Nonanalyticity in the electron distribution function and nonlinear propagation of electromagnetic waves in semiconductors

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The propagation of electromagnetic waves in a plasma may be nonlinear if a mechanism operates to make the charge-carrier distribution function nonanalytic (i.e., if this function acquires discontinuous derivatives). A nonlinear wave equation is derived for the propagation of an electromagnetic wave in a semiconductor in a strong static electric field. This strong field stimulates spontaneous emission of optical phonons by electrons. As a result, the electron distribution function becomes nonanalytic at a certain set of points corresponding to energies which are multiples of the energy of these optical phonons. This result in turn causes a nonlinear response to a high-frequency electric field and, correspondingly, a nonlinear behavior of a wave. The creation and propagation of dark envelope solitons of an electromagnetic wave in a semiconductor are described.

1. INTRODUCTION

Such effects as modulational instability, self-focusing, and second-harmonic generation are quite familiar in media in which the electric displacement is a nonlinear function of the electric field strength.[1] This nonlinearity mechanism might be called the "dielectric" mechanism. In a conducting medium, yet another nonlinearity mechanism can operate: a "current" mechanism. That this is true can be seen easily from the equation describing the propagation of an electromagnetic wave in a plasma:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2}+\frac{c^2}{\varepsilon_0}\operatorname{rot}\operatorname{rot}\mathbf{E}=-\frac{4\pi}{\varepsilon_0}\frac{\partial \mathbf{J}}{\partial t}, \qquad (1)$$

where $\mathbf{E}$ is the electric field, $\mathbf{J}$ is the current density, and ε_0 is the static dielectric constant of the medium. If the current $\mathbf{J}$ in the system is a nonlinear function of the field $\mathbf{E}$, the behavior of an electromagnetic wave is nonlinear.[2,3]

In this paper we examine the current mechanism for the nonlinear propagation of electromagnetic waves in a semiconductor. We will see that the current may become a nonlinear function of the field as a result of nonanalyticity of the nonequilibrium steady-state electron distribution function. This result applies to any plasma. The choice of a semiconductor specifies the nature of the nonanalyticity of the distribution function. In the case considered here, the nonanalyticity stems from spontaneous emission of optical phonons by electrons in a static electric field.

The kinetic equation for an electron gas in static and high-frequency electric fields is solved in the high-frequency limit in Sec. 2. The electron distribution function is found and then used to calculate the current density $\mathbf{J}$. The right side of Eq. (1) is calculated in Sec. 3 for the case in which the semiconductor is in a static electric field and a quantizing magnetic field parallel to it. The quantizing magnetic field, which is not of fundamental importance to the analysis, is introduced partly to simplify the calculations. Another reason is that conditions corresponding to high-energy electron runaway and spontaneous emission of optical phonons by electrons[4] are more favorable for shaping an electron distribution function with a discontinuous derivative near the energy of an optical phonon. A nonlinear Schrödinger equation is constructed for the slowly varying field amplitude $\mathbf{E}$ in Sec. 4. In this equation, the signs of the dispersive and nonlinear terms correspond to repulsion of the particles of a Bose gas or defocusing. In this case, dark solitons[5,6] (see also Ref. 7), i.e., regions from which the electromagnetic field has been displaced (a condensate), can propagate through the medium. The creation and propagation of solitons is analyzed in Sec. 5 by the inverse scattering method.[6] In Sec. 6 we discuss the conditions which would be required for experimental observation of envelope solitons of an electromagnetic wave in a semiconductor.

2. KINETIC EQUATION

Let us examine the ordinary kinetic equation for electrons with an isotropic parabolic energy spectrum $\varepsilon_\mathbf{p}=\mathbf{p}^2/(2m)$ in a static field $\mathbf{E}_0$ and an oscillatory field $\mathbf{E}$. The electrons are interacting with scatterers. We write this equation in the form

$$\hat{L}f=-e\mathbf{E}\frac{\partial f}{\partial \mathbf{p}}, \qquad (2)$$

$$\hat{L}=\frac{\partial}{\partial t}+\mathbf{v}_\mathbf{p}\frac{\partial}{\partial \mathbf{r}}+e\mathbf{E}_0\frac{\partial}{\partial \mathbf{p}}-\hat{I}, \qquad (3)$$

where $\mathbf{v}_\mathbf{p}=\partial\varepsilon_\mathbf{p}/\partial\mathbf{p}$, and $\hat{I}$ is the collision integral. In the absence of the oscillatory field $\mathbf{E}$, we describe the spatially uniform steady state set up by the static field $\mathbf{E}_0$ by the distribution function $f_0(\mathbf{p})$. We restrict the discussion to the high-frequency limit, in which the conditions

$$\omega\gg k\bar{v},\quad \nu_m,\quad |e|E_0/\bar{p} \qquad (4)$$

hold. Here k and ω determine the spatial and time scales of the system, $\bar{p}\equiv m\bar{v}$ is the average electron momentum, and ν_m is the highest of the rates at which charge carriers collide with the scatterers. We also assume that the field $\mathbf{E}$ is turned on adiabatically, so that at $t=-\infty$ we have $\mathbf{E}=0$ and $f=f_0(\mathbf{p})$. In lowest order in the small parameters determined by the inequalities (4), we then have $\hat{L}=\partial/\partial t$, and the solution of Eq. (2) under the initial conditions is

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$$f(\mathbf{p},t)=f_0\left(\mathbf{p}-e\int_{-\infty}^{t}dt'\mathbf{E}\right)=\exp\left(-e\int_{-\infty}^{t}dt'\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)f_0(\mathbf{p}). \qquad (5)$$

It is important to note that the form of the solution (5) presupposes that the function $f_0(\mathbf{p})$ is real and analytic, as can be seen clearly from the second equality in (5), since the displacement operator

$$\hat{T}=\exp\left(-e\int_{-\infty}^{t}dt'\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)$$

is defined for analytic functions. In other words, the Taylor series

$$f(\mathbf{p},t)=\hat{T}f_0(\mathbf{p})=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-e\int_{-\infty}^{t}dt'\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)^n f_0(\mathbf{p}) \qquad (6)$$

must converge. For piecewise-analytic functions, the validity of (5) and (6) is thus disrupted near points in momentum space at which the derivatives of the function $f_0(\mathbf{p})$ are discontinuous. Discontinuous functions of this sort are the subject of the present paper.

If the function $f_0(\mathbf{p})$ is analytic everywhere, we easily find a well-known result for the right side of Eq. (1):

$$\frac{4\pi}{\varepsilon_0}\frac{\partial J}{\partial t}=\omega_p^2\mathbf{E}, \qquad (7)$$

where $\omega_p=(4\pi e^2n/\varepsilon_0 m)^{1/2}$ is the plasma frequency, and the carrier density is a constant ($n=\text{const}$). In other words, Eq. (1) describes a linear propagation of high-frequency waves in the plasma in this case. If, on the other hand, the function $f_0(\mathbf{p})$ is nonanalytic at a certain set of points in momentum space, then Eq. (7) does not hold, since the substitution

$$\mathbf{p}'=\mathbf{p}-e\int_{-\infty}^{t}dt'\mathbf{E}$$

cannot be made throughout space in the equation for the current, and the wave equation (1) becomes nonlinear. This conclusion is valid for any plasma, but a special analysis of the nonanalyticity of the distribution function $f_0(\mathbf{p})$ will have to be carried out in each specific case. The simplest examples of a nonanalyticity are corners in the distribution function (discontinuities in its first derivatives at a certain set of points in momentum space). Substitution of the distribution function f as in series (6) into the expression

$$\mathbf{J}(t)=\frac{e}{mV}\sum_{\mathbf{p}}\mathbf{p}f(\mathbf{p},t) \qquad (8)$$

for the current density (V is the volume of the system), in the region in which the function $f_0(\mathbf{p})$ is analytic, shows that discontinuities in the derivatives of the steady-state distribution function lead to nonzero terms which are nonlinear in the field $\mathbf{E}$. The lowest degree of nonlinearity is determined by the number of continuous derivatives of $f_0(\mathbf{p})$. One might say that the electromagnetic field determines the smoothness of the electron distribution function. The discontinuities in the derivatives should be finite (more precisely, the corresponding integrals should converge). If the field $\mathbf{E}$ is not very strong,

$$\frac{|e|E}{\omega}\ll\bar{p}, \qquad (9)$$

and we assume below that this is the case, writing f as a series makes it possible to calculate the current in the necessary order in the small parameter determined by condition (9).

To again call attention to the smoothness properties of the function $f_0(\mathbf{p})$, we will outline an iterative procedure for solving the kinetic equation (2) in the form of a series

$$f=f_0+f_1+f_2+\dots, \qquad (10)$$

in which f_n is given by the following expression in the situation described above:

$$f_n=(-e)^n\int_{-\infty}^{t}dt_1E_{k_1}\dots\int_{-\infty}^{t_{n-1}}dt_nE_{k_n}\frac{\partial}{\partial p_{k_1}}\dots\frac{\partial}{\partial p_{k_n}}f_0(\mathbf{p})$$

$$=\frac{1}{n!}\left(-e\int_{-\infty}^{t}dt'\,\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)^n f_0(\mathbf{p}). \qquad (11)$$

This procedure may be thought of as a method for constructing both an exact solution, (5), and an approximate one, under condition (9). In addition, each iteration presupposes explicit account of the differentiability properties of the function $f_0(\mathbf{p})$ in the calculation of the current density $\mathbf{J}$.

3. NONLINEAR WAVE EQUATION IN A SEMICONDUCTOR

We turn now to a specific system in which an electromagnetic wave exhibits nonlinear behavior as a result of nonanalyticity of the steady-state electron distribution function. One such system is a semiconductor in a strong static electric field, which is susceptible to spontaneous emission of optical phonons by electrons and thus gives rise to abrupt changes in slope (corners) on the electron energy distribution, at points corresponding to energies which are multiples of the energy of optical phonons.[4,8–10]

To simplify the problem and to arrange conditions more favorable for the onset of this nonlinearity, we consider a semiconductor in an electric field $\mathbf{E}_0$ and a quantizing magnetic field $\mathbf{H}$ parallel to the electric field. We assume that the electrons are scattered by acoustic and nonpolar optical phonons. If the electric field is strong, the steady-state electron distribution function will be very far from equilibrium, because two competing mechanisms will determine the behavior of the electrons: a high-energy electron runaway and a spontaneous emission of optical phonons by electrons.[4,9] The spontaneous emission of phonons prevents the electrons from reaching high energies, as they would in a quantizing magnetic field (because the electron spectrum is one-dimensional) if the electron scattering were quasielastic. As a result, sharp changes in slope appear on the distribution function.[4,9] Inoue *et al.*[10] have experimentally observed an electron energy distribution with a corner near the energy of an optical phonon in the absence of a magnetic field; they also studied it numerically, by the Monte Carlo method.

Below we consider the propagation of a high-frequency transverse wave $\mathbf{E}$ which is polarized linearly along the fields $\mathbf{E}_0$ and $\mathbf{H}$. We assume that the electrons occupy only the zeroth Landau level (this is the quantum limit). We also assume $\omega_c>\omega_{0,\mathbf{q}}$, where ω_c is the cyclotron frequency, and $\omega_{0\mathbf{q}}$ is the frequency of longitudinal optical phonons. Under these conditions the kinetic equation has the form of Eq. (2), and the corresponding distribution function has the form (5), with $\mathbf{E}=\{0,0,E\}$ and $\mathbf{p}=\{0,0,p_z\}$ (since the motion of an electron along the quantizing magnetic field is one-dimensional). In the case at hand, the quantizing magnetic field is manifested only in the coefficients of the expansion of $\partial J_z/\partial t$ in powers of E.

The steady-state distribution function $f(p_z)$ was derived in the quantum limit in Ref. 4 without consideration of the dispersion of optical phonons. As a result, infinite discontinuities occurred in the derivatives of the function $f(p_z)$ at the points corresponding to the emission of optical phonons (see also Ref. 9). When dispersion is incorporated, the discontinuities in the derivatives become finite. The distribution function $f(p_z)$ is derived in the Appendix for the case in which the frequency of an optical phonon is

$$\omega_{0\mathbf{q}}=\omega_0-\frac{\hbar\mathbf{q}^2}{2m_p}, \qquad (12)$$

where $\mathbf{q}$ and m_p are the wave vector and "mass" of the phonon. That derivation is carried out for strong fields $\mathbf{E}_0$, whose strength satisfies the condition[4]

$$E_0>E_h\equiv\frac{u_lH}{2c}\frac{l}{l_{ac}}\frac{\hbar\omega_c}{\bar{\varepsilon}}; \qquad (13)$$

here $\bar{\varepsilon}$ is the average energy of an electron, u_l is the sound velocity, l is the magnetic length, and l_{ac} is the mean free path of an electron with respect to scattering by longitudinal acoustic phonons at $\mathbf{H}=0$. The distribution function is

$$f(p_z)=\begin{cases} f_1(p_z), & |p_z|\leqslant p_0, \\ f_2(p_z), & |p_z|\geqslant p_0, \end{cases} \qquad (14)$$

where $p_0{}^2\equiv 2m\hbar\tilde{\omega}_0$, and $\tilde{\omega}_0=\omega_0(1-m/m_p)$ $(m/m_p\ll 1)$. The quantity $\hbar\tilde{\omega}_0$ is the energy at which an electron becomes capable of emitting an optical phonon. The function (14) is continuous at the points $\pm p_0$, but its derivatives are discontinuous; as a result, the behavior of an electromagnetic wave is nonlinear.

As we mentioned earlier, we assume that the strength of the field $\mathbf{E}$ is limited by the inequality (9). Expanding $\partial J_z/\partial t$ in E, and retaining terms of up to third order, we find a wave equation for a linearly polarized wave $\mathbf{E}(\mathbf{r},t)$:

$$\frac{\partial^2E}{\partial t^2}-\frac{c^2}{\varepsilon_0}\Delta E+\omega_p{}^2E=\alpha E\int_{-\infty}^{t}dt'E-\beta E\left(\int_{-\infty}^{t}dt'E\right)^2, \qquad (15)$$

$$\alpha=\frac{8\pi e^2\hbar\tilde{\omega}_0}{\varepsilon_0 m}g(\hbar\tilde{\omega}_0)\left[\varphi_2'(p_0)-\varphi_1'(p_0)\right], \qquad (16)$$

$$\beta=\frac{8\pi}{\varepsilon_0}\left(\frac{e^2\hbar\tilde{\omega}_0}{m}\right)^2g(\hbar\tilde{\omega}_0)\left[F_2''(\hbar\tilde{\omega}_0)-F_1''(\hbar\tilde{\omega}_0)\right], \qquad (17)$$

where $g(\varepsilon)$ is the density of states at the zeroth Landau level, $F(\varepsilon)$ and $\varphi(p_z)$ are the parts of the function $f(p_z)$ which are respectively even and odd in p_z, and the prime means differentiation with respect to the argument. For $|p_z|>p_0$, the functions F and φ vary more rapidly than they do for $|p_z|<p_0$, so the conditions $\alpha,\beta>0$ hold. In the case of a strong field $\mathbf{E}_0$ [see (13) and the Appendix], we find the following expressions for α and β:

$$\alpha=\frac{2\gamma\omega_p{}^2}{bb_1},\quad \beta=\frac{\alpha\gamma}{2}, \qquad (18)$$

where

$$\gamma=\frac{\pi^{1/2}|e|\mu}{2^{1/2}m^{1/2}\varepsilon_d{}^{1/2}},\quad \mu=\frac{E_{0p}}{E_0},\quad \varepsilon_d=\frac{\hbar^2l^{-2}}{m_p}\left(1-\frac{m}{m_p}\right). \qquad (19)$$

The energy ε_d in (18) and (19) is related to the dispersion of the optical phonons $(\varepsilon_d\ll\hbar\tilde{\omega}_0)$. The field E_{0p} is an upper limit on the external fields E_0 $(\mu^2\gg 1)$ at which the quantum limit holds. In addition, b and b_1 are ~ 1 (these quantities are defined in the Appendix). The derivation of expressions (18) allowed for the circumstance that for $E_0>E_h$ the quantities α and β are determined primarily by the derivatives of f_2, $|\varphi_2'(p_0)|\gg|\varphi_1'(p_0)|$ and $F_2''(\hbar\tilde{\omega}_0)\gg F_1''(\hbar\tilde{\omega}_0)$, in accordance with the inequalities $\varepsilon_d{}^{1/2}Z^2/\tilde{\omega}_0{}^{5/2}\ll 1$ and $\varepsilon_d/\tilde{\omega}_0\ll 1$, respectively, $(Z\equiv_{\bar{\varepsilon}}E_h/E_0)$.

4. EQUATION FOR THE ENVELOPE

The standard way to solve nonlinear wave equations like (15) is to write the wave in terms of a rapidly oscillating carrier and a slowly varying envelope (the spatial and temporal scales of the envelope are considerably larger than those of the carrier). The method of multiscale expansion is used to represent the wave in this manner.[11] In accordance with that method, we introduce slow variables $T_n=\varepsilon^nt$ and $\mathbf{R}_n=\varepsilon^n\mathbf{r}$, where $\mathbf{r}=\{x,y,z\}$, $\mathbf{R}_n=\{X_n,Y_n,Z_n\}$ and $\varepsilon\ll 1$, and we expand the field $E(\mathbf{r},t)$ in a power series in ε:

$$E=\varepsilon E_1+\varepsilon^2E_2+\ldots. \qquad (20)$$

Assuming that T_n and $\mathbf{R}_n$ are independent variables, we substitute expansion (20) into Eq. (15). Equating terms of a common order in ε, we then obtain a system of linear equations for E_n. We assume that the rapid variations of the wave occur only along the x axis. We can then write E_1 in the following form [the equation for E_1 is of the form (15), with vanishing right side]:

$$E_1=A(\mathbf{R}_1,T_1,\ldots)e^{i\vartheta}+\text{c.c.}, \qquad (21)$$

where A is a slowly varying complex amplitude, $\vartheta=kx-\omega t$, $\omega^2=\omega_p{}^2+c_m^2k^2$, $c_m{}^2=c^2/\varepsilon_0$, and c.c. means the complex conjugate. Substituting (21) into the ε^2 equation, we find that secular terms (i.e., terms which grow as $t\to\infty$) appear in this equation. Requiring that there be no secular term, we find an equation for A:

$$\left(v_g\frac{\partial}{\partial X_1}+\frac{\partial}{\partial T_1}\right)A=0, \qquad (22)$$

where $v_g=\partial\omega/\partial k=c_m{}^2k/\omega$ is the group velocity. From Eq. (22) we find

$$A=A(X,Y_1,Z_1,\mathbf{R}_2,T_2,\ldots), \qquad (23)$$

where $X=X_1-v_gT_1$. Now substituting (21) into the ε^3 equation, using (23), and again requiring that the secular terms be removed, we find our final equation for the slow dimensionless amplitude $\psi=A/E_d$:

$$i\left(\frac{\partial\psi}{\partial T_2}+v_g\frac{\partial\psi}{\partial X_2}\right)+\frac{v_g'}{2}\frac{\partial^2\psi}{\partial X^2}+\frac{v_g}{2k}\left(\frac{\partial^2\psi}{\partial Y_1{}^2}+\frac{\partial^2\psi}{\partial Z_1{}^2}\right)-2\Omega\psi|\psi|^2=0. \qquad (24)$$

Here $v_g'=\partial v_g/\partial k>0$, $\Omega>0$, and

$$\Omega=\frac{\pi\mu^2}{8bb_1}\frac{\omega_p{}^2}{\omega}\left(1-\frac{2}{3bb_1}\right),\quad E_d=\frac{m^{1/2}\varepsilon_d{}^{1/2}\omega}{|e|},$$

where E_d and Ω are a characteristic electric field and a characteristic frequency. The first term in the expression for Ω corresponds to the term in (15) which is cubic in E, and the second to that which is quadratic $(2/3bb_1<1)$. We also note that the integrals of $\exp\{\pm i\omega t\}$ in (15) are to be understood in the sense

$$\lim_{\delta\to 0}\int_{-\infty}^{t} d\tau \exp\{\pm i\omega\tau+\delta\tau\}.$$

Using the substitutions $T_2=\tau_2$ and $X_2=\xi_2+v_g\tau_2$, we can eliminate the variable X_2 from Eq. (24). It is also convenient to introduce the dimensionless variables $\tau=\tau_2\Omega$, $\xi=X/L$, $\eta=Y_1/L_\perp$ and $\zeta=Z_1/L_\perp$, where $L^2=v_g^2/2\Omega$ and $L_\perp^2=v_g/2k\Omega$. As a result, the nonlinear Schrödinger equation (24) becomes

$$i\frac{\partial\psi}{\partial\tau}+\frac{\partial^2\psi}{\partial\xi^2}+\Delta_\perp\psi-2\sigma\psi|\psi|^2=0, \tag{25}$$

where $\sigma=1$ and $\Delta_\perp=\partial^2/\partial\eta^2+\partial^2/\partial\zeta^2$. In the one-dimensional case $[\psi=\psi(\xi,\tau)]$, the signs of the dispersive and nonlinear terms in (25) correspond to stability of a monochromatic wave with respect to self-modulation or to the absence of envelope solitions for the most common boundary conditions: $\psi\to 0$ as $\xi\to\pm\infty$. For $|\psi|\to$const, however, solitons exist in the ordinary sense in the limits $\xi\to\pm\infty$ (i.e., they have a finite energy, they recover their shape after an interaction, etc.).[6,7]

5. CREATION AND PROPAGATION OF SOLITONS

We rewrite Eq. (25) for the one-dimensional case $(\psi=A/E_d)$,

$$i\frac{\partial\psi}{\partial\tau}+\frac{\partial^2\psi}{\partial\xi^2}-2\sigma\psi|\psi|^2=0, \tag{26}$$

and we impose the initial conditions

$$\psi|_{\tau=0}=\begin{cases}\psi_0 & |\xi|\leqslant\delta_0,\\ \psi_c, & |\xi|>\delta_0,\end{cases} \tag{27}$$

where $\sigma=1$, $\delta_0\equiv a/L$, $\psi_0\equiv A_0/E_d$, $\psi_c\equiv A_c/E_d$, and $0\leqslant\psi_0<\psi_c$. Conditions (27) constitute a dark pulse of length $2a$ in a coordinate system which is moving at the group velocity of the monochromatic wave, v_g. Since τ and ξ are slow variables, the length $2a$ and the duration of the dark pulse, $\tau_d=2a/v_g$, must be greater than the length $\lambda_c=2\pi/k$ and the period $T_c=2\pi/\omega$, respectively, of the monochromatic wave ($2a\gg\lambda_c$ and $\tau_d\gg T_c$). In a spatial interpretation of the variable τ, Eq. (26) with conditions (27) for $\psi_0=0$ describes a steady-state diffraction by a band in a nonlinear defocusing medium.[6]

Equation (26) was studied in detail in Refs. 6 and 7 for the case $\sigma>0$ by the inverse scattering method, so we will restrict the discussion here to some points concerning the experimental situation to which the initial conditions (27) correspond. We write the one-soliton solution of Eq. (26) with the boundary conditions $|\psi|\to\psi_c$ as $\xi\to\pm\infty$:

$$\psi(\xi,\tau)=|\psi(\theta)|e^{i\Phi(\theta)}, \tag{28}$$

$$|\psi(\theta)|^2=\psi_c^2(1-\nu^2\operatorname{sech}^2\theta), \tag{29}$$

$$\Phi(\theta)=\arcsin\frac{\sin 4\delta_0\lambda}{|\psi(\theta)|[1+\exp(2\theta)]}, \tag{30}$$

$$\theta=\nu(\xi-v_s\tau-\xi_c)=\frac{1}{L_s}(X-V_s\tau-X_c). \tag{31}$$

Here $v_s=2\lambda$ is the velocity of the solition ($V_s=\lambda u$, $u=2L\Omega$), ξ_c is the position of the center of the soliton at $\tau=0$ ($X_c=\xi_cL$), $L_s=L/\nu$ is the size of the soliton, the quantity $\nu^2=\psi_c^2-\lambda^2$ characterizes the amplitude of the soliton, and the parameter λ is an eigenvalue of the equation

$$\frac{\partial\varphi}{\partial\xi}=U(\lambda,\psi)\varphi, \tag{32}$$

where

$$\varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\quad U=i\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+i\begin{pmatrix}0 & -\bar\psi\\ \psi & 0\end{pmatrix} \tag{33}$$

for $\psi=\psi|_{\tau=0}$ (Ref. 12). If $\psi|_{\tau=0}$ is given by (27), it is a simple matter to show that we have

$$\varphi=C_1e^{i\zeta\xi}\begin{pmatrix}1\\ \mu_0\end{pmatrix}+C_2e^{-i\zeta\xi}\begin{pmatrix}\mu_0\\ 1\end{pmatrix},\quad |\xi|<\delta_0, \tag{34}$$

$$\varphi=C_3e^{\nu\xi}\begin{pmatrix}1\\ \mu_c\end{pmatrix},\quad \xi<-\delta_0, \tag{35}$$

$$\varphi=C_4e^{-\nu\xi}\begin{pmatrix}\mu_c\\ 1\end{pmatrix},\quad \xi>\delta_0, \tag{36}$$

where $\zeta=(\lambda^2-\psi_0^2)^{1/2}$, $\mu_0=(\lambda-\zeta)/\psi_0$ and $\mu_c=(\lambda+i\nu)/\psi_c$. Joining the function φ at the points $\xi=\pm\delta_0$, we find an equation for λ:

$$\frac{(\mu_c-\mu_0)^2}{1-\mu_0\mu_c(2-\mu_c)}=\exp(4i\nu\delta_0). \tag{37}$$

In the case $\psi_0=0$ (Ref. 6), this equation takes the simpler form

$$\cos\left(2\delta_0\frac{\lambda}{\psi_c}\right)=\pm\frac{\lambda}{\psi_c}. \tag{38}$$

According to (38), under the condition $\delta_0=(a/L)\gg 1$, about $N=4\delta_0/\pi$ pairs of solitons are created. In the case $\delta_0\ll 1$, we have one pair of solitons:

$$\lambda_{1,2}\approx\pm\psi_c\frac{(1+8\delta_0^2)^{1/2}-1}{4\delta_0^2}. \tag{39}$$

Their velocities ($v_{sj}=2\lambda_j$, $j=1,2$) are close to the limit, while their amplitudes are small (in the limit $\delta_0\to 0$ we have $|v_{sj}|\to 2\psi_c$, and $\nu_j\to 0$).

6. DISCUSSION AND ESTIMATES

We have found a mechanism which makes the current a nonlinear function of a high-frequency electric field [see condition (4)], and we have derived a nonlinear wave equation for a semiconductor in a static electric field and a quantizing magnetic field parallel to the electric field. A slightly different nonlinearity mechanism was proposed in Refs. 2 and 3, where a study was made of the propagation of electromagnetic waves in a medium with a model elastic-collision operator: $\nu(v^2)=0$ for $v^2<v_0^2$ and $\nu(v^2)=\infty$ for $v^2>v_0^2$, where $\mathbf{v}$ is the velocity of an electron, and v_0 is a velocity which determines the differences in the nature of the collisions for high- and low-energy electrons.[13] In that model, the distribution function is found to be Maxwellian[2,3] for $v^2>v_0^2$, and those terms in the expression for the current which are nonlinear in the field arise essentially because of the limitation imposed on the range of integration by the quantity v_0.

It was suggested in Refs. 2 and 3 that this model applies to electrons in gases which exhibit a strong Ramsauer effect, as was assumed in Ref. 13, where this collisional model was proposed for studying echo effects in plasmas. It can be seen from the experimental dependence $\nu(v^2)$ given in Ref. 13 that the choice of v_0 is exceedingly arbitrary and that the $\nu(v^2)$ model could hardly apply in cases in which derivatives of the distribution function play an important role [in Ref. 13, for example, v_0 was chosen in a region in which the function $\nu(v^2)$ increases monotonically with increasing v].

It is also important to note here that the sign of the nonlinear terms in wave equation (15) [and, in particular, the sign of σ in Eq. (26)] is determined by the sign of the difference between the derivatives of the distribution function to the left and right of the point at which they are discontinuous [see (15)–(17)]. It is clear, for example, that the nature of the nonlinear behavior of the wave in the nonlinear Schrödinger equation (26) depends strongly on the sign of σ ($\sigma<0$ in Ref. 3).

In the preceding section we discussed the one-dimensional nonlinear Schrödinger equation, so the stability of the solitons with respect to transverse perturbations remained an open question. We simply note that this problem was taken up in Ref. 14 for Eq. (25) in the case $\sigma<0$, but the case $\sigma>0$ requires separate study.

The results of this study are of interest from the standpoint of nonequilibrium kinetics and also from the standpoint of the nonlinear optics of semiconductors. The wave equation (15) (and the corresponding envelope solitons) constitute only one example of a nonlinear behavior of an electromagnetic wave. As other examples we might cite nonlinear diffraction and higher-harmonic generation.

To get an idea of the conditions under which one might observe nonlinear propagation of electromagnetic waves in a semiconductor, we consider some estimates based on the following parameter values (which correspond to InSb) of the semiconductor: $m=0.017m_e$, $\hbar\omega_0=2.4\cdot10^{-3}$ eV, $m_p=10^4 m$, $n=10^{16}$ cm^{-3}, and $\varepsilon_0=16$, where m_e is the mass of a free electron, and $m_p\sim\hbar\omega_0/u_l^2$. We take the frequency and the wavelength in vacuum to be $\omega_v=1.74\cdot10^{13}$ rad/s and $\lambda_v=10.81$ μm, respectively (a CO_2 laser); we assume a magnetic field $H=100$ Oe (for InSb, strong-quantization conditions hold at $H>20$ kOe); we take the static electric field E_0 to have a strength such that the condition $\mu=5$ holds [$E_h\ll E_0\ll E_{op}$; for polar scattering we would have (Ref. 9) $E_{op}\sim500$ V/cm]; and we assume $T=20$–80 K. The typical parameter values of this system are then $\omega_p=1.54\cdot10^{13}$ rad/s, $\omega_c=2.8\omega_0$, $l=8.12\cdot10^{-7}$ cm, $E_h\sim1$ V/cm [see (13)], $bb_1=3.4$ ($E_0=100$ V/cm), $\varepsilon_d=2.8\cdot10^{-4}\hbar\omega_0$, $E_d=1.4$ kV/cm, $\Omega=3\cdot10^{12}$ s^{-1}, $L=2\cdot10^{-5}$ cm, $v_g\approx c_m$, and $u=1.2\cdot10^8$ cm/s. For a dark pulse as in (27) ($\psi_0=0$) of duration $\tau_d=10$ ps ($2a=7.5\cdot10^{-3}$ cm), about 250 pairs of solitons would be produced [$N\approx4a/(\pi L)$].

It can be seen from Eq. (38) that small values of λ/ψ_c correspond to low-velocity solitons and that their velocities are determined approximately by the zeros of the left side of (38). We find a smallest wavelength $|\lambda|\equiv\lambda_{\min}\approx\psi_c/N$, for the solitons, while their lowest velocity is $V_{\min}=\lambda_{\min}u\approx5\cdot10^5$ cm/s for $\psi_c=1$ ($\psi_c\equiv A_c/E_d$). Finally, we note that the number of pairs of solitons, N, and their velocities $V_i=\lambda_i u$ ($i=1, 2,\ldots,2N$) can vary over wide ranges, depending on the values of L, Ω, and ψ_c, which in turn depend on the parameters of the semiconductor and of the external fields.

APPENDIX

The electron distribution function was derived in Ref. 4 for a semiconductor in an electric field $\mathbf{E}_0$ and a quantizing magnetic field $\mathbf{H}$ parallel to the electric field for the case in which the electrons were scattered by acoustic and nonpolar optical phonons. In this Appendix, we incorporate the dispersion of optical phonons [their frequency is given by (12)]. Dispersion must be taken into account so that the discontinuities in the derivatives of the distribution function $f(p_z)$ at those points in momentum space at which optical phonons are emitted will remain finite. In this situation, the phonon dispersion makes a negligible contribution to the electrical conductivity[4] and also to the linear response and to current fluctuations,[15] since the corresponding integrals converge and the relative number of electrons with an energy above that of an optical phonon is small.

As in Ref. 4, we distinguish two regions of electron energies: (1) $\varepsilon<\hbar\omega_{0\mathbf{q}}$ and (2) $\varepsilon\geqslant\hbar\omega_{0\mathbf{q}}$. We are assuming here that scattering by acoustic phonons dominates in the first region, and scattering by optical phonons in the second. If $\omega_{0\mathbf{q}}=\omega_0$, derivatives of the distribution function do not exist in the limit $\varepsilon\to\hbar\omega_0+0$. To make the discontinuities in the derivatives finite, it is sufficient to incorporate the dispersion of the optical phonons in a δ-function describing energy conservation in the collision integral representing the collisions of electrons with optical phonons, for the term responsible for the transition of an electron with energy ε out of the second region. The kinetic equation for $\varepsilon\geqslant\hbar\omega k_{0\mathbf{q}}$ becomes

$$\frac{\partial f(p_z)}{\partial p_z}=\mu\frac{\pi^{1/2}}{p_d}e^r[1-\Phi(r^{1/2})]f(p_z). \tag{A1}$$

Here $\Phi(x)$ is the error function,

$$r=\frac{p_z^2-p_0^2}{p_d^2},\quad p_0^2=2m\hbar\tilde{\omega}_0,\quad \tilde{\omega}_0=\omega_0\left(1-\frac{m}{m_p}\right),$$

$$p_d^2=\frac{\hbar^2}{l^2}\frac{2m}{m_p}\left(1-\frac{m}{m_p}\right),\quad \mu=\frac{E_{op}}{E_0},$$

$$E_{op}=\frac{\hbar\omega_c}{|e|L_{op}},\quad L_{op}=\frac{2\pi\hbar^3\rho\omega_0}{m^2D_t^2K^2},$$

ρ is the density of the crystal, D_t is the constant of the interaction of electrons with nonpolar optical phonons, K is the reciprocal-lattice constant, and $\Phi(x)\to1$ as $x\to\infty$ and $\Phi(x)\to2x/\pi^{1/2}$ as $x\to0$. At $p_z>p_0$, Eq. (A1) has a growing solution. Since electrons are moved out of this region by the field $\mathbf{E}_0$, we assume $f(p_z)=0$ at $p_z\geqslant p_0$. For $p_z\leqslant-p_0$ we have

$$f(p_z)\approx f(-p_0)\begin{cases}\exp\left[\dfrac{\pi^{1/2}\mu}{p_d}(p_0+p_z)\right], & r^{1/2}\ll1, \quad \text{(A2)}\\[2ex] \dfrac{C_d}{p_0^{\mu}}|p_z+(p_z^2-p_0^2)^{1/2}|^{\mu}, & r^{1/2}\gg1, \quad \text{(A3)}\end{cases}$$

where the constant $f(-p_0)$ is found from the condition that the function $f(p_z)$ be continuous at the point $p_z=-p_0$ (Ref. 4), and C_d can be found by joining (A2) and (A3) [$C_d\approx\exp(-\pi^{1/2}\mu)$]. If the phonon dispersion is ignored, expression (A3) is an exact solution of Eq. (A1) (Ref. 9). The function (A3) must fall off fairly rapidly with increasing ε if we are to restrict the discussion to the quantum limit.

To satisfy this requirement, we impose the following inequality, which places an upper limit on the field E_0:

$$\mu^2 \gg 1. \quad (A4)$$

We also write the function $f(p_z) \equiv F(\varepsilon) + \varphi(p_z)$ in the region $|p_z| \leqslant p_0$ for the case of a strong field $\mathbf{E}_0$ (Ref. 4):

$$F(\varepsilon) = C\left[1 - \frac{\delta}{b}\ln\left(1 + \frac{\varepsilon^2}{Z^2}\right)\right], \quad (A5)$$

$$\varphi(p_z) = -eE_0\tau(p_z)\frac{\partial F(\varepsilon)}{\partial p_z}. \quad (A6)$$

Here C is a normalization constant,

$$\delta = \frac{E_{ac}}{E_0}, \quad E_{ac} = \frac{\hbar\omega_c}{4|e|l_{ac}} \quad l_{ac} = \frac{\pi\hbar^4\rho u_l^2}{m^2E_D^2T},$$

$$\tau(p_z) = \frac{(2m)^{1/2}l_{ac}}{\hbar\omega_c}\varepsilon^{1/2},$$

$$b = 1 + \delta\ln\left[1 + \left(\frac{\hbar\tilde{\omega}_0}{Z}\right)^2\right], \quad Z = \frac{\hbar\omega_c}{2}\frac{u_lH}{cE_0}\frac{l}{l_{ac}},$$

$$C = \frac{n}{b_1g(\hbar\tilde{\omega}_0)\hbar\tilde{\omega}_0},$$

$$b_1 = \frac{2(1+4\delta)}{b}, \quad f(-p_0) = 2F(\hbar\tilde{\omega}_0) = \frac{2C}{b},$$

T is the lattice temperature, $g(\varepsilon)$ is the density of states at the zeroth Landau level, E_D is the strain-energy constant, and u_l is the sound velocity. The region of strong fields $\mathbf{E}_0$ is determined by the inequality (13) or

$$Z^2 \ll \bar{\varepsilon}^2. \quad (A7)$$

We have also noted that the function $F(\omega)$ falls off fairly rapidly in the second region, over a distance $\varepsilon_1 = \hbar\tilde{\omega}_0/\mu^2$, and that the contribution of the region $\varepsilon > \hbar\tilde{\omega}$ to the normalization is small. The distribution function $f(p_z)$, given by (A2), (A3), (A5), and (A6), makes it possible (in particular) to derive expressions (18) for the coefficients α and β in wave equation (15).

We conclude this Appendix with a brief discussion of the analytic properties of the electron distribution function. As we mentioned earlier, the interaction of electrons with dispersionless optical phonons gives rise to a corner on the distribution function and to infinite discontinuities in its derivatives for $\varepsilon = \hbar\omega_0$ (Refs. 4 and 9). There is accordingly the question of the extent to which the physical system will actually exhibit this new property of the solution of the kinetic equation. In other words, how would the corner be affected by an additional account of collisional, quantum, etc., effects?

It is clearly difficult to answer this question fully, so we will point out two specific mechanisms which affect the nature of the corner. One of them—the dispersion of optical phonons—was described above. The other is the collisional broadening associated with a scattering of electrons by acoustic phonons. This broadening was discussed in Ref. 16 in a study of the energy distribution of photoexcited electrons in a quantizing magnetic field. While the two mechanisms differ in essential ways, they act in the same direction: They lower the energy at which the emission of an optical phonon becomes possible (in the case at hand, $\omega_0 \to \tilde{\omega}_0$), and they reduce the size of the discontinuity in the derivatives of the distribution function. On the other hand, there is no qualitative change in the nature of the discontinuity (the corner is simply "renormalized"). These two examples of course do not exhaust the topic, but their analysis suggests that the existence of a "nonanalytic" mechanism will not be qualitatively affected (neutralized) by other, "analytic" mechanisms.

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