Nonlinear modulated waves in dispersive hydrodynamics

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The method of slow modulations is used to study nonstationary flows in dispersive hydrodynamics. The modulation equations are presented in a new form having a clear physical meaning and easy to integrate. Analytic solutions are obtained for the asymptotic modulation-theory regimes of nonlinear wave generation and soliton formation. The modulation equations for Korteweg–de Vries hydrodynamics are considered, as are also equation systems for plasma ion-sound and magnetosonic waves having no Riemann invariants.

1. INTRODUCTION

Investigation of nonlinear stationary flows in dispersive hydrodynamics is not only important for many problems of rarefied-plasma dynamics, hydrodynamics of waves on water, nonlinear waves in dielectrics, electroacoustic waves, and others, but is also in itself of theoretical interest. Indeed, neglecting small dispersive terms, the flow is described, as in ordinary hydrodynamics, by the Euler equations. The most important feature of Euler dynamics is phase conjugation, which leads to the appearance of a shock wave (Ref. 1, Sec. 101). In ordinary hydrodynamics the major role is played here by dissipative processes that lead to establishment of a shock-wave front of finite width.

In dispersive hydrodynamics with total absence of dissipation, the motion following the onset of a singularity has principally a different character.[2] What appears here is a region that expands continuously with time and is filled with undamped small-scale nonlinear oscillations. It is called a nondissipative shock wave (NSW).[3] The NSW region can be described by using Whitham's method[4] based on averaging, over the oscillations, the integrals of the initial equations. The resultant system of averaged equations is quite complicated. It can be considerably simplified if Riemann invariants exist, when simple analytic solutions that describe the NSW can be obtained.[3,5,6] By now, Riemann invariants have been found for the system of averaged equations corresponding to the initial Korteweg–de Vries (K-dV) equations,[4] the nonlinear Schrödinger equation (NSE),[7] the sine-Gordon equations, and related ones. The general question of the connection between the existence of Riemann invariants of averaged equations and total integrability of the initial equation has been raised in Ref. 8.

It is clear, however, that Riemann invariants exist only for a rather limited class of nonlinear equations with dispersion. Considerable interest attaches therefore to an investigation of nonlinear oscillations in dispersive hydrodynamics in the general case. This is the subject of the present paper.

The investigation is greatly facilitated by the fact that the averaged equations can be reduced to a form that has a clear physical meaning and is easy to interpret. This is done in Sec. 2, where the averaged equations are represented in the form of a hydrodynamic system of modulation equations for the parameters of the motion of the medium and of the excited waves, viz., the average density n, the average flow velocity v, the energy density A^2 of the oscillations, and the wave density (i.e., the wave number) k. The modulation equations are equivalent to Whitham's corresponding system[4] but are substantially simpler.

The solutions of the averaged equations must be joined to the solutions of the Euler equations on the boundaries of the region of the nonlinear oscillations, since the latter equations are valid as before outside the NSW region. It is important here that the number of averaged equations always exceeds that of the Euler equations. The junction points are therefore singular points—points where the characteristics merge. It follows hence that solution of these equations—even numerical—calls for an analytic investigation capable of determining the behavior of the solution in the vicinities of the singular points. This is the subject of Secs. 3 and 4. The modulation equations are joined here with the Euler equations in the following manner (in accordance with Refs. 3, 5, and 6). The oscillation amplitude vanishes, $A\to 0$, on one of the NSW boundaries. The solution of the modulation equations near this boundary comprises, in fact, the general theory of nonlinear generation of waves in dispersive hydrodynamics (Sec. 3). The wave number vanishes, $k\to 0$, on the other boundary. This is the NSW soliton front, whose structure is studied in Sec. 4.

2. MODULATION EQUATIONS

a. Hydrodynamic form of modulation equations

According to Ref. 4, to obtain the averaged equations it is necessary to represent the initial system in the form of conservation law, i.e., in divergent form. A completely integrable system has an infinite number of conservation laws. In the general case considered here, however, the number of conservation laws is restricted. For example, only four integrals each can be obtained for systems describing ion-sound waves in a nonisothermal plasma or else magnetosonic waves in a cold plasma moving across a magnetic field. This corresponds exactly to the differential order of the initial system and is equal to the number of unknown functions in the system of averaged equations describing slow modulation. The modulation system turns out therefore to be closed.

To obtain the modulation system we must average the initial equations, represented in conservative form, over the period of the stationary wave. A particularly important problem here is the choice of the slowly varying variables. Recognizing that in NSW problems the modulation equations must be joined to the Euler equations on the boundaries of the oscillating region, it is convenient to choose the de-

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sired functions to be the ordinary hydrodynamic variables: the average density n, the average velocity v, and also the parameter A^2 having the meaning of the oscillation energy density, and the wave number (wave density) k. After transformations which will be illustrated below with specific examples, it is possible to represent the modulation equations in the following hydrodynamic form:

$$\frac{\partial n}{\partial t}+\frac{\partial (nv)}{\partial x}+\frac{\partial A^2}{\partial x}=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{n}\frac{\partial P}{\partial x}+\frac{\partial \mathscr{E}^2}{\partial x}=0, \tag{2}$$

$$\frac{\partial A^2}{\partial t}+\frac{\partial}{\partial x}(VA^2)+A^2\frac{\partial v}{\partial x}+\mathscr{E}^2\frac{\partial n}{\partial x}=0, \tag{3}$$

$$\frac{\partial k}{\partial t}+\frac{\partial (kU)}{\partial x}=0. \tag{4}$$

We have introduced here the functions

$$P=P(n),\ \mathscr{E}^2=\mathscr{E}^2(n,k,A^2),\ V=v+V_1(n,k,A^2),$$
$$U=v+U_1(n,k,A^2). \tag{5}$$

They will be expressed in detail below.

The system (1)–(5) has a lucid physical meaning. The first two equations are the Euler equations of the hydrodynamics of an ideal liquid [$P(n)$ is the usual pressure]. They contain the additional terms $\partial A^2/\partial x$ and $\partial\mathscr{E}^2/\partial x$ that determine the influence of the excited oscillations on the density and velocity of the hydrodynamic flow. The terms $A^2\partial v/\partial x$ and $\mathscr{E}^2\partial n/\partial x$ in the energy-transport equation (3) for the nonlinear waves indicate the influence exerted on the oscillations by changes of the hydrodynamic-flow parameters. Finally, Eq. (4) is the conservation law for the number of waves, and is always satisfied if the oscillations are adiabatic and single-phase.

In the linear limit $\mathscr{E}^2\sim A^2\sim a^2\to 0$ (where a is the amplitude of the linear oscillations), U becomes the usual phase velocity and V becomes the group velocity V_0 of the linear waves. The hydrodynamic equations (1) and (2) coincide then automatically with the Euler equations, and Eq. (3) goes over into the oscillation-energy transport equation known from the linear theory:

$$\frac{\partial a^2}{\partial t}+\frac{\partial}{\partial x}(V_0a^2)=0. \tag{6}$$

b. K–dV modulation equation

We consider the procedure for obtaining modulation equations in hydrodynamic form, using the K–dV equations as the example. Although the K–dV equation is integrable, it is expedient to consider first just this example, since the equations obtained in this case are simple enough but contain many features of more complicated systems. The important difference from the general hydrodynamic case is that the K–dV modulation system consists of three equations, in accord with the differential order of the initial equation.

Thus, the K–dV equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0 \tag{7}$$

reduces for stationary waves $u(x-Ut)$ to the form

$$\left(\frac{du}{d\xi}\right)^2={}^1/_3Q(u,C,B,U),\quad \xi=x-Ut,$$
$$Q(u,C,B,U)=-u^3+3Uu^2+6Bu-6C, \tag{8}$$

where C, B, and U are arbitrary constants, U having the meaning of the phase velocity.

It is convenient to use as the intermediate equations the roots $b_1\geqslant b_2\geqslant b_3$ of the polynomial $Q(u)$:

$$Q(u,C,B,U)=(b_1-u)(b_2-u)(b_3-u).$$

Then

$$U={}^1/_3(b_1+b_2+b_3),\quad B=-{}^1/_6(b_1b_2+b_1b_3+b_2b_3),$$
$$C=-{}^1/_6b_1b_2b_3. \tag{9}$$

The general solution of (8) takes the form[3]

$$u(x,t)=\frac{2a}{m}dn^2\left(\left(\frac{a}{6m}\right)^{1/2}(x-Ut),m\right)+b_3. \tag{10}$$

where $a=(b_1-b_2)/2$ is the amplitude of the oscillations; $m=(b_1-b_2)/(b_1-b_3)$ is the modulus of the elliptic function ($0\leqslant m\leqslant 1$).

The phase velocity U and the wave number k are expressed in terms of the introduced variables a, m, and b_3 by the equations

$$U=2a\frac{2-m}{3m}+b_3, \tag{11}$$

$$k=2\pi\left(\oint({}^1/_3Q(u))^{-1/2}du\right)^{-1}=\frac{\pi}{K(m)}\left(\frac{a}{6m}\right)^{1/2}, \tag{12}$$

where $K(m)$ is a complete elliptic integral of the first kind.

Let us write down for the K–dV equation the first three conservation laws which we need to obtain the modulation system:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(u^2/2+u_{xx})=0, \tag{13}$$

$$\frac{\partial}{\partial t}(u^2/2)+\frac{\partial}{\partial x}(u^3/3+uu_{xx}-u_x^2/2)=0, \tag{14}$$

$$\frac{\partial}{\partial t}(u^3/3-u_x^2)+\frac{\partial}{\partial x}(u^4+u^2u_{xx}+2u_tu_x+u_{xx}^2)=0. \tag{15}$$

Averaging Eq. (13) over the period of the stationary wave (10), we get

$$\frac{\partial\bar u}{\partial t}+\frac{\partial}{\partial x}(\bar u^2/2+A^2)=0, \tag{16}$$

where $A^2=(\overline{u^2}-\bar u^2)/2$ is the energy density of the oscillations. Averaging of the second integral yields

$$\frac{\partial}{\partial t}\left(\frac{\overline{u^2}}{2}\right)+\frac{\partial}{\partial x}\left(C+\frac{1}{2}U\overline{u^2}\right)=0. \tag{17}$$

We express the variables C, U, and $\overline{u^2}$ in terms of a, m, and $\eta\equiv\bar u$ with the aid of (9)–(11). According to (10),

$$\eta=b_3+\frac{2a}{m}\mu(m), \tag{18}$$

where $\mu(m)=E(m)/K(m)$, and $E(m)$ is a complete elliptic integral of the second kind. We have then for b_1, b_2, and b_3

$$b_1=\eta+\frac{2a}{m}(1-\mu),\qquad b^2=\eta+\frac{2a}{m}(1-m-\mu),\qquad b_3=\eta-\frac{2a}{m}\mu. \tag{19}$$

From (11) we obtain

$$U=\eta+\frac{2a}{m}\left(\frac{2-m}{3}-\mu\right). \tag{20}$$

Equation (9) yields

$$C=-\frac{1}{6}\left(\left(\eta-\frac{2a}{m}\mu\right)^3+\left(\eta-\frac{2a}{m}\mu\right)^2\frac{2a}{m}(2-m)\cdot+\frac{4a}{m^2}(1-m)\left(\eta-\frac{2a}{m}\mu\right)\right). \tag{21}$$

Finally,

$$\overline{u^2}=\eta^2+\frac{4a^2}{3m^2}(m-1+2(2-m)\mu-3\mu^2). \tag{22}$$

Then

$$A^2=\frac{2a^2}{3m^2}(m-1+2(2-m)\mu-3\mu^2). \tag{23}$$

Subtracting from (17) the product of (16) by $\bar u$ we have, taking (20)–(23) into account, an equation for A^2:

$$\frac{\partial A^2}{\partial t}+\frac{\partial}{\partial x}(VA^2)+A^2\frac{\partial\eta}{\partial x}=0,$$

where

$$V=\eta+\frac{C+{}^1/_2\overline{u^2}-{}^1/_3\eta^3-2\eta A^2}{A^2}=\eta+\frac{4a}{m}\left(\frac{1}{3}-\frac{m}{6}-\mu-\mu^2\frac{1-m/2-\mu}{m-1+2(2-m)-3\mu^2}\right). \tag{24}$$

It is convenient to choose in place of the third averaged conservation law the wave-number conservation law, which is a consequence of the three averaged equations (13)–(15) and is always valid under quasistationary conditions.

The K–dV modulation system takes ultimately the form

$$\frac{\partial\eta}{\partial t}+\eta\frac{\partial\eta}{\partial x}+\frac{\partial A^2}{\partial x}=0,$$
$$\frac{\partial A^2}{\partial t}+\frac{\partial}{\partial x}(VA^2)+A^2\frac{\partial\eta}{\partial x}=0, \tag{25}$$
$$\frac{\partial k}{\partial t}+\frac{\partial(kU)}{\partial x}=0.$$

The variables V and U are expressed here in parametric form in terms of A^2, η, and k by Eqs. (12), (20), (23), and (24).

c. Nonlinear ion-sound waves in a nonisothermal plasma

As an example of a system whose with non-Riemannian modulation equations we consider the equations describing nonlinear flow in a two-temperature ($T_e\gg T_i$) plasma[2,9]

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0,\qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial x}=0,$$
$$\frac{\partial^2\varphi}{\partial x^2}=e^\varphi-\rho. \tag{26}$$

Here ρ is the ion density, u the ion hydrodynamic velocity, and φ the electric potential; all the variables are made nondimensional by normalization to their characteristic values.

The conservation laws of the system (26) are

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0,$$
$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(u^2/2+\varphi)=0,$$
$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+e^\varphi-\varphi_x^2/2)=0, \tag{27}$$
$$\frac{\partial}{\partial t}(\rho u^2/2-e^\varphi+\varphi e^\varphi+\varphi_x^2/2)+\frac{\partial}{\partial x}(\rho u^3/2+\rho u\varphi-\varphi\varphi_{xt})=0.$$

Averaging the integrals of (27) over the period of the stationary wave[10] and introducing the variables $n\equiv\bar\rho$, $v\equiv\bar u$, k, and $A^2=\overline{\rho u}-\bar\rho\bar u$ we arrive at the system (1)–(5) with $P(n)=n$, in which the dependence of the coefficients on the sought variables is determined in parametric form by the equations

$$\mathscr{E}^2=\overline{{}^1/_2u^2+\varphi}-{}^1/_2\bar u^2-\ln\bar\rho=\ln(\alpha f_{-1}{}^{-1})-f_{+1}{}^2/2,$$
$$U=v+f_{+1}, \tag{28}$$
$$V=v+f_{+1}-\frac{f_{-1}{}^{-1}(f_{+1}+\beta)+\ln(\alpha f_{-1}{}^{-1})+1-f_{+1}{}^2/2}{f_{+1}-f_{-1}{}^{-1}},$$

where

$$f_{\pm1}(\alpha,\beta)=\frac{\oint(-2\psi)^{\pm1/2}Q^{-1/2}(\psi)\,d\psi}{2\pi f_0(\alpha,\beta)},$$
$$f_0(\alpha,\beta)=\frac{1}{2\pi}\oint Q^{-1/2}(\psi)\,d\psi, \tag{29}$$
$$Q(\psi)=(-2\psi)^{1/2}+\alpha e^\psi+\beta.$$

The variables α and β are connected with n, k, and A^2 by the expressions

$$A^2=n(f_{+1}-f_{-1}{}^{-1}),\qquad k=(2n)^{1/2}f_{-1}^{-1/2}/f_0. \tag{30}$$

As before, the wave-number conservation law, which is the consequence of the four averaged conservation laws (27), has been introduced in place of the fourth averaged law.

d. Nonlinear dynamics of a cold magnetized plasma (magnetic sound)

Nonlinear hydrodynamic motions of a cold rarefied plasma across a magnetic field are described by the system[4,9]

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0,\qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{B}{\rho}\frac{\partial B}{\partial x}=0,$$
$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial B}{\partial x}\right)=B-\rho. \tag{31}$$

Here ρ and u are the dimensionless hydrodynamic density and velocity, and B is the magnetic field intensity.

The conservation laws for the system (31) are[4]

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0,$$

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(u^2/2+B+\left(\frac{1}{\rho}B_x\right)^2\Big/2\right)=0,$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+B^2/2)=0,$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+\frac{\rho}{2}\left(\frac{B_x}{\rho}\right)^2+\frac{1}{2}B^2\right)$$

$$+\frac{\partial}{\partial x}\left(\frac{1}{2}\rho u^3+\frac{1}{2}\rho u\left(\frac{B_x}{\rho}\right)^2\right.$$

$$\left.+B\left(\rho u-\left(\frac{B_x}{\rho}\right)_t\right)\right)=0. \quad (32)$$

These equations, averaged over the period of the stationary wave,[4] lead to the system (1)–(5) for the variables n, v, A^2, and k. In this case

$$P(n)={}^1/_2n^2 \quad \mathscr{E}^2=\frac{1}{2}\left(\left(\frac{\alpha}{\beta}\right)^2+\gamma\right)-\beta^2 f_{-1}-\frac{1}{2\beta^2}f_{+1}^2,$$

$$U=v+f_{+1}/\beta, \quad (33)$$

$$V=v+\frac{2}{\beta}f_{+1}+\frac{\alpha-{}^1/_2f_{-1}(\alpha^2+\beta^2(\gamma+f_{-1})+f_{+1}^2)}{\beta(f_{-1}f_{+1}-1)}.$$

Here

$$f_{\pm1}(\alpha,\beta,\gamma)=\frac{\oint(\alpha-\psi^2/2)^{\pm1}Q^{-1/2}(\psi)\,d\psi}{2\pi f_0(\alpha,\beta,\gamma)}$$

$$f_0(\alpha,\beta,\gamma)=\frac{1}{2\pi}\oint Q^{-1/2}(\psi)\,d\psi, \quad (34)$$

$$Q(\psi)=\frac{-\psi^4+4\alpha\psi^2-8\beta^2\psi+4\beta\gamma}{(\alpha-\psi^2/2)^2}.$$

The variables α, β, and γ are connected with the sought n, A^2, and k by the equations

$$k=\beta/f_0,\quad n=\beta^2 f_{-1},\quad A^2=\beta(f_{-1}f_{+1}-1). \quad (35)$$

3. NONLINEAR GENERATION OF MODULATED WAVES

a. Linear waves above a hydrodynamic background

In the limit of infinitesimally small amplitudes ($A\to0$, $\mathscr{E}\sim A$) the thermodynamic equations (1) and (2) are decoupled and the modulation system takes the form

$$\frac{\partial n}{\partial t}+\frac{\partial(nv)}{\partial x}=0,\quad \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{n}\frac{\partial P(n)}{\partial x}=0, \quad (36)$$

$$\frac{\partial A^2}{\partial t}+\frac{\partial}{\partial x}(V_0A^2)=0,\quad \frac{\partial k}{\partial t}+\frac{\partial \omega_0}{\partial x}=0. \quad (37)$$

Here $\omega_0=kU_0=\omega_0(k,n,v)$ is the linear dispersion relation, and $V_0=\partial\omega_0/\partial k$ is the linear group velocity. It is important that the changed hydrodynamic variables enter in the equation for the wave quantities A and k; the equations for n and v, on the other hand, are solved independently. For constant n and v we have from (37) the usual system for a linear wave packet in a homogeneous medium. Equations (36) and (37) describe thus infinitesimal-amplitude oscillations accompanying hydrodynamic flows in a dispersive medium.

The transition to the system (36) and (37) was made under the assumption that $\partial n/\partial x$ and $\partial v/\partial x$ are small. In a situation close to phase conjugation, however, this assumption is violated and the terms $A^2\partial v/\partial x$ and $\mathscr{E}^2\partial n/\partial x$ should be taken into account in Eq. (3) for A^2, which takes the form

$$\frac{\partial A^2}{\partial t}+\frac{\partial}{\partial x}(V_0A^2)+A^2\left(\frac{\partial v}{\partial x}+\left(\frac{\partial\mathscr{E}^2}{\partial A^2}\right)_{A=0}\frac{\partial n}{\partial x}\right)=0. \quad (38)$$

It follows from (38) that in the region of a compression wave ($\partial n/\partial x<0$, $\partial v/\partial x<0$) an increase takes place in the oscillation amplitude and can be interpreted as instability development with a growth rate proportional to the moduli of the density and velocity gradients. Development of this instability produces in the nonlinear stage a nondissipative shock wave (NSW) described by the complete system (1)–(5). It should be noted that the stability of the toppling hydrodynamic profile is determined by the sign of the sum

$$\frac{\partial v}{\partial x}+\left(\frac{\partial\mathscr{E}^2}{\partial A^2}\right)_{A=0}\frac{\partial n}{\partial x}.$$

In the case of a hydrodynamic simple wave defined by constancy of one of the Riemann invariants,[1] the density and velocity gradients are of the same sign, so that phase conjugation of such a wave in a dispersive medium always leads to instability development. In the general hydrodynamic situation, however, when $\partial n/\partial x$ and $\partial v/\partial x$ are independent, phase conjugation of the compression wave need not be accompanied by a growth of the amplitude of the linear oscillations. We present in conclusion expressions for $\delta=(\partial\mathscr{E}^2/\partial A^2)_{A=0}$ in the case of ion-sound (see the Appendix) and magnetosonic waves, obtained from the small-amplitude expansions of $\mathscr{E}^2(A^2,n,k)$ [see (28)–(30), (33)–(35)]:

$$\delta_{\text{i.s.}}=\frac{k^2/n}{2n(1+k^2/n)^{3/2}},\quad \delta_{\text{m.s.}}=\frac{k^2/2n+1}{2n^{1/2}(1+k^2/4n)^{3/2}}.$$

b. Small modulation system

To describe nonlinear generation of NSW we must consider the region of sufficiently small (but finite) amplitudes (i.e., small A^2 and $\mathscr{E}^2$). If generation is against a homogeneous hydrodynamic background, the hydrodynamic variables can be eliminated from (1)–(4) and the system itself can be replaced by two simpler equations whose solutions can be easily investigated analytically. These equations, which describe universally the behavior of weakly nonlinear modulated waves, will be called hereafter a small modulation system (SMS). Systems of this type are encountered in problems dealing with evolution of a nonlinear wave packet whose amplitude and frequency profiles are specified at $t=0$ (see, e.g., Refs. 11–13), and were investigated under the assumption that k has a small scatter about a constant value k_0.

In the present problem of nonlinear generation, in contrast to evolutional problems, the assumption that the wave is quasimonochromatic no longer holds. Indeed, the linear group velocity of oscillations of infinitesimal amplitude should coincide with the geometric velocity of the NSW front on which these oscillations are generated, i.e., $\omega_0'(k_0) = x_g'(t)$, where $\omega = \omega_0(k)$ is the linear dispersion equation, k_0 is the wave number at the generation point, and $x = x_g(t)$ is the law of motion of this point. Understandably, $x_g(t)$ is, generally speaking, an arbitrary function defined by the initial hydrodynamic profile, so that $x_g'(t)$ and correspondingly k_0 can vary significantly in the course of generation. In addition, the nonlinear correction to the group velocity, which makes a negligible contribution to the solution of the packet-evolution problem,[13] turns out to be substantial in generation problems, and must therefore be taken into account in the amplitude-transport equation. Allowance for the hydrodynamic variables in an SMS calls for introducing the concept of an effective dispersion equation.

We investigate thus a modulation system in the small-amplitude limit (we retain in the equations terms that guarantee the first nonvanishing order in the solution): $A^2 \sim a^2 \ll 1$. We consider for simplicity the procedure of obtaining SMS using as an example the modulation K–dV system (25) (another example, ion-sound waves in a plasma, is discussed in the Appendix).

Using the expansions of elliptic integrals for $m \ll 1$ (Ref. 14), we obtain for $\mu = E(m)/K(m)$

$$\mu(m) = 1 - \frac{m}{2} - \frac{m^2}{16} - \frac{m^3}{32} - \frac{41}{2^{11}} m^4 + \dots .$$

For the wave number k we have

$$k^2 = \frac{2a}{3m}\left(1 - \frac{m}{2} - \frac{3}{32} m^2 + \dots\right). \tag{39}$$

The expansions for the phase (20) and group (24) velocities take then the form

$$U = \eta - k^2 + \frac{1}{24}\frac{a^2}{k^2} + \dots ,$$

$$V = \eta - 3k^2 + \frac{1}{48}\frac{a^2}{k^2} + \dots . \tag{40}$$

The amplitude a is connected with the oscillation energy density A^2 by the relation (23), which takes for small a the form

$$A^2 = \frac{a^2}{4}\left(1 - \frac{1}{9\cdot 2^5}\frac{a^2}{k^4} + \dots\right). \tag{41}$$

The modulation system (25) can thus be represented in the small-amplitude limit in the form

$$\begin{gathered}\frac{\partial\eta}{\partial t} + \eta\frac{\partial\eta}{\partial x} + \frac{\partial A^2}{\partial x} = 0,\\ \frac{\partial A^2}{\partial t} + \frac{\partial}{\partial x}\left(\left(\frac{\partial\omega_0}{\partial k} + V_2(k)A^2\right)A^2\right) + A^2\frac{\partial\eta}{\partial x} = 0,\\ \frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(\omega_0(k,\eta) + \omega_2(k)A^2) = 0.\end{gathered} \tag{42}$$

where

$$\begin{aligned}&\omega_0(k,\eta) = k\eta - k^3,\\ &\omega_2(k) = 1/6k,\quad V_2(k) = 1/12k^2.\end{aligned} \tag{43}$$

It can be seen that in the leading order

$$\frac{\partial A^2}{\partial t} = -\frac{\partial\omega_0}{\partial k}\frac{\partial A^2}{\partial x}. \tag{44}$$

Using (44), we can integrate the first equation of (42) accurate to $O(A^2)$. We seek its solution in the form of the expansion

$$\eta = \eta_0 + A^2\eta_2(k), \tag{45}$$

where $\eta_0 = \text{const}$.

Substituting (45) in the first equation of (42) and using (44), we get

$$\eta_2 = -\frac{1}{3k^2}. \tag{46}$$

Thus, excluding the hydrodynamic variable from the system (42), we arrive at a closed system of two equations:

$$\begin{aligned}&\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(\omega_0(k) + A^2\tilde{\omega}_2(k)) = 0,\\ &\frac{\partial A^2}{\partial t} + \frac{\partial}{\partial x}((\omega_0'(k) + \tilde{V}_2(k)A^2)A^2) = 0.\end{aligned} \tag{47}$$

Here $\omega = \omega_0(k) + \tilde{\omega}_2(k)A^2$ is the effective nonlinear dispersion relation and $V_z(k)$ is the effective nonlinear correction to the group velocity. On our case of the K–dV equation

$$\tilde{\omega}_2(k) = -\frac{1}{6k},\quad \tilde{V}_2(k) = -\frac{5}{12k^2}. \tag{48}$$

We emphasize that $\tilde{V}_2(k) \neq \tilde{\omega}_2'(k)$.

Clearly, this procedure of eliminating hydrodynamic variables can be used also in a general dispersion-hydrodynamics case (see the Appendix), so that the above analysis based in the K–dV equation does not make less general Eqs. (47), which are valid in the low-amplitude limit for any dispersive hydrodynamics.

The system (47) written with accuracy that ensures a first nonvanishing order in the solution, and equivalent with this accuracy to the complete modulation system, takes the form

$$\begin{gathered}\frac{\partial k}{\partial t} + \omega_0'(k)\frac{\partial k}{\partial x} + \tilde{\omega}_2(k)\frac{\partial A^2}{\partial x} = 0,\\ \frac{\partial A^2}{\partial t} + (\omega_0'(k) + 2A^2\tilde{V}_2(k))\frac{\partial A^2}{\partial x} + \omega_0''(k)A^2\frac{\partial k}{\partial x} = 0.\end{gathered} \tag{49}$$

The term $2A^2\tilde{V}_2(k)\partial A^2/\partial x$ was retained in the second equation, since in the general solution, as will be shown below, $A \sim |x - x_g|^{1/2}$, where x_g is the coordinate of the generation point, and the contribution of the indicated term to the solution is of the same order as the term that follows.

We shall call (49) and its modifications a "small modulation system." Note that a system of similar type, in a simplified model form, was considered in Ref. 13 as a direct generalization of the equations of motion of a linear packet to include the nonlinear case. It must also be emphasized that in the considered case of small amplitudes we have $A \sim a$, where a is the exact amplitude of the oscillation. The covariational variable A is more convenient in dispersive-hydrodynamics problems, and its connection with the amplitude is easily determined in each specific case [for the K–dV equation this is Eq. (41)]. Introducing in (49) the new variables

$$w = \omega_0'(k),\quad I = A^2|c(w)|,\quad c(w) = \omega_0''(k)\tilde{\omega}_2(k), \tag{50}$$

we arrive at the following SMS modification:

$$\frac{\partial I}{\partial t}+\frac{\partial (Iw)}{\partial x}+If(w)\frac{\partial I}{\partial x}=0,$$
$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}+\sigma\frac{\partial I}{\partial x}=0, \tag{51}$$

where

$$f(w)=(2\tilde{V}_2(w)+c'(w))/c(w), \quad \sigma=\mathrm{sgn}\, c(w).$$

If we now replace the independent variable x by z, such that $dz=dx-f(\omega)dt$, then the SMS (51) takes the form of the system of Euler-hydrodynamics equations with $\gamma=2$. It must be remembered, however, that the SMS describes a wave structure and the variables I and w are in no way connected with the hydrodynamic density and velocity.

The parameter σ is indicative of the SMS structure. If $\sigma=1$, the system is hyperbolic. For $\sigma=-1$ the system is elliptic and its hydrodynamic modification corresponds to hydrodynamics with negative pressure $P=I^2/2$. Problems with NSW generation always correspond to the hyperbolic case, so that the condition $\sigma=1$ can be regarded as the condition for nonlinear generation of modulated waves. Note that in problems dealing with evolution of a weakly nonlinear wave packet this is the condition of stability of a wave to perturbations propagating along the wave's direction of motion.

The system (51) has the Riemann form

$$\frac{\partial r_\pm}{\partial t}+V_\pm(r)\frac{\partial r_\pm}{\partial x}=0, \tag{52}$$

where the Riemannian coordinates $r_\pm$ and characteristic velocities $V_\pm$ are expressed in terms of the initial variables I and w by the equations

$$r_\pm=w\pm 2I^{1/2}+O(I), \quad V_\pm=w\pm I^{1/2}+O(I). \tag{53}$$

The invariants and velocities are presented here with accuracy $O(I^{1/2})$, which permits the solutions to be investigated in the form of simple waves. The SMS term connected with the nonlinear correction to the group velocity makes a contribution to (52) on the order of $O(I)$ and is thus immaterial in the investigation of simple waves. It does, however, make a nontrivial contribution to the general SMS solution. The lines $r_\pm=$ const are two families of SMS characteristics; the families coincide at $I=0$.

Let us consider nonlinear generation of a modulated SMS wave. Let the motion of the point where the wave is generated be given by $x=x_g(t)$. As applied to the NSW theory, $x_g(t)$ is the law of motion of the NSW front on which the generation takes place. The following conditions are then satisfied on the curve $x=x_g(t)$:

$$I=0, \quad w=x_g'(t). \tag{54}$$

The first of these conditions is that the amplitude vanish on the investigated boundary. The second condition means that the signal group velocity on the curve $x=x_g(t)$ coincide with the geometric velocity of the generation point, and is equivalent to equality of the boundary to a multiple characteristic at $I=0$ (Fig. 1). In the hydrodynamic analog of (51) we encounter thus the original boundary-value problem of finding the gas flow in which the density vanishes on a specified curve $x_g(t)$.

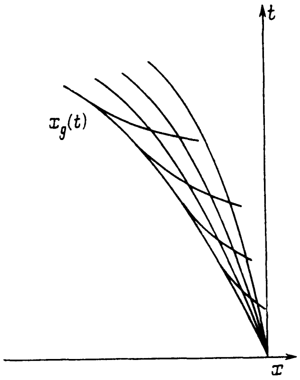

FIG. 1. SMS characteristics near the boundary $x_g(t)$.

c. SMS solutions

We consider first self-similar SMS solutions that satisfy the conditions (54) on the boundary. Such solutions describe generation of a simple centered NSW realizable as a result of decay of an initial discontinuity. A problem of this type was considered for Riemannian systems in Refs. 3, 10, and 11. The solution of the modulation equations depends in this case only on $\tau=x/t$, and the SMS (52), (53) is transformed into a system of ordinary differential equations. Its nontrivial solution is obvious: one of the invariants is a constant, and for the other we have $V=\tau$.

To be specific, we consider the case of negative dispersion. It follows then from the NSW structure that the invariant to be assumed constant must be the one ensuring an increase of the group velocity with increasing distance from the generation point, i.e., $r=r_0=$ const. Thus, the connection between w and I in the wave is given by

$$w=r_0+2I^{1/2}. \tag{55}$$

Next, $V_+=\tau$. We then obtain with the aid of (53) and (55) simple equations that describe the behavior of I and w near the generation point:

$$I^{1/2}=\tfrac{1}{3}(\tau-r_0), \quad w=\tfrac{1}{3}r_0+\tfrac{2}{3}\tau. \tag{56}$$

The conditions (54) take in the self-similar case the form

$$I(\tau_g)=0, \quad w(\tau_g)=\tau_g, \tag{57}$$

where τ_g is the self-similar generation coordinate and is assumed known. It follows then from (57) that $r_0=\tau_g$.

The determination of the behavior of the quantities A^2 and k which are of physical interest reduces to a simple recalculation of Eqs. (56) with the aid of the normalization (50). Generalizing the results to the case of positive inversion, we get ultimately

$$A=C(\tau_g)\,|\tau-\tau_g|, \tag{58}$$

where

$$C(\tau_g)=\frac{1}{3[\omega_0''(k_0)\tilde{\omega}_2(k_0)]^{1/2}}$$
$$k=k_0+\frac{2}{3\omega_0''(k_0)}(\tau-\tau_g), \tag{59}$$

where k_0 is the root of the equation $\omega_0'(k_0)=\tau_g$.

The dependences of the self-similar coordinate of the generation point on the initial values of the hydrodynamic variables are given in Refs. 3 and 9. We emphasize that the generation mechanism considered is in principle nonlinear, as evidenced by the factor $\tilde{\omega}_2(k_0)$ in expression (58) for $C(\tau_g)$.

We present the $C(\tau_g)$, $k_0(\tau_g)$, and $\omega_0''(k_0)$ dependences for a K–dV equation and for systems (26) and (31) that describe ion-sound and magnetosonic waves in a plasma.

For the K–dV equation (43)

$$C=\frac{1}{3}, \quad k_0=\left(\frac{\eta_0-\tau_g}{3}\right)^{1/2}, \quad \omega_0''(k_0)=-6k_0. \tag{60}$$

For ion-sound waves (see the Appendix and also Ref. 10)

$$C_{\text{i.s.}}=\frac{1}{3}\left(\frac{n_0}{3\tau_0^*}\right)^{1/2}\frac{4-3(\tau_0^*)^2+2(\tau_0^*)^{4/3}+(\tau_0^*)^{2/3}}{4(1+(\tau_0^*)^{4/3}+(\tau_0^*)^{2/3})}$$
$$+\frac{2(\tau_0^*)^{8/3}-3(\tau_0^*)^2+6(\tau_0^*)^{4/3}+15(\tau_0^*)^{2/3}-24}{24}\Bigg)^{-1/2}, \tag{61}$$
$$k_0=\left[n_0\frac{1-(\tau_0^*)^{2/3}}{(\tau_0^*)^{2/3}}\right]^{1/2}, \quad \omega_0''(k_0)=-\frac{3k_0}{n_0(1+k_0^2/n_0)^{5/2}}.$$

Here $\tau_0^* = \tau_g - v_0$, n_0 and v_0 are the constant values of the hydrodynamic variables at the generation point.

Finally, self-similar wave generation of waves in a cold magnetized plasma is given by

$$C_{\text{m.s.}}=\frac{2n_0^{1/4}}{\ }(\tau_0^*)^{5/6} \tag{62}$$
$$\times\left(\frac{(\tau_0^*)^{4/3}+4(\tau_0^*)^{2/3}}{2}+\frac{28(\tau_0^*)^{2/3}-13(\tau_0^*)^{4/3}-18}{1+(\tau_0^*)^{4/3}+(\tau_0^*)^{2/3}}\right)^{-1/2}$$
$$k_0=2\left[n_0\frac{1-(\tau_0^*)^{2/3}}{(\tau_0^*)^{2/3}}\right]^{1/2},$$
$$\omega_0''(k_0)=-\frac{3}{2^{3/2}}\frac{k_0/n_0^{1/2}}{(1+k_0^2/4n_0)^{5/2}},$$

where

$$\tau_0^*=\frac{\tau_g-v_0}{n_0^{1/2}}.$$

We consider now SMS solutions that are not simple waves and describe the general process of nonlinear wave generation resulting from phase conjugation of arbitrary monotonic hydrodynamic profiles (Figs. 2a and 2b).

The initial hydrodynamic profile is shown in Fig. 2 dashed. The arrow indicates the direction of motion of the generation point. The SMS describes a region near this point, which moves in both cases from the very beginning against a homogeneous hydrodynamic background. To obtain SMS solutions that are not simple waves ($\partial(I;w)/\partial(x;t)\neq 0$), it is convenient to use a hodograph transformation which linearizes the initial equations. The SMS takes on the Iw hodograph plane the form

$$\frac{\partial x}{\partial w}-(w+If(w))\frac{\partial t}{\partial w}+I\frac{\partial t}{\partial I}=0,$$
$$\frac{\partial x}{\partial I}-w\frac{\partial t}{\partial I}+\frac{\partial t}{\partial w}=0. \tag{63}$$

The conditions (54) are transformed into

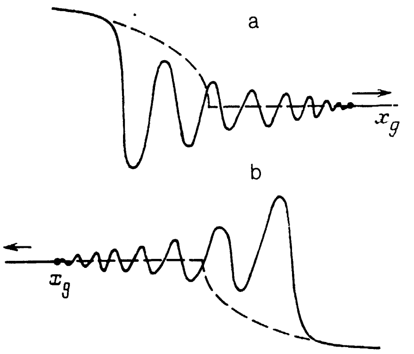

FIG. 2. Nonlinear generation in phase conjugation of a monotonic hydrodynamic profile: a—positive dispersion, b—negative dispersion.

$$\text{for } I=0: \quad x=x_0(w), \quad t=t_0(w), \tag{64}$$

where $x_0(w)$ and $t_0(w)$ specify parametrically the equation, assumed known, of the boundary $x_0 = x_g(t_0)$. The function $t_0(w)$ is obtained by inverting the equalities $w = x_g'(t_0)$, $x_0(w) = x_g(t_0(w))$. Using the condition that I be small, we seek the solution of the system (53) in the form of the expansions

$$x=x_0(w)+Ix_1(w)+\dots,$$
$$t=t_0(w)+It_1(w)+\dots. \tag{65}$$

Substituting (65) in (63), we obtain

$$t_1(w)={}^1/_2 f(w)t_0'(w)+{}^1/_2 t_0''(w),$$
$$x_1(w)={}^1/_2 wf(w)t_0'(w)+{}^1/_2 wt_0''(w)-t_0'(w). \tag{66}$$

Returning to the initial independent variables, we have

$$I=-x_g''(t)(x-x_g(t)), \tag{67}$$

$$w=x_g'(t)+\frac{1}{2}(x-x_g(t))\left(x_g''(t)f(x_g'(t))-\frac{x_g'''(t)}{x_g''(t)}\right). \tag{68}$$

We emphasize that the function f, which is connected with the nonlinear correction to the group velocity, does not enter into the formula for I. It follows from (67) that in media with negative dispersion $[x > x_g(t)]$ we have $x_g''(t) \leqslant 0$, since I is always positive. In media with positive dispersion we obtain similarly $x_g''(t) \geqslant 0$. Thus, $x_g''(t)$ and $x_g'(t)$ are of the same sign, i.e., the generation-point motion is accelerated in phase conjugation of monotonic hydrodynamic profiles.

In terms of the wave variables k and A, the solution of (67) and (68) takes the form

$$A=C(t)|x-x_g(t)|^{1/2}, \tag{69}$$

where

$$C(t)=|x_g''(t)/(\omega_0''(k_0)\tilde{\omega}_2(k_0))|^{1/2}, \quad \omega_0'(k_0)=x_g'(t).$$

The formula for k

$$k=k_0(t)-(x-x_g(t))k_1(t)$$

is obtained from $\omega_0'(k) = w(x,t)$ with the aid of (68).

An important difference between this solution and the self-similar equations (58) and (59) is that the amplitude has here a square-root increase with distance from the generation point, whereas in the decay of the initial burst the dependence of the amplitude on the distance to the generation point was linear. The wave number decreases linearly in both cases. Note that uniqueness of the obtained solution is ensured by the fact that the singular point $I = 0$, $w = x_g(t)$ on the hodograph plane is of the saddle type. The solution obtained is thus a segment of the separatrix joining the singular points on the leading and trailing edges. This case differs in principle from the case, analyzed in Ref. 5, of quasisimple waves in which the generation points on the hodograph plane was a node from which an infinite number of trajectories emerges. A local construction of the solution is impossible here, and to calculate the separatrix trajectory one must resort to the conditions on the soliton front.

4. STRUCTURE OF SOLITON FRONT

a. Soliton waves

Before investigating the structure of the soliton front of NSW, we consider the evolution of the system of noninteracting solitons formed as a result of evolution of a localized large-scale perturbation. These structures were named soliton waves[5] and are described by a solution of the modulation system as $k \to 0$. The quasistationarity condition

$$\frac{\partial k}{\partial t} + \frac{\partial (kU_s)}{\partial x} = 0, \tag{70}$$

where U_s is the soliton velocity in the wave, has in this case the meaning of a conservation law for the number of solitons. If the soliton motion takes place on a homogeneous hydrodynamic backround ($n = \text{const}$, $v = \text{const}$) the evolution of the soliton-wave velocity profile is described by the Hopf equation

$$\frac{\partial U_s}{\partial t} + U_s \frac{\partial U_s}{\partial x} = 0. \tag{71}$$

Soliton motion on a variable hydrodynamic profile, however, is accompanied by nonlinear interaction of the solitons with the hydrodynamic flows, and the equations describing the soliton wave become more complicated. This situation is realized, for example, as a result of evolution of two hydrodynamic perturbations that differ in scale (Fig. 3a). Clearly, the perturbation A decays rapidly into solitons that will move over the slowly evolving profile B (Fig. 3b).

Let us consider a modulation system of general form in the limit as $k \to 0$. Understandably, the oscillation energy density A^2 should also vanish in this case. However, the quantity

$$\zeta = A^2/k, \tag{72}$$

which is proportional to the energy of one oscillation—the soliton, remains finite as $k \to 0$ in the limit of interest to us. In fact, analysis of the asymptotic equations for A^2 in various specific cases [see (23), (30), (35)] shows that $A^2 \sim k$ in the soliton limit ($m \to 1$). The value of $\mathscr{E}^2$, just as in the case of small amplitude, is then proportional to A^2, so that it is convenient to write it in the form

$$\mathscr{E}^2 = f(k, n, \zeta) k\zeta, \tag{73}$$

where $f(0, n, \zeta) = O(1)$.

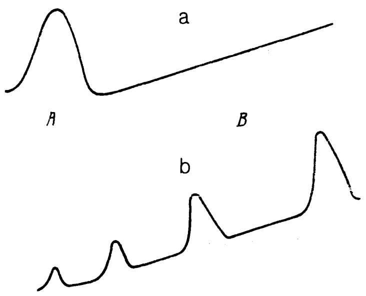

FIG. 3. Formation of soliton wave against the background of an evolving hydrodynamic flow: a—initial hydrodynamic profile, b—soliton wave.

Then, introducing in place of A^2 the variable ζ, we effect in the system (1)–(5) a transition to a soliton wave, $k \to 0$. For the solitons the phase velocity is equal here to the group velocity, so that we must put $V = U$ in the equations. As a result we obtain a general equation for a soliton wave moving over a variable hydrodynamic background:

$$\frac{\partial \zeta}{\partial t} + U_s \frac{\partial \zeta}{\partial x} + \zeta \left(\frac{\partial v}{\partial x} + f(0, n, \zeta) \frac{\partial n}{\partial x} \right) = 0, \tag{74}$$

where $U_s = v + U_1(0, n, \zeta)$ and the behavior of n and v, just as in the linear limit, is determined by the usual hydrodynamic equations (36). It can be seen that at $n = \text{const}$ and $v = \text{const}$ Eq. (74) goes over into the Hopf equation (71) for U_1.

Let us analyze soliton K–dV waves with the aid of the described technique. The transition $k \to 0$ in the modulation equations (25) leads, after a trivial change of variables, to the following system for the soliton wave:

$$\begin{gathered} \frac{\partial \zeta}{\partial t} + U_s(\eta, \zeta) \frac{\partial \zeta}{\partial x} + \zeta \frac{\partial \eta}{\partial x} = 0, \\ \frac{\partial \eta}{\partial t} + \eta \frac{\partial \eta}{\partial x} = 0, \end{gathered} \tag{75}$$

where

$$U_s(\eta, \zeta) = \eta + {}^2/_3 \zeta^{2/3}.$$

Since (75) is a hyperbolic system of second order, it can be represented in Riemannian form. Introducing the variable $r = \eta + \zeta^{2/3}$ we arrive at the system[1)]

$$\begin{gathered} \frac{\partial r}{\partial t} + \left(\frac{\eta}{3} + \frac{2r}{3} \right) \frac{\partial r}{\partial x} = 0, \\ \frac{\partial \eta}{\partial t} + \eta \frac{\partial \eta}{\partial x} = 0, \end{gathered} \tag{76}$$

the solution of which is

$$\eta = \eta_0(x - \eta t), \qquad r = r_0\left(x - {}^2/_3 rt - {}^1/_3 \int_0^t \eta(x, t')\, dt' \right), \tag{77}$$

where $\eta_0(x)$ and $r_0(x)$ are the initial profiles of the functions $\eta(x, t)$ and $r(x, t)$.

Examination of soliton waves in more general cases of dispersion hydrodynamics with the aid of Eqs. (74) and (36) also leads to a second-order system if it is assumed that the soliton evolution takes place on a background of a hydrodynamic simple wave in which $v = v(n)$. This makes it possible to reduce the system to a Riemann form independently of the properties of the complete modulation system, and obtain solutions similar to (77) after integrating it by the method of characteristics.

b. Soliton front of NSW

We confine ourselves in the analysis of a soliton-front structure to the case of soliton motion against a uniform hydrodynamic background. The equations (70) and (71) of the soliton wave have in this case a single characteristic $dx/dt = U_s$. The degeneracy is due to the fact that no account is taken in the system of the soliton interaction. In the NSW theory the characteristics converge only directly on the soliton boundary $x_s(t)$ (Ref. 3). The system (70), (71) can therefore not describe the behavior of a group of waves that are close to solitary near $x_s(t)$ (we shall call them hereafter weakly interacting solitons).

The dispersion relation for noninteracting solitons is $\omega = kU_s$. To lift the degeneracy in the system (70), (71) it is necessary to take account in the dispersion equation of the next higher terms in the expansion of $U = \omega/k$ for small k [when solitons are considered it is convenient to call $U(k)$ the dispersion equation, inasmuch as in the soliton limit as $k \to 0$ the frequency ω also vanishes, whereas the phase velocity remains finite]. This will correspond in fact to allowance for soliton interaction in the wave.

Let us consider in general form stationary waves in dispersion hydrodynamics. The equation of the stationary wave can always be reduced to the form

$$\left(\frac{d\varphi}{d\xi}\right)^2 = Q(\varphi, U, k), \qquad \xi = x - Ut. \tag{78}$$

We assume for simplicity that the number of constants defining the stationary wave is equal to two (here U and k). The function $-Q(\varphi)$ forms the potential well shown in Fig. 4a. The oscillations take place in the region between the roots b_1 and b_2 of the function $Q(\varphi)$. The soliton wave corresponds to a case when the roots b_2 and b_3 merge (Fig. 4b). The case corresponding to the considered situation of weakly interacting solitons is that of close roots b_2 and b_3 (Fig. 4c).

The wave number k is determined by the relation[4]

$$k = \frac{2\pi}{\lambda} = 2\pi\left[\oint \frac{d\varphi}{Q^{1/2}(\varphi)}\right]^{-1}. \tag{79}$$

The function $Q(\varphi)$ can be represented in the form

$$Q(\varphi) = (b_1-\varphi)(b_2-\varphi)(b_3-\varphi)/F^2(\varphi), \tag{80}$$

where $F(\varphi)$ is a definite-sign analytic function in the region $b_2 \leqslant \varphi \leqslant b_1$.

It follows from the representation (80) that as $b_2 \to b_3 = b$ the integral (79) has a logarithmic singularity. Indeed, expanding the function $F(\varphi)$ in a series near $\varphi = b$:

$$F(\varphi) = F(b) + F'(b)(\varphi - b) + {}^1/_2 F''(b)(\varphi - b)^2 + \ldots,$$

we have for k the expansion

$$k = \frac{F(b)}{\ln(16/m_1)} + O\left(\frac{m_1}{\ln(16/m_1)}\right). \tag{81}$$

We have introduced here the parameter m_1 customarily used to describe the amplitude of a stationary wave. In problems where $Q(\varphi)$ is an exact cubic parabola (K–dV, NSE) we have $m_1 = 1 - m$, where m is the elliptic-function parameter.

The phase velocity U always enters in simple and explicit manner in the equations for a stationary wave, and is therefore an analytic function of the roots of the $Q(\varphi)$ curve. In the case of close roots b_2 and b_3 the expansion of U near U_s is therefore in powers of the parameter m_1:

$$U = U_s + f(U_s) m_1 + O(m_1^2). \tag{82}$$

Using (81) and recognizing that in the considered case the parameter of the stationary wave can be expressed in terms of U and k, we obtain a dispersion relation for weakly interacting solitons:

$$U = U_s + f(U_s)\exp\left(-\frac{G(U_s)}{k}\right) + \ldots, \tag{83}$$

where

$$G(U_s) = F(b(U_s)).$$

In dispersion hydrodynamics, an expansion of general form for the phase velocity contains hydrodynamic variables and terms proportional to k, but for small k the hydrodynamics can be excluded from the modulation system and the effective dispersion takes the form (83).

We consider now the procedure for integrating the hydrodynamic equations and obtaining a system describing the evolution of weakly interacting solitons, using as an example the K–dV modulation equations. The dispersion-hydrodynamics systems go over into a K–dV equation in the case of weak nonlinearity, and therefore the expansions for the quantities in a wave of arbitrary amplitude will be analogous to the corresponding K–dV expansions. This is important, since the expansions for small k are non-analytic. We shall use the hydrodynamic form of the modulation equations,

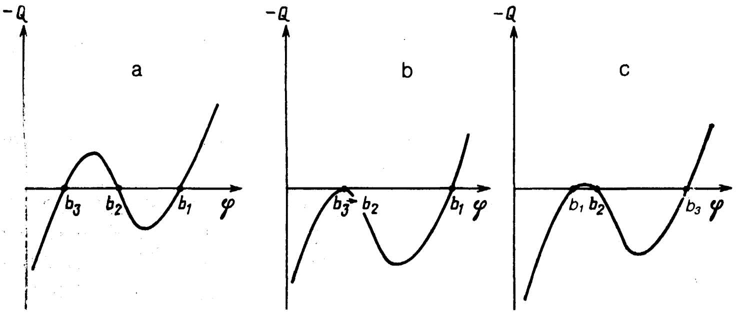

FIG. 4. Potential curve indicative of stationary waves.

and therefore the procedure of excluding the hydrodynamic variables should have a universal character and can be used for arbitrary dispersion systems.

Let us use the K–dV modulation equations (25) obtained in Sec. 2b and let us replace A^2 in them by the variable $\zeta = A^2/k$, which is more convenient for the investigation of a soliton front. In terms of the variables η, k, and ζ the K–dV modulation system takes the form

$$\frac{\partial\eta}{\partial t}+\eta\frac{\partial\eta}{\partial x}+\frac{\partial}{\partial x}(k\zeta)=0,$$
$$\frac{\partial k}{\partial t}+\frac{\partial(kU)}{\partial x}=0, \qquad (84)$$
$$\frac{\partial\zeta}{\partial t}+U\frac{\partial\zeta}{\partial x}+\zeta\frac{\partial\eta}{\partial x}+\frac{1}{k}\frac{\partial}{\partial x}(k\zeta(V-U))=0.$$

The $U(\eta, k, \zeta)$ and $V(\eta, k, \zeta)$ dependences are specified by the parametric equations (12), (20), (23), and (24). Using the expansions of complete elliptic integrals for $m_1 \ll 1$ (Ref. 14), we get for small k the expansions:

$$U=\eta+\frac{2a}{3}-\frac{2}{\pi}(6a)^{1/2}k+\frac{a}{48}\exp\left(-\frac{\pi(2a/3)^{1/2}}{k}\right)$$
$$+O\left(k\exp\left(-\frac{\pi(2a/3)^{1/2}}{k}\right)\right), \qquad (85)$$

$$V-U=-\frac{(6a)^{1/2}}{\pi}k-\frac{a}{16}\exp\left(-\frac{\pi(2a/3)^{1/2}}{k}\right)$$
$$+O\left(k\exp\left(-\frac{\pi(2a/3)^{1/2}}{k}\right)\right). \qquad (86)$$

We have retained here for convenience the amplitude a as an intermediate parameter. It is connected with the variable ζ by the relation

$$\zeta=\frac{2a^{3/2}}{\pi 6^{1/2}}-\frac{3a}{\pi^2}k-\frac{a^2}{24k}\left(\frac{2}{3}\right)^{1/2}\exp\left(-\frac{\pi(2a/3)^{1/2}}{k}\right)$$
$$+O\left(\exp\left(-\frac{\pi(2a/3)^{1/2}}{k}\right)\right). \qquad (87)$$

Using the wave-number conservation law and expanding (85)–(87), we can integrate the first equation of the system (84) at the required accuracy. We seek its solution in the form of an expansion near a constant value of the hydrodynamic variables, which can be assumed to be zero without loss of generality, as can be readily verified by a suitable choice of the reference frame. As a result we get

$$\eta=\frac{2(6a)^{1/2}}{\pi}k+\frac{(6a)^{1/2}}{8\pi}k\exp\left(-\frac{\pi(2a/3)^{1/2}}{k}\right)$$
$$+O\left(k^2\exp\left(-\frac{\pi(2a/3)^{1/2}}{k}\right)\right). \qquad (88)$$

Substituting (88) in (85) we obtain an effective dispersion relation of form (83). Here

$$U_s=\frac{2a}{3}, \quad G(U_s)=\pi U_s^{1/2}, \quad f(U_s)=\frac{1}{32}U_s. \qquad (89)$$

We emphasize that expressions (89) were obtained for a K–dV equation. They will naturally be different for other systems.

The expansion (88) permits the hydrodynamic variable to be eliminated from the system (84). As a result, the modulation system describing the motion of weakly interacting solitons (we call it for brevity the soliton system) takes the form

$$\frac{\partial k}{\partial t}+\frac{\partial(kU)}{\partial x}=0,$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\frac{g(U)}{k^3}\exp\left(-\frac{2G(U)}{k}\right)\frac{\partial k}{\partial x}=0. \qquad (90)$$

In the derivation of (90) it is convenient to obtain first a system for the variables k and U_s, and only then use (83) and (89) to change to the variables k and U. The coefficient $g(U)$ preceding the exponential term in the second equation of (90) makes no contribution in the first nonvanishing order of the solution, which is obtained with logarithmic accuracy, so that there is no need for its specific form. What is important is that $g(U) > 0$. This ensures a hyperbolic soliton system.

Equations (90) are of hydrodynamic form with an exponential dependence of the pressure on the density. The role of density is played here by the wave number (soliton density); the equation of state is

$$P(k, U)\approx\frac{1}{k}\exp\left(-\frac{2G(U)}{k}\right)+O\left(\exp\left(-\frac{2G(U)}{k}\right)\right). \qquad (91)$$

We emphasize that the soliton-system solutions in first nonvanishing order are determined only by the form of the function $G(U)$.

The modulation system (90) can be represented at the accuracy required in the Riemann form

$$\frac{\partial r_\pm}{\partial t}+V_\pm(r)\frac{\partial r_\pm}{\partial x}=0, \qquad (92)$$

where

$$r_\pm=U\pm k\exp\left(-\frac{G(U)}{k}\right), \quad V_\pm=U\pm\frac{1}{k}\exp\left(-\frac{G(U)}{k}\right). \qquad (93)$$

The lines $r_\pm$ = const are two families of characteristic that become equal at $k=0$.

We formulate now the problem of the NSW soliton front. Let the law of motion of the soliton boundary $x_s(t)$ be given. The conditions

$$k=0, \ U=x_s'(t), \qquad (94)$$

are satisfied then on the curve $x = x_s(t)$ and are analogous to the conditions at the nonlinear generation point (54) on the opposite NSW front. The picture of the coalescence of the characteristics on $x = x_s(t)$ is also similar to the one shown in Fig. 1, but the coalescence is governed by a different law.

c. Solutions of soliton system

Just as before, we consider first self-similar solutions of the system (92) and (93), defined by a constant value of one of the invariants and by the equality $V = \tau$, where V is the characteristic velocity corresponding to the second invariant and $\tau = x/t$ is the self-similar variable. If the dispersion in the system is negative, we must, starting from the NSW structure, set constant the invariant ensuring the decrease of group velocity with increase of the wave number. The self-similar solution of the system (77) takes thus the form

$$r_+=r_s=\text{const},\ V_-=\tau. \tag{95}$$

Using (93) and (94) we get, with logarithmic accuracy,

$$k=\frac{G(\tau_s)}{\ln(1/|\tau_s-\tau|)}, \tag{96}$$

$$U=\tau_s-\frac{\tau_s-\tau}{\ln(1/|\tau_s-\tau|)}. \tag{97}$$

We have carried out here a generalization to the case of positive dispersion; τ_s is the self-similar coordinate of the NSW soliton boundary. It can be seen that the phase velocity of the waves that are close to solitary are determined near the NSW soliton boundary only by its self-similar coordinate. It should be noted that the accuracy of the soliton system is not enough to guarantee a numerical coefficient for the small term $(\tau_s - \tau)/\ln(1/|\tau_s - \tau|)$ in (97). However, substituting (96) and (97) in the exact wave-number conservation law we easily verify that this coefficient is in fact unity.

Consider now those system-(90) solutions which are not simple waves ($\partial(k;U)/\partial(x;t)\neq 0$). Just as in the case of generation, it is convenient here to use the hodograph transformation $(k;U)\to(x;t)$ and seek the solutions of the linearized equations in series form:

$$x=x_0(U)+\frac{1}{k}\exp\left(-\frac{2G(U)}{k}\right)x_1(U)+\ldots;$$
$$t=t_0(U)+\frac{1}{k}\exp\left(-\frac{2G(U)}{k}\right)t_1(U)+\ldots, \tag{98}$$

where $x_0(U)$ and $t_0(U)$ specify parametrically the soliton-boundary motion: $x_0 = x_s(t_0)$. The sought solution is

$$k=\frac{2G(x_s'(t))}{\ln(1/x_s''(t)(x_s(t)-x))} \tag{99}$$

$$U=x_s'(t)+x_s''(t)(x_s(t)-x)q(t),$$
$$q(t)=\frac{G'(h)}{G(h)},\quad h=x_s'(t). \tag{100}$$

The uniqueness of the solution, just as in the generation problem for SMS, is ensured by the saddle-like character of the point $k=0$, $U=x_s'(t)$ (Ref. 5). Clearly, the system (90) describes not only the K–dV case, for which it has been derived, but is valid also in other problems of dispersive hydrodynamics. Indeed, the behavior of the considered solutions of the soliton system is determined by the factor $\exp(-2G(U)/k)/k^3$ in the second equation. The form of this factor is universal, since the expansions of U, V, and ζ should in general be to the same powers of k and $\exp(-G(U)/k)$ as in (85)–(87). This follows from the cubic behavior of the $Q(\varphi)$ curve near the soliton boundary. The solution of the soliton system for different dispersive-hydrodynamic systems is determined in first nonvanishing order only by the $G(U)$ dependence, which takes for the equations considered in this paper the form

$$G_{\text{i.s.}}(U)=\pi n_s^{1/2}(U^2-1)^{1/2}/U, \tag{101}$$

$$G_{\text{m.s.}}(U)=2\pi n_s^{1/2}(U^2-n_s)^{1/2}/U, \tag{102}$$

where n_s is the density to the right of the NSW, and $v_s = 0$.

APPENDIX. DERIVATION OF SMS FOR NONLINEAR ION-SOUND WAVES IN A TWO-TEMPERATURE PLASMA

Modulated nonlinear ion-sound waves are described by the system (1)–(5) with coefficients determined by (28)–(30). In the case of small amplitudes we have the expansions[10]

$$\omega=kU=\omega_0(k,n,v)+A^2\omega_2(k,n),$$

where

$$\omega_0(k,n,v)=kv+W_0(k,n),\ W_0(k,n)=k(1+k^2/n)^{-1/2},$$
$$\omega_2(k,n)=k\frac{24-2\gamma^4+3\gamma^3-6\gamma^2-15\gamma}{24n(1-\gamma)},\quad \gamma=\frac{1}{1+k^2/n}. \tag{A1}$$

For $\mathscr{E}^2$ we have

$$\mathscr{E}^2=\frac{1}{k}\frac{\partial\omega_0}{\partial n}A^2. \tag{A2}$$

The nonlinear group velocity can be represented in the form

$$V=\frac{\partial\omega_0}{\partial k}+V_2(k,n)A^2. \tag{A3}$$

To find the quadratic corrections to the group velocity it is necessary to expand the numerator in Eq. (28) for the group velocity, accurate to $O(A^4)$. In the case of ion-sound waves this leads to extremely cumbersome equations, and therefore do not present here the expression for $V_2(k,n)$.

With (A1)–(A3) taken into account, the system (1)–(5) takes in the small-amplitude limit the form

$$\frac{\partial n}{\partial t}+\frac{\partial(nv)}{\partial x}+\frac{\partial A^2}{\partial x}=0,$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{n}\frac{\partial n}{\partial x}+\frac{1}{k}\frac{\partial\omega_0}{\partial n}\frac{\partial A^2}{\partial x}=0,$$

$$\frac{\partial A^2}{\partial t}+\frac{\partial}{\partial x}\left(\left(\frac{\partial\omega_0}{\partial k}+V_2A^2\right)A^2\right)+A^2\frac{\partial v}{\partial x}+\frac{1}{k}A^2\frac{\partial\omega_0}{\partial n}\frac{\partial n}{\partial x}=0,$$

$$\frac{\partial k}{\partial t}+\frac{\partial}{\partial x}(\omega_0(k,n,v)+A^2\omega_2(k,n))=0. \tag{A4}$$

We emphasize that all the coefficients are expressed in terms of k, n, v, and A^2 in a simple and lucid manner.

According to Sec. 2a, we seek the solutions of the first two equations in the series form

$$n=n_0+A^2n_2,\ v=v_0+A^2v_2, \tag{A5}$$

where $n_0 = \text{const}$ and $v_0 = \text{const}$.

Substituting (A5) in the hydrodynamic equations (A4) of the system and using the third equation in zeroth order [see (44)], we obtain

$$n_2=-\frac{n_0(\partial W_0/\partial n)_{n=n_0}-(\partial W_0/\partial k)_{n=n_0}}{1-(\partial W_0/\partial k)^2_{n=n_0}},$$
$$v_2=-\frac{1-n_0[(\partial W_0/\partial n)(\partial W_0/\partial k)]_{n=n_0}}{1-(\partial W_0/\partial k)^2_{n=n_0}}. \tag{A6}$$

Eliminating with the aid of (A5) and (A6) the hydrodynamic variables from (A4) we arrive at an SMS that contains the effective dispersion relation

$$\omega(k, A^2)=\omega_0(k, n_0, v_0)+A^2\tilde{\omega}_2(k, n_0),$$

where

$$\tilde{\omega}_2(k, n_0) = -\frac{k}{n_0}\left(\frac{4+2\gamma_0^2-3\gamma_0^3+\gamma_0}{4(1-\gamma_0^3)}-\frac{24-2\gamma_0^4+3\gamma_0^3-6\gamma_0^2-15\gamma_0}{24(1-\gamma_0)}\right),$$
$$\gamma_0=1/(1+k^2/n_0). \qquad \text{(A7)}$$

[1)] The system (76) can be obtained directly from the Riemann system of Ref. 3, in which we must put $r_2 = r_3 \equiv r$ and $r_1 \equiv \eta$.

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