O(2,1)$ algebra gives rise to those potentials for which the Schrödinger equation reduces to the confluent hypergeometric equation (the harmonic oscillator, Coulomb, and Morse potentials). Further, the algebra serves as a dynamical symmetry algebra in the sense of Ref. 2, i.e., one of the generators of the algebra serves as the Hamiltonian of the corresponding quantum system. However such an interpretation is impossible in those cases when the Schrödinger equation reduces to the full hypergeometric equation (potentials of the Pöschl-Teller or Eckart type). Indeed, the spectra of these potentials are quadratic (or reduce to quadratic), while in a Lie algebra the discrete spectrum of any generator can only be linear.

A somewhat artificial way out of this situation has been proposed: take as the Hamiltonian not a generator of the Lie algebra but the corresponding Casimir operator. This method suffers from the deficiency that the properties of dynamical symmetry are lost: since the Casimir operator cannot be changed by the action of the generators of the given representation the dynamical connection between representations disappears. We may list as another deficiency of this approach the circumstance that some of the parameters of the corresponding potentials are restricted to integer values.

Nevertheless this method was used to classify the exactly solvable problems in the cases where the starting algebras were taken to be $SU(1,1)$ or $SU(2)$ (Refs. 3, 4). The algebraic method was applied in Refs. 5 and 6 to find the transition coefficients in the scattering problem (see also Ref. 7).

Naturally this question presents itself: could all the exactly solvable problems be stated in the framework of an algebra so as to preserve the idea of the dynamical symmetry method, i.e., so that one of the generators of the algebra would serve as the Hamiltonian?

The answer to this question turns out to be affirmative; the present work is devoted to the corresponding construction. The principal idea consists of giving up on Lie algebras. In their place we propose to make use of so-called quadratic algebras.

We recall that quadratic algebras were first introduced in two pioneering papers by Sklyanin.[8] In contrast to Lie algebras the commutation relations for the generators K_i of these algebras contain quadratic combinations of these generators. In other words the general form of the commutation relations of a quadratic algebra is as follows:

$$[K_i, K_j] = \sum_{l,m} a_{ijlm} K_l K_m + \sum_s b_{ijs} K_s + c_{ij}. \tag{1.1}$$

The quantities a_{ijlm}, b_{ijs}, and c_{ij} enter in the role of structure constants of the algebra. If $a_{ijlm} = 0$ holds we return to ordinary Lie algebras.

The generators of a quadratic algebra, in contrast to a Lie algebra, do not form a linear space. This shortcoming is, however, compensated by the possibility of constructing representations of these algebras in analogy to the Lie algebras. For this reason it turns out to be possible to use quadratic algebras to clarify the situation for various exactly solvable many-particle problems.

Until now, however, quadratic algebras, as well as the related so-called quantum algebras, have found applications only in rather complicated and exotic problems of mathematical physics (see, for example, Ref. 9). In Ref. 10 it was shown that a quadratic algebra of a simple structure—the so-called Wilson-Racah algebra—can be successfully used to analyze "hidden" symmetries of the $6j$- and $3j$-symbols.

In the present work we investigate a quadratic algebra with three generators and simplest choice of nonlinearity, which we shall call the Jacobi algebra. Within its framework we find a natural interpretation of all exactly solvable problems with a quadratic (discrete or continuous) spectrum in the spirit of the dynamical symmetry idea. The Lie case, corresponding to a potential with a linear spectrum, is obtained by a simple degeneracy (contraction) of the Jacobi algebra.

This paper is organized as follows: we first give the definition of the Jacobi algebra and construct its ladder representation, which encompasses both the discrete and the continuous spectrum. We then show that the Jacobi algebra is the symmetry algebra of the hypergeometric equation. This fact allows a choice of a realization of the Jacobi algebra in which one of its generators has the structure of a nonrelativistic one-dimensional Hamiltonian. This realization permits the construction of the spectrum and the wave functions on the basis of the commutation relations of the algebra. Lastly we show how in the case of the continuous spectrum the Jacobi algebra permits the determination of the coefficients of the scattering matrix. The present work extends the results obtained previously and communicated in the preprint, Ref. 11.

2. THE JACOBI ALGEBRA AND ITS LADDER REPRESENTATION

Let the three operators K_0, K_1, K_2 generate the quadratic algebra, in which linearity is broken "minimally":

 0038-5646/91/020205-05$03.00

$$[K_0, K_1]=K_2, \tag{2.1.a}$$

$$[K_1, K_2]=aK_1^2+bK_1+c_0, \tag{2.1.b}$$

$$[K_2, K_0]=a\{K_0, K_1\}+bK_0+dK_1+c_1, \tag{2.1.c}$$

where

$$\{K_0, K_1\}=K_0K_1+K_1K_0.$$

The structure constants a, b, d, c_0, c_1 are assumed to be real. It is not hard to show that the Jacobi identity holds and that the operators K_0 and K_1 can be simultaneously Hermitian.

The Casimir operator Q, which commutes with all the three generators, has the form

$$Q=K_2^2+(a^2+d)K_1^2+a\{K_0, K_1^2\} +b\{K_0, K_1\}+(2c_1+ab)K_1+2c_0K_0. \tag{2.2}$$

We shall call the algebra defined by the commutation relations (2.1) the Jacobi algebra—it is a special case of the more general Wilson-Racah quadratic algebra with three generators, studied in Refs. 10 and 11.

An important property of the Jacobi algebra is the ladder property, which means the following. Let $|\lambda\rangle$ be an eigenstate of the operator K_0 to the eigenvalue λ:

$$K_0|\lambda\rangle=\lambda|\lambda\rangle. \tag{2.3}$$

It is possible to form a linear combination of the generators of the algebra such that the state

$$|\tilde{\lambda}\rangle=(\xi(\lambda)K_0+\eta(\lambda)K_1+\zeta(\lambda)K_2)|\lambda\rangle \tag{2.4}$$

is again an eigenstate of K_0 but with a different eigenvalue $\tilde{\lambda}$. In (2.4) the coefficients in the linear combination depend on the spectral parameter λ. Only in the case of a Lie algebra ($a=0$) are the coefficients independent of λ.

This ladder property makes it possible to construct a representation of the Jacobi algebra in which the operator K_0 is diagonal:

$$K_0\psi_s=\lambda_s\psi_s, \tag{2.5.a}$$

and the operators K_1 and K_2 are tridiagonal in this basis:

$$K_1\psi_s=A_{s+1}\psi_{s+1}+A_s\psi_{s-1}+B_s\psi_s, \tag{2.5.b}$$

$$K_2\psi_s=g_{s+1}A_{s+1}\psi_{s+1}-g_sA_s\psi_{s-1}, \tag{2.5.c}$$

where

$$g_s=\lambda_s-\lambda_{s-1}. \tag{2.6}$$

Making use of the commutation relations (2.1) and Eq. (2.2) for the Casimir operator (in each given representation Q is a constant), we obtain explicit expressions for the matrix elements in the ladder representation (2.5):

$$\lambda_s=-\frac{a}{2}(s-\sigma)(s-\sigma+1)-\frac{d}{2a}, \tag{2.6.a}$$

$$g_s=-a(s-\sigma), \tag{2.6.b}$$

$$B_s=\frac{b\lambda_s+c_1}{g_sg_{s+1}}, \tag{2.6.c}$$

$$4g_{s-1/2}g_{s+1/2}A_s^2=g_{s-1}g_{s+1}B_sB_{s-1}+c_0(\lambda_s+\lambda_{s-1})-Q, \tag{2.6.d}$$

where σ is an arbitrary parameter.

The reality requirement for the spectrum λ_s imposes constraints on the parameters. Two versions are possible: either we have $s=n$, an integer, and σ is an arbitrary real parameter, or the following relation holds

$$s-\sigma=-ik-1/2,$$

where k is an arbitrary real parameter.

In the first case the spectrum is discrete and quadratic in n:

$$\lambda_n=-1/2a(n-\sigma)(n-\sigma+1)-d/2a,$$

while in the second case the spectrum is continuous and quadratic in k:

$$\lambda_k=1/2a(k^2+1/4)-d/2a.$$

In this manner, in contrast to Lie algebras, the spectrum of the generator K_0 for fixed parameters of the algebra (by parameters we mean values of the structure constants and the Casimir operator) can be both discrete and continuous—one representation contains both spectra. Properly speaking, this is how the basic idea of dynamical symmetry for all states of the system is realized.

In the case of the discrete spectrum the equation for the eigenvalues of the operator K_1

$$K_1\psi_n=A_{n+1}\psi_{n+1}+A_n\psi_{n-1}+B_n\psi_n=x\psi_n \tag{2.7}$$

under the condition $A_0=0$ (the spectrum is bounded from below) gives rise to a system of orthogonal polynomials

$$P_n(x)=\frac{\psi_n(x)}{\psi_0(x)}, \tag{2.8}$$

for which A_n and B_n are the coefficients in a three-term recursion relation that determines their properties.[12] It follows from the explicit form of these coefficients, Eqs. (2.6.c) and (2.6.d), that we can conclude that the $P_n(x)$ are the classical Jacobi polynomials. It is for this reason that we have named the algebra the Jacobi algebra.

We note that on the basis of the more general Wilson-Racah quadratic algebra one can give an algebraic interpretation to all classical orthogonal polynomials (see Ref. 11).

3. REALIZATION OF THE JACOBI ALGEBRA BY DIFFERENTIAL OPERATORS: THE HYPERGEOMETRIC EQUATION

In the preceding Sec. we have constructed the ladder representation in some abstract basis ψ_s. We now equip the functions in this basis with an argument x and find a realization of the Jacobi algebra in the form of differential operators acting on the functions $\psi_s(x)$.

For the operator K_1 we take the operation of multiplication by x

$$K_1\psi(x)=x\psi(x), \tag{3.1}$$

while we choose the operator K_0 in the form of a second-order differential operator

$$K_0=q(x)\frac{d^2}{dx^2}+v(x), \tag{3.2}$$

where $q(x)$ and $v(x)$ are functions to be determined.

In accordance with (2.1.a) the operator K_2 takes the form

$$K_2=2q(x)\frac{d}{dx}. \tag{3.3}$$

Making use of the commutation relations (2.1) and the Casimir operator Q we find

$$2q(x)=-(ax^2+bx+c_0), \tag{3.4}$$

$$2v(x)=\tilde{q}(x)/q(x). \tag{3.5}$$

Here $\tilde{q}(x)$ is a polynomial in x of second degree:

$$2\tilde{q}(x)=-(\tilde{a}x^2+\tilde{b}x+\tilde{c}), \tag{3.6}$$

where

$$\tilde{a}=-d,\quad \tilde{b}=-2c_1,\quad \tilde{c}=ac_0+Q.$$

The eigenvalue equation for K_0

$$K_0\psi(x)=\lambda\psi(x)$$

can be written in the form

$$\psi''(x)+\frac{v(x)-\lambda}{q(x)}\psi(x)=0. \tag{3.7}$$

Equation (3.7) is the hypergeometric equation[12] in the form without a first derivative. Thus the basis functions $\psi_s(x)$ are expressible in terms of the hypergeometric function ${}_2F_1(\alpha, \beta, \gamma, x)$. In particular for a discrete spectrum $(\alpha = -n)$ we obtain the Jacobi polynomials, once again justifying the naming of this algebra.

In two particular cases Eq. (3.7) reduces to the confluent hypergeometric equation. In the first case $a = 0$ and we have a Lie algebra. In the second case $q(x)$ is a perfect square

$$q(x)=a(x-x_0)^2.$$

It is not hard to show that in this case too the Jacobi algebra can be reduced to a Lie algebra.

In this way the Jacobi algebra is the dynamical symmetry algebra of the hypergeometric equation. In particular, all the properties of hypergeometric functions can be obtained directly from the properties of the Jacobi algebra. We do not stop to discuss this question (which could be the subject of a separate investigation) but go on to an application of the Jacobi algebra—the determination of exactly solvable potentials.

4. THE JACOBI ALGEBRA AND EXACTLY SOLVABLE POTENTIALS: DISCRETE SPECTRUM

In the preceding section we have constructed a realization of the Jacobi algebra by differential operators, with the eigenfunctions of the operator K_0 in this realization being the hypergeometric functions. It follows that the Jacobi algebra is the dynamical symmetry algebra for potentials which permit the reduction of the Schrödinger equation to the hypergeometric form by a change of variable (in short—hypergeometric potentials). As in the case of Lie dynamical symmetry,[1,2] the "quadratic" symmetry makes it possible to determine the spectrum, the wave functions and other properties of the system by purely algebraic means.

Let the Hamiltonian $H = p^2 + U(x)$ of a quantum system in the potential $U(x)$ be taken as the operator K_0 of the Jacobi algebra. For K_1 we take the operation of multiplication by a certain function $\varphi(x)$:

$$K_0=-\frac{d^2}{dr^2}+U(x),$$

$$K_1=\varphi(x), \tag{4.1}$$

$$K_2=-2\varphi'(x)\frac{d}{dx}-\varphi''(x).$$

We proceed to the analysis of the realization (4.1). Making use of the commutation relations (2.1) and the Casimir operator Q we obtain

$$\varphi'^2(x)=q(\varphi)/2, \tag{4.2.a}$$

$$U(x)=q_1(\varphi)/2q(\varphi), \tag{4.2.b}$$

where

$$q(\varphi)=a\varphi^2+b\varphi+c_0, \tag{4.3.a}$$

$$q_1(\varphi)=\frac{aq(\varphi)}{4}-d\varphi^2-2c_1\varphi+\frac{ac_0}{4}+\frac{3}{16}b^2+Q. \tag{4.3.b}$$

In the following, unless explicitly stated to the contrary, we assume $a\neq 0$. By shifting the operator K_1 by a constant we can achieve $b = 0$. The absolute values of the parameters a and c_0 affect only the scale factors of the function $\varphi(x)$ and its argument and therefore they can be fixed by taking, for example, $|a| = |c_0| = 2$.

Depending on the combinations of signs three versions are possible.

1. $a = c_0 = 2$.

In this case we have

$$\varphi(x)=\operatorname{sh} x, \tag{4.4.a}$$

$$U(x)=\frac{(1-d)\operatorname{sh}^2 x-2c_1\operatorname{sh} x+Q+2}{4\operatorname{ch}^2 x} \tag{4.4.b}$$

(the so-called "cosh" soliton potential).

2. $a = -c_0 = 2$.

In this case we obtain

$$\varphi(x)=\operatorname{ch} x, \tag{4.5.a}$$

$$U(x)=\frac{(1-d)\operatorname{ch}^2 x-2c_1\operatorname{ch} x+Q-2}{4\operatorname{sh}^2 x}. \tag{4.5.b}$$

3. $a = -c_0 = -2$.

In this case the potential is of trigonometric form

$$\varphi(x)=\sin x, \tag{4.6.a}$$

$$U(x)=\frac{(1-d)\sin^2 x-2c_1\sin x+Q-2}{4\cos^2 x}. \tag{4.6.b}$$

Lastly one must consider separately the case $c_0 = 0$ (keeping as before $a = 2$). Here we have the Morse potential

$$\varphi(x)=e^x, \tag{4.7.a}$$

$$U(x)=\frac{1}{4}(1-d-2c_1e^{-x}+Qe^{-2x}). \tag{4.7.b}$$

We note that the potentials (4.4)–(4.6) are modified Pöschl-Teller potentials. The fact that the wave functions for these potentials are expressible in terms of the hypergeometric functions follows directly from the Jacobi dynamical symmetry.

In previous treatments[3,4] of these potentials the Hamiltonian H was interpreted as the square of the angular momentum in certain special realizations of the $SU(2)$ and $SU(1,1)$ algebras. Such an approach suffers from two essential shortcomings: first, it is necessary for each potential to begin by specifying the form of the corresponding Lie algebra and its specific realization; second, the parameters of these potentials must satisfy certain integer requirements, characteristic of irreducible representations of Lie algebras.

In our approach these difficulties are absent as a direct consequence of the construction of the Jacobi algebra. In addition the Jacobi algebra makes it easy to determine various characteristics of these potentials, which are "not seen" in other algebraic approaches. In particular, starting from just the one basis representation (2.5) we can find both the discrete spectrum and the scattering matrix in the continuous spectrum. We illustrate the effectiveness of the method using the potential (4.4) as an example, confining ourselves in this section to the case of the discrete spectrum. To ensure the standard asymptotic limits $U(\pm\infty)=0$ we set $d=1$ and reparametrize the potential in the form

$$U(x)=\frac{\beta^2-\alpha^2+{}^1/_4-2\alpha\beta\,\mathrm{sh}\,x}{\mathrm{ch}^2 x}, \tag{4.8}$$

where α and β are parameters connected with Q and c_1 by the relations

$$Q=4(\beta^2-\alpha^2)-1,\quad c_1=4\alpha\beta.$$

It follows from (2.6) that the energy spectrum has the form

$$E_n=-(n-\sigma)(n-\sigma+1)-{}^1/_4. \tag{4.9}$$

The value of the parameter σ will be found from the requirement that in the ground state the relation $A_0=0$ should hold. From the explicit form of A_n [see (2.6.d)]

$$4g_{n-1/2}g_{n+1/2}A_n^2=\frac{(4\alpha^2-g_n^2)(4\beta^2+g_n^2)}{g_n^2} \tag{4.10}$$

it follows that $A_0=0$ for

$$\sigma=\alpha>0. \tag{4.11}$$

Thus the spectrum has the form

$$E_n=-(n-\alpha+{}^1/_2)^2, \tag{4.12}$$

with the index n ranging from 0 up to the maximum value $[\alpha-1/2]$ (here the square brackets denote the integer part of the number). For $\alpha=N+1/2$ the last level of the discrete spectrum coincides with the boundary of the continuous spectrum ($E_N=0$). We note that the spectrum does not depend on the parameter β.

5. CONTINUOUS SPECTRUM: SCATTERING MATRIX

The Jacobi algebra can also be used with success to analyze the states in the continuum.

By itself the algebraic approach to scattering is not new. Thus, as early as 1967 Zwanziger applied the $O(3,1)$ algebra (the dynamical symmetry algebra of the hydrogen atom in the case of the continuum) to calculate the scattering amplitude in the Coulomb potential.[7] A similar method was applied in Ref. 6 to the determination of the coefficients of the scattering matrix in the case of a one-dimensional "soliton" potential. In these approaches the Lie algebra acts on the space of states with fixed energy. Therefore the class of admissible one-dimensional potentials is restricted by the requirement that one of the parameters of the potential be an integer.

In our approach recursion relations are obtained for elements of the scattering matrix in potentials of a general form that satisfy the Jacobi algebra. Here the shifts in the recursion relations are in energy. As far as we know this approach has not been tried previously.

We show how the method works for the "cosh soliton potential" (4.4). We look for the wave function with the asymptotic limits

$$\psi_k(x)=\begin{cases} C_k e^{ikx}+D_k e^{-ikx}, & x\to-\infty, \quad (5.1)\\ G_k e^{ikx}, & x\to\infty, \quad (5.2)\end{cases}$$

where $E=k^2$.

Let us act on this wave function with the operator K_1 [in the ladder representation (2.5) we replace the shift of the parameter s by 1 with a shift of the parameter k by i]:

$$K_1\psi_k(x)=A_{k+i}\psi_{k+i}(x)+A_k\psi_{k-i}(x)+B_k\psi_k(x), \tag{5.3}$$

where

$$A_k^2=\frac{(\beta^2+(ik+{}^1/_2)^2)(\alpha^2-(ik+{}^1/_2)^2)}{4ik(ik+1)(ik+{}^1/_2)^2}. \tag{5.4}$$

On the other hand, for the potential (4.4) the relation

$$K_1\psi_k(x)=\mathrm{sh}\,x\,\psi_k(x) \tag{5.5}$$

holds. Therefore for $x\to-\infty$ we have

$$A_{k+i}C_{k+i}e^{ikx}+A_kD_{k-i}e^{-ikx}=-\frac{1}{2}(C_ke^{ikx}+D_ke^{-ikx}); \tag{5.6.a}$$

similarly, for $x\to+\infty$ we have

$$A_kG_{k-i}={}^1/_2G_k. \tag{5.6.b}$$

We introduce the reflection and transmission amplitudes

$$r_k=\frac{D_k}{C_k},\quad t_k=\frac{G_k}{C_k}. \tag{5.7}$$

From (5.6) we obtain the following recurrence equations for the determination of these amplitudes:

$$\frac{r_{k+i}}{r_k}=-\frac{t_{k+i}}{t_k}=4A_{k+i}^2$$
$$=\frac{(ik+i\beta-{}^1/_2)(ik-i\beta-{}^1/_2)(ik+\alpha-{}^1/_2)(ik-\alpha-{}^1/_2)}{(-ik)(ik-1)(ik-{}^1/_2)^2}. \tag{5.8}$$

Equation (5.8) is an extremely simple difference equation, whose solution can be written in the form of gamma functions. Exploiting analyticity properties of the amplitude t_k, as well as the fact that all its poles lie in the upper half-plane and are connected with the states in the discrete spectrum, we obtain the explicit expression for the transmission amplitude:

$$t_k=\frac{\Gamma({}^1/_2-\alpha-ik)\Gamma({}^1/_2+\alpha-ik)\Gamma({}^1/_2+i\beta-ik)\Gamma({}^1/_2-i\beta-ik)}{\Gamma^2({}^1/_2-ik)\Gamma(-ik)\Gamma(1-ik)}. \tag{5.9}$$

In determining the normalization factor in front of t_k we made use of the asymptotic property: $t_k \to 1$ for $k \to \infty$. We note that for $\beta = 0$ we obtain the familiar expression[6] for the transmission amplitude in the case of a "simple" soliton potential.

The expression for the reflection amplitude r_k has a more complicated structure: it consists of two terms, which contain "redundant" poles in comparison with the transmission amplitude, with, evidently, each term satisfying the recursion relation (5.8). To determine the coefficients of these terms it is necessary to make use of the normalization condition and the asymptotic behavior of r_k as $k \to 0$. Taking these considerations into account we obtain

$$r_k = \frac{\pi \Pi(ik)\Gamma(ik)}{\Gamma(1/2+\alpha)\Gamma(1/2-\alpha)\Gamma(1/2+i\beta)\Gamma(1/2-i\beta)\Gamma^2(1/2-ik)\Gamma(-ik)} + \frac{i\pi \Pi(ik)\Gamma(1/2+ik)}{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(i\beta)\Gamma(1-i\beta)\Gamma(1/2-ik)\Gamma(-ik)\Gamma(1-ik)}, \tag{5.10}$$

where

$$\Pi(ik) = \Gamma(1/2-\alpha-ik)\Gamma(1/2+\alpha-ik)\Gamma(1/2-i\beta-ik)\Gamma(1/2+i\beta-ik).$$

6. CONCLUSION

We have shown in this paper that the quadratic Jacobi algebra is the dynamical symmetry algebra of exactly solvable (hypergeometric) potentials.

Starting from the commutation relations of the algebra one can obtain both the spectrum of the bound states and the scattering matrix for states in the continuum. We have discussed the case in which the Hamiltonian in the Schrödinger equation is directly a generator of the Jacobi algebra. The corresponding potentials are found to be generalized Pöschl-Teller type potentials (Morse potentials in the confluent case) and have a quadratic spectrum. If instead the operator K_0 is chosen to be the Hamiltonian multiplied from the left by some function (i.e., we consider the spectral problem not for the energy but for one of the potential parameters), then we obtain potentials of the Eckart type (Coulomb type in the confluent case). We did not dwell here on the latter case since its analysis is no different from the one we studied.

Consequently all possible cases of dynamical symmetry of the Schrödinger equation can be described by one algebra. The nontrivial point is that that algebra is quadratic. To our knowledge quadratic algebras have not been used previously to analyze exactly solvable potentials.

We note that the Jacobi algebra clarifies the mysterious connection between exactly solvable classical and quantum problems, and also provides a geometric interpretation of all exactly solvable potentials (see Ref. 11).

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Translated by Adam M. Bincer