Relaxation of high-frequency acoustic phonons

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In this paper we propose a model that allows us to describe the kinetics of phonon systems with anharmonic three-phonon interactions. Using this kinetic model, we carry out a numerical investigation of the system's time history as it approaches equilibrium when an initial phonon distribution is specified that is substantially out of equilibrium. We compare our results with the usual self-similar (i.e., scaling) approximate solutions for such systems, and also determine the limits of applicability of the self-similar approximation.

INTRODUCTION

One of the most interesting and important problems in the physics of phonons is how a system of nonequilibrium phonons relaxes in the course of establishing an equilibrium temperature for the crystal. It is by no means unusual to encounter situations in which the state of the phonon system of a solid is far from equilibrium or, in other words, the distribution function for the phonons departs significantly from the Planck distribution. This situation occurs, for example, during the relaxation of the electron subsystem of a solid when the latter is excited by an intense laser pulse. Such an excitation leads to the generation of high-frequency acoustic phonons over a wide frequency range, with energies that are comparable to the frequency of a Debye phonon Ω_D. In this case the initial distribution of acoustic phonons will have a maximum that, as a rule, occurs at a rather high frequency, i.e., on the order of half the frequency of the lowest optical branch. Since the interaction of the various electronic excitations (free current carriers, excitons, etc.) with this nonequilibrium system of phonons will differ from their interaction with thermalized phonons, in order to describe these processes correctly it is necessary to take into account the evolution of the phonon system as it relaxes toward an equilibrium distribution with finite temperature.

A number of experimental investigations, in particular Refs. 1 and 2, have tracked the early stages of this relaxation process under weak excitation of the phonon system when the spontaneous decay of the phonons dominates. As was shown theoretically in Ref. 3, under these conditions there exists an approximate solution to the system of kinetic equations that describes the behavior of the phonons: the so-called self-similar (or "scaling") solution. The question of whether or not such a self-similar solution can evolve from a given initial phonon distribution remains open. Furthermore, there exist only qualitative estimates for the time period during which the system can be described by such a solution;[4] this is because a complete description of the relaxation process, including the final stage during which the phonon temperature is established, must take into account both processes that lead to decay of the high-frequency acoustic phonons and processes that lead to their amplification. Analytic calculations that would lead to such a description do not appear to be possible.

In this connection there is considerable interest in devising numerical experiments, based on models of the relaxation processes in a system of phonons, that would allow us to follow the evolution of the phonon distribution with time for various initial conditions. This type of experiment was in fact carried out by the authors of Ref. 5 within the simplified "generation" scheme first discussed in Refs. 6 and 7 (in which the continuous phonon spectrum was replaced by a set of levels Ω_D, $\Omega_D/2$, $\Omega_D/2^2$, etc.) and in Ref. 8 for an equidistant subdivision of the spectrum and explicit inclusion of processes of decay and scattering by impurities. Here we will use a variant of the equidistant-spacing model that best corresponds to a true continuum, and will take into account all three-phonon processes, i.e., leading both to decay and amplification;[9] this implies that in our model the system can undergo thermalization as $t\to\infty$.

In this paper we will present the results of calculations that were carried out for the case of low initial crystal temperature T_0 and high levels of excitation, where the finite temperature T that characterizes the total energy of the system greatly exceeds the initial temperature and reaches values comparable to Ω_D. We show that for $T\ll\Omega_D$ the system of nonequilibrium phonons can evolve into the state described by the self-similar solution at a certain stage of the relaxation, and obtain bounds on the existence of the latter, whereas for $T\approx\Omega_D$ the amplification of phonons turns out to be important at the very beginning of the relaxation process[9] and the solution has no self-similar stage at all.

DESCRIPTION OF THE MODEL

In order to describe all the relaxation processes of a system of nonequilibrium phonons up to the time when a finite equilibrium temperature T is established in the system, we have carried out numerical calculations of the kinetics associated with changes in the occupation numbers of phonons with various frequencies. In our reference physical model we consider phonons belonging to two modes with an isotropic Debye spectrum. The dispersion of the longitudinal l-phonons is taken to be $\Omega = u_l|\mathbf{k}|$, that of the transverse t-phonons $\Omega = u_t|\mathbf{k}|$ (this latter mode is assumed to be doubly degenerate in view of the two polarizations of the t-phonons). Here u_l and u_t are the longitudinal and transverse sound velocities, respectively. For convenience in the numerical calculations we set u_t equal to one, while u_l is measured in units of u_t and is labelled s, and is chosen so that $s = u_l/u_t > 1$. The frequency of the highest Debye phonon is also assumed to equal unity, i.e., $\Omega_D = 1$; we assume that this frequency is the same for both phonon branches, and use it to provide the scale for energy. In this model we also assume that time is scaled by the lifetime of the Debye phonon.

We will take into account all the normal three-phonon processes that are possible in such a system and that satisfy the law of conservation of energy and momentum, i.e., we

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will not discuss umklapp processes. This approximation is reasonable for $T \ll 1$. We will also assume that the processes of mutual conversion of *l*- and *t*-mode phonons at impurities are rapid in comparison to the anharmonic processes that control the behavior of the system. This allows us to equate the phonon occupation number functions $n_l(\Omega,t)$ and $n_t(\Omega,t)$ for the longitudinal and transverse branches respectively (which are in general different functions), and to deal only with a single function $n(\Omega,t)$. This assumption is valid, for example, in semiconductors at low temperatures where scattering by impurities dominates over phonon–phonon interactions.

Thus, our interest will center on the time dependence of the occupation number $n(\Omega,t)$ of phonons of various frequencies Ω for a spatial homogeneous nonequilibrium initial distribution $n(\Omega,0) = n_0(\Omega)$. We study the process by which the nonequilibrium function $n_0(\Omega)$ at $t = 0$ is transformed as $t \to \infty$ into the equilibrium function $n_T(\Omega)$ corresponding to a Planck distribution with a finite temperature T, where the latter is specified by the total energy of the initial phonon distribution $n_0(\Omega)$.

In order to solve numerically the kinetic equations that determine the behavior of our physical model of the phonon system, we need to specify the continuous function of frequency $n(\Omega,t)$ on a grid with a certain finite step size. This operation is equivalent to replacing the continuous phonon spectrum by a set of levels that are equidistant in frequency: $\Omega_i = i/N$, $1 \leqslant 1i \leqslant N$, where i is the index of the level and N is the number of levels under discussion. In calculations that follow we have set $N = 128$. Here our calculations differ significantly from those of Ref. 5 based on the generation model, since these authors artificially limited the three-phonon processes to decays that involved halving the initial phonon energy, a procedure that significantly distorts the entire energy relaxation picture. In particular, we will see in what follows that those processes in which a phonon decays into two other phonons with nearly equal energies make up a comparatively small fraction of all the decay processes.

Thus, the mathematical model that underlies our numerical calculations assumes that the phonon spectra are a set of δ-spheres in **k**-space with radii

$$r_l(i) = \frac{1}{Ns} i, \quad r_t(i) = \frac{1}{N} i \tag{1}$$

for the longitudinal and transverse phonon branches respectively. In order that this choice correspond to the Debye spectrum of the original physical model the density of states for these levels is chosen in the following way:

$$\rho_\alpha(i, \mathbf{k}) = \rho_\alpha \delta(|\mathbf{k}| - r_\alpha(i)), \tag{2}$$

where the mode type α can be l or t, and $\rho_l = 1/s$, $\rho_t = 2$.

In this mathematical model, the kinetic equation that takes into account three-phonon anharmonic processes leading to transitions $i \rightleftarrows j + k$ between the levels (where i, j, k are the level indices) gives rise to a nonlinear system of N equations whose mth equation has the form

$$\dot{n}_m(t) = \frac{1}{(\rho_l + \rho_t) m^2 R} \Big(-\frac{1}{2} \sum_{j=1}^{N-m} R_{m \rightleftarrows j + (m-j)} F_1 + \sum_{j=1}^{N-m} R_{(m+j) \rightleftarrows m + j} F_2 \Big), \tag{3}$$

$$F_1 = n_m(t)\,[1 + n_j(t)]\,[1 + n_{(m-j)}(t)] - [1 + n_m(t)]\, n_j(t)\, n_{(m-j)}(t),$$

$$F_2 = n_{(m+j)}(t)\,[1 + n_j(t)]\,[1 + n_m(t)] - [1 + n_{(m+j)}(t)]\, n_j(t)\, n_m(t).$$

The expression in brackets is the difference between the numbers of phonons of frequency m/N created and annihilated per unit time. The first sum in Eq. (3) takes into account processes whereby phonons of frequency m/N (the mth levels) decay to phonons with frequency j/N and $(m - j)/N$, $(1 \leqslant j \leqslant m - 1)$, along with processes in which the latter are amplified, leading to the creation of phonons of frequency m/N. The factor 1/2 eliminates the repetition of terms in the first sum. The second sum takes into account the processes of amplification of phonons of frequency m/N by the remaining phonons, and also the decay of phonons of higher frequencies leading to the creation of phonons of frequency m/N.

It is not difficult to see from Eq. (3) what the physical meaning of the transition constant $R_{i \rightleftarrows j + k}$ is. Thus, for $n_i = 1$, $n_j = n_k = 0$ the number of decays per unit time of phonons of frequency i/N to phonons with frequencies j/N and k/N is directly proportional to $R_{i \rightleftarrows j + k}$. Taking into account that the density of states at level i is proportional to i^2, the quotient $R_{i \rightleftarrows j + k}/i^2$ will be proportional to the probability of decay of a phonon of frequency i/N into phonons with frequencies j/N and k/N per unit time. In order to calculate $R_{i \rightleftarrows j + k}$ we need to sum over the contributions of all three types of processes that satisfy the laws of conservation of energy and momentum:

$$i_l \leftrightarrow j_t + k_t, \quad \gamma = 1, \qquad i_l \leftrightarrow j_l + k_t, \quad \gamma = 2, \qquad i_l \leftrightarrow j_t + k_l, \quad \gamma = 3, \tag{4}$$

$$R_{i \rightleftarrows j+k} = \sum_{\gamma=1}^{3} W_\gamma(i \rightleftarrows j + k). \tag{5}$$

As an example we present a calculation of $W_\gamma(i \rightleftarrows j + k)$ for the first process, i.e., $\gamma = 1$. The expressions for the two remaining processes are analogous insofar as it is accurate to interchange the indices l and t. For $W_l(i \rightleftarrows j + k)$ we have

$$W_1(i \rightleftarrows j + k) = \iiint d\mathbf{k}_i\, d\mathbf{k}_j\, d\mathbf{k}_k\, \rho_l(i, \mathbf{k}_i)\, \rho_t(j, \mathbf{k}_j)\, \rho_t(k, \mathbf{k}_k) \times \frac{|M_{l \rightleftarrows t+t}(\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_k)|^2}{ijk} \delta(\mathbf{k}_i - \mathbf{k}_j - \mathbf{k}_k)\, \delta_{i,(j+k)}. \tag{6}$$

Here $\delta_{i,j}$ is the Kronecker delta symbol, while $M_{l \rightleftarrows t + t}(\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_k)$ are the matrix elements of the corresponding three-phonon process. We have assumed that the latter are proportional to the product of the magnitudes of the phonon wave vectors that participate in the process.[10] For example, for the first of these processes

$$M_{l \rightleftarrows t+t}(\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_k) = C_{l \rightleftarrows t+t}\, r_l(i)\, r_t(j)\, r_t(k). \tag{7}$$

The integral in Eq. (6) is calculated analytically, which by taking into account Eq. (7) leads to the following expression for $W_1(i \rightleftarrows j + k)$:

$$W_1(i \rightleftarrows j + k) = C^2_{l \rightleftarrows t+t}\, \rho_l \rho_t \rho_t \frac{[r_l(i)\, r_t(j)\, r_t(k)]^3}{ijk} \times \theta(r_l(i) - |r_t(j) - r_t(k)|)\, \theta(|r_t(j) + r_t(k)| - r_l(i))\, \delta_{i,(j+k)}. \tag{8}$$

Finally, the normalization constant R in the denominator of Eq. (3) is chosen so that the unit of time is the lifetime of a Debye phonon ($\Omega_D = 1$). From the expression for the lifetime of a phonon of frequency i/N due to these decay processes,

$$\tau_D^{-1}(i)=\frac{1}{(\rho_l+\rho_t)\,i^2R}\,\frac{1}{2}\sum_{j=1}^{i-1}R_{i\rightleftarrows j+(i-j)},\quad i>1, \tag{9}$$

it follows that for $\tau_D(N) = 1$ we must set

$$R=\frac{1}{(\rho_l+\rho_t)N^2}\,\frac{1}{2}\sum_{j=1}^{N-1}R_{N\rightleftarrows j+(N-j)}. \tag{10}$$

Incidentally, we note that the dependence of the lifetime Eq. (9) on the frequency of the ith phonon follows the well-known law $\tau_D \sim \Omega^{-5}$, since $\tau_D(i) \sim i^{-5}$.

The numerical calculations were carried out using the fourth-order Runge–Kutta method with error control at each step. An IBM AT personal computer used.

DECAY CHARACTERISTICS

Let us now discuss how the probability for decay of a phonon of frequency i/N into two phonons with frequencies j/N and k/N depends on the ratio of these latter phonon frequencies. Figure 1a shows the dependence of $R_{i\rightleftarrows j+k}$ on the ratio j/i (the upper curve), and also its component $W_\gamma(i\rightleftarrows j+k)$ for the three decay processes Eq. (4). It is clear that although the decay process involving strict halving of the phonon energy has the largest decay probability, the probabilities for decay into two phonons with different frequencies are also rather substantial. The cutoffs of the functions in Fig. 1a are due to the law of conservation of momentum [the θ-function in Eq. (8)]; the sharpness of these cutoffs is a consequence of our assumption of an isotropic dispersion for the phonons.

Generally speaking, Eq. (8) should contain different coefficients $C_{l\rightleftarrows t+t}$ and $C_{l\rightleftarrows l+t}$. These coefficients characterize the ratio of the lifetimes associated with the $l\rightleftarrows t+t$ and $l\rightleftarrows l+t$ decays. It follows from Eqs. (8) and (9) that

$$\frac{\tau_{l\rightleftarrows t+t}}{\tau_{l\rightleftarrows l+t}}\approx\frac{C^2_{l\rightleftarrows l+t}}{C^2_{l\rightleftarrows t+t}}\,\frac{1}{2s^3}. \tag{11}$$

The functions in Fig. 1a, as well as all the results of the calculations which follow, were obtained for $C_{l\rightleftarrows t+t} = C_{l\rightleftarrows l+t} = 1$. Estimates of the ratio of the lifetimes associated with these decays for specific materials can be found in Ref. 11. The longitudinal mode velocity in these calculations was taken to be $s = 1.5$, which gives a lifetime ratio of ≈ 0.148 according to Eq. (8). It is clear that the results of our general model agree reasonably well with Ref. 11. As s increases, the cutoffs clamp the decay probability at about one-half, and the lifetime ratio Eq. (11) decreases, while as s decreases we observe the opposite behavior. Note that a test calculation of functions of the type shown in Fig. 1a for a specific choice of parameters corresponding to CaF_2 gives values close to those obtained in Ref. 8.

In order to verify the generality of the model, i.e., to determine to what extent the results and derivations obtained in this paper can be applied to the study of various materials, we investigated the stability of the solution with respect to variation of the physical parameters of the model. In particular, we varied the form of the probability functions associated with the decay of a phonon into two others, i.e., the coefficient $R_{i\rightleftarrows j+k}$ as a function of j/i for fixed i.

It is not difficult to transform the factor in Eq. (8), which is a smooth varying function of i, j, k, into the following form:

$$\frac{[r_t(i)\,r_t(j)\,r_t(k)]^3}{ijk}=i^6((j/i)\,(1-(j/i)))^2. \tag{12}$$

Then the decay functions used in the calculations have the form

$$i^6[\,(j/i)\,(1-(j/i))\,]^\beta,\ \beta=2.$$

In addition to the function with $\beta = 2$, we also carried out calculations for $\beta = 1$ and $\beta = 4$ without including the θ-function. Figure 1b shows all three decay probability functions: the function given in Eq. (8) as well as two approximations using the smooth functions given above for $\beta = 1$ and $\beta = 4$. It is clear that there are not qualitative differences in the behavior of these three functions.

EVOLUTION OF THE NONEQUILIBRIUM PHONON DISTRIBUTION

The results of calculations for an initial phonon distribution of "δ-function" from [i.e., $n_i(0) = 0, i \neq N$] serve to illustrate our model. The excitation energy corresponds to a finite temperature of $T = 0.04$. For convenience the results are presented in the form of a "modal" temperature (i.e., the temperature of each mode):

$$T(\Omega,t)=\frac{\Omega}{\ln[1+1/n(\Omega,t)]}\equiv T_\Omega(t).$$

[For the case of an equilibrium distribution $n(\Omega,t)$ the temperature satisfies $T(\Omega,t) = \text{const} = T$.] Figure 2 shows the

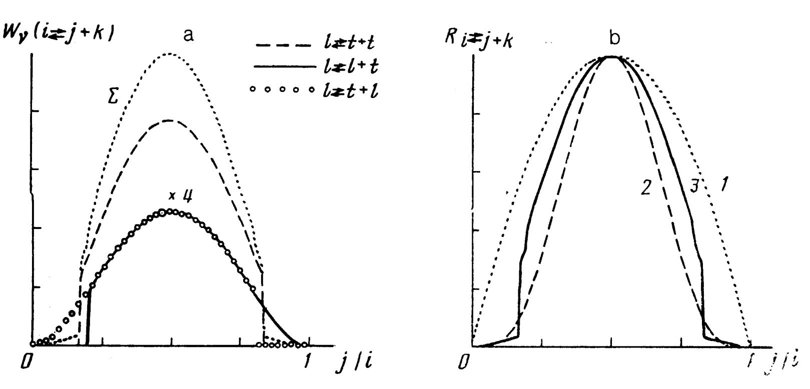

FIG. 1. Decay characteristics for three-phonon processes: (a)—dependence of the decay probability $R_{i\rightleftarrows j+k}$ of a phonon of frequency i/N into two phonons with frequencies j/N and k/N as a function of the ratio j/i (the upper curve, Σ), along with its component $W_\gamma(i\rightleftarrows j+k)$ for the three allowed processes (see text); (b)—decay characteristics of phonons when the decay function is approximated by the smooth function $[(j/i)(l-(j/i))]^\beta$. l—$\beta = 1$, 2—$\beta = 4$, 3—calculated decay probability for $\beta = 2$ (corresponds to curve Σ of Fig. 1a).

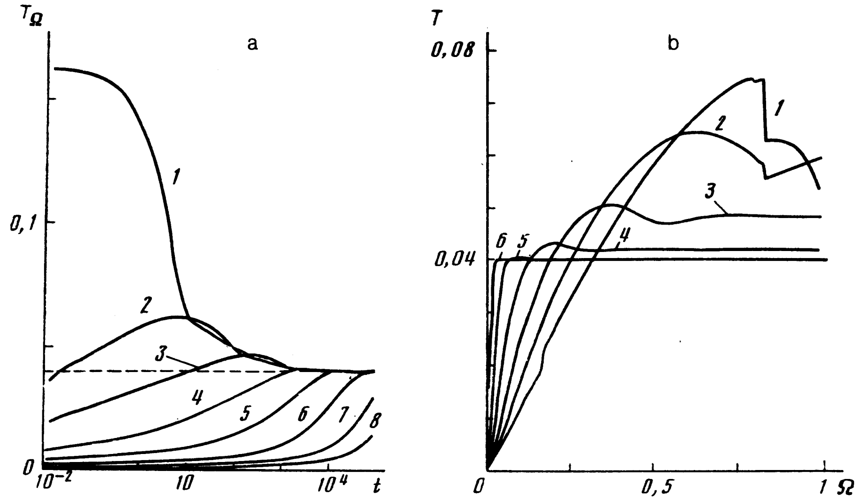

FIG. 2. Results of calculations for δ-type initial conditions $[n_i(0) = 0, i \neq N]$ for the phonon distribution. The excitation energy corresponds to a finite temperature $T = 0.04$; *a*—time dependence of the phonon modal temperature T_Ω for various values of frequency $\Omega = 1$ (*1*), 1/2 (*2*), 1/4 (*3*), 1/8 (*4*), 1/16 (*5*), 1/32 (*6*), 1/64 (*7*), 1/128 (*8*), ($i = 128, 64, \ldots, 1$ respectively); b—frequency dependence of the modal temperature at various times $t = 1$ (*1*), 10 (*2*), 10^2 (*3*), 10^3 (*4*), 10^4 (*5*), 10^5 (*6*).

behavior of this modal temperature for phonons with frequencies 1, 1/2,..., 1/128 ($i = 128, 64, \ldots, 1$ respectively). We see that first the modal temperatures of phonons with frequencies 1 and 1/2 become equal; soon after, the temperatures of those with 1, 1/2 and 1/4 become equal, etc. For this case the maximum of the phonon distribution shifts in a stepwise way into the region of low frequencies. The evolution of the phonon distribution can be followed more clearly if we plot the modal temperatures as a function of phonon frequency at different times (Fig. 2b). Here we see the gradual diffusion of the thermalized interval into the region of lower frequencies; the rate of this shift slows significantly as the frequencies decrease. As $t \to \infty$ the system reaches equilibrium at a temperature $T_\Omega = T = 0.04$ for all Ω.

THE ESTABLISHMENT AND BOUNDARIES FOR EXISTENCE OF THE SELF-SIMILAR SOLUTION

The possibilities of this model have not been exhausted by the description given above. In particular, we will now investigate the question of self-similarity. A theoretical analysis of the kinetic equations that describe the behavior of a system of phonons in the early stages of its evolution was given in Ref. 3. In studying the linearized version of a system of equations similar to Eq. (3) for a spatially homogeneous excitation. Kazakovtsev and Levinson found a self-similar solution for $n(\Omega,t)$ of the form

$$n(\Omega, t) = A\Omega^{-3} t^{1/5} f_{sc}(\eta), \quad \eta = \Omega t^{1/5}, \quad \Omega \ll 1, \tag{13}$$

where A is a constant that depends on the overall energy of the system and $f_{sc}(\eta)$ is a certain universal function which does not even depend on the injected energy. In their further work[4] they obtained order-of-magnitude estimates of the time interval within which solutions of the form (13) can exist; solutions for the real occupation numbers ceased to be of this form as processes involving phonon amplification were switched on, which occurred when some characteristic $n(\Omega,t)$ attained the value unity. However, the simple fact that a self-similar solution exists still does not answer the question of whether or not a specific initial condition, which will not itself be self-similar as a rule, can eventually evolve toward such a solution. Because our model allows us to obtain the explicit time dependence of the occupation numbers and does take into account phonon amplification processes, it is possible to use it to answer both of these questions, i.e., to study how an initial condition evolves towards a solution of the form (13) and to determine the time interval within which a real solution remains self-similar.

In order to answer these questions, we must calculate the function $f(\eta)$ using our model in order to obtain the functions $n(\Omega,t)$:

$$f(\eta) = \frac{f^*(\eta)}{\int_0^\infty f^*(\eta)\, d\eta}, \quad f^*(\eta) = \begin{cases} n(\Omega, t)\Omega^3/t^{1/5}, & \eta = \Omega t^{1/5} \\ 0, & \eta > t^{1/5}. \end{cases} \tag{14}$$

The customary normalization, which corresponds to conservation of energy, has the form

$$E = A\int_0^\infty f(\eta)\, d\eta = \text{const}$$

and allows us to eliminate the coefficient A. The function analogous to (14) for a system of linearized kinetic equations corresponding to the set (3) ceases to depend on time for $t \gg 10$, which allows us to identify it with the scaling function $f_{sc}(\eta)$ and to conclude that the solution of the linearized system has achieved self-similarity. Figure 3 shows the scaling function obtained in this fashion for various profiles of the decay probability (Fig. 1b). It is clear that the differences between the various scaling functions are considerably smaller than the differences in the decay probability.

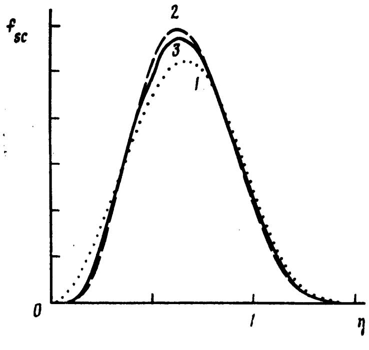

FIG. 3. Form of the scaling function for the various phonon decay characteristics shown in Fig. 1a. The labelling of these curves corresponds to that in Fig. 1b.

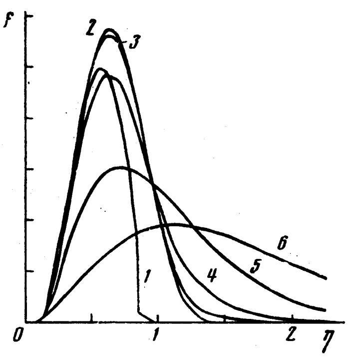

FIG. 4. Form of the function $f(\eta)$ at various instants of time t. The labelling of the curves corresponds to the labelling in Fig. 2b.

Figure 4 shows the results of calculating the function $f(\eta)$ at various instants of time by solving the full nonlinear system of equations (3). In this case the overall phonon energy was chosen so that the final (equilibrium) temperature is $T = 0.04$. The figure shows clearly that during the time interval $t = 10$ to 100 the shape of the function $f(\eta)$ becomes stable and closely resembles the self-similar function $f_{sc}(\eta)$ (Fig. 3). However, during other time intervals $f(\eta)$ differs considerably from $f_{sc}(\eta)$ and is time-dependent (Fig. 4).

TRANSITION OF THE SOLUTION TO THE SYSTEM OF KINETIC EQUATIONS TO SELF-SIMILAR FORM

We conclude that the solution of system (3) has reached the self-similar form when the functions $f(\eta)$ and $f_{sc}(\eta)$ coincide. Let us define a quantitative measure of how they differ as follows:

$$S(t)=\int_0^\infty |f(\eta)-f_{sc}(\eta)|\,d\eta. \tag{15}$$

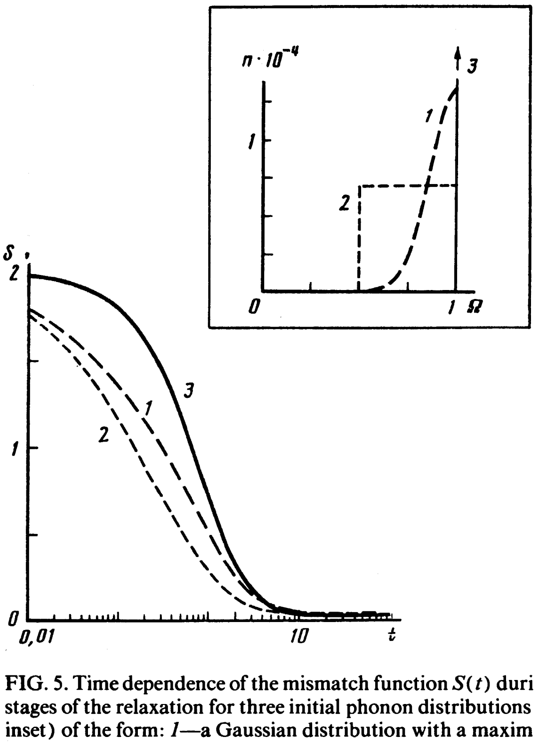

FIG. 5. Time dependence of the mismatch function $S(t)$ during the initial stages of the relaxation for three initial phonon distributions $n(\Omega,0)$ (see inset) of the form: *1*—a Gaussian distribution with a maximum at $\Omega = 1$ and a half-width at half-height of 0.156; *2*—a "half-step" distribution $[n(\Omega,0) = 0, \Omega < 1/2; n(\Omega,0) = \text{const} \neq 0, 1/2 < \Omega < 1]$; and *3*—a δ-distribution $[n_i(0) = 0, i \neq N]$. The finite temperature for all these distributions was $T = 0.04$; $S = 0$ corresponds to complete coincidence of the functions $f(\eta)$ and $f_{sc}(\eta)$, $S = 2$ implies the functions are not correlated.

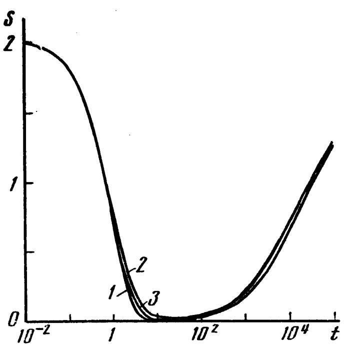

FIG. 6. Time dependence of the function $S(t)$ for a δ-type initial phonon distribution for the various phonon decay characteristics shown in Fig. 1b. The labelling of the curves corresponds to the labelling of Fig. 1b, and $T = 0.04$.

We will illustrate the transition of the solution of self-similar form for the following three different initial distributions $n(\Omega,0)$ (see the inset to Fig. 5): 1—a Gaussian distribution with a maximum at $\Omega = 1$ and a half-width of 0.156; 2—half of a step function [i.e., $n(\Omega,0) = 0$ for $\Omega < 1/2$, and $n(\Omega,0) = \text{const} \neq 0$ for $1/2 < \Omega < 1$); and 3—the delta distribution described above. The finite temperature of all these distributions was $T = 0.04$. It is clear that the broader initial distributions resemble $f_{sc}(\eta)$ more and therefore the self-similar solution is established more rapidly. It is also clear from Fig. 5 that until $t \approx 10$, the distribution function $n(\Omega,t)$ corresponds to the scaling function $f_{sc}(\eta)$ independent of the type of initial distribution.

In addition, we investigated the question of the stability of the solution with respect to various decay laws. In Fig. 6 we show the behavior of $S(t)$ for the δ-type initial distribution. $T = 0.04$, for various profiles of the decay probability (Fig. 1b). The difference in $S(t)$ for these three laws is even smaller than the differences in the corresponding scaling function $f_{sc}(\eta)$ (Fig. 3).

Thus, we can conclude that the solution to the system (3) will approach the self-similar solution for various initial conditions and decay laws.

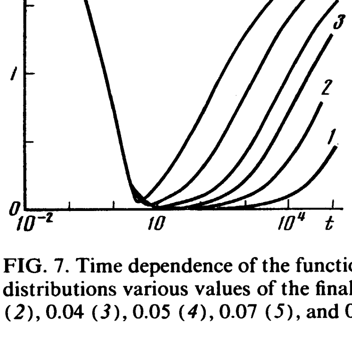

FIG. 7. Time dependence of the functions $S(t)$ for δ-type initial phonon distributions various values of the final temperature: $T = 0.02$ (*1*), 0.03 (*2*), 0.04 (*3*), 0.05 (*4*), 0.07 (*5*), and 0.1 (*6*).

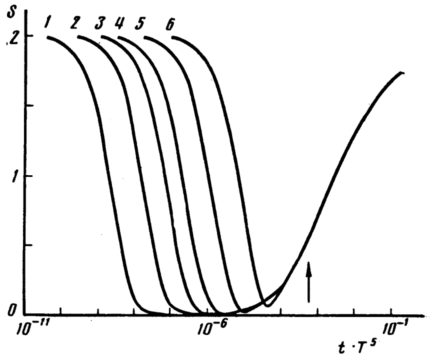

FIG. 8. Time dependence of the function $S(t)$ for δ-type initial phonon distributions plotted against the coordinate tT^5 for various values of the final temperature T. The labelling of the curves corresponds to the labelling of Fig. 7. The arrow denotes that value $tT^5 = 6 \times 10^{-4}$ (see text).

DISRUPTION OF THE SELF-SIMILAR SOLUTION

Figure 7 illustrates the disruption of the self-similar solution and the dependence on total system energy of the time within which the latter exists: the behavior of $S(t)$ for various excitation energies, i.e., values of the temperature T, is shown for the solution described above (see Fig. 2) corresponding to an initial δ-distribution. It is clear that for $T \leqslant 0.1$ the transition of the solution $n(\Omega,t)$ to the self-similar form is practically independent of the system energy. However, in qualitative agreement with Ref. 3, we find that the time within which the self-similar approximation exists (i.e., the time of activation of the amplification processes) depends strongly on this energy.

According to our calculations (see Fig. 3), a phonon distribution that satisfies the scaling solution has a maximum energy density at a frequency

$$\tilde{\Omega}_{sc}(t) \approx 0{,}64 t^{-1/5}. \tag{16}$$

On the other hand, it is well-known that the maximum of the energy density for an equilibrium distribution with temperature T is

$$\tilde{\Omega}_T = 2{,}82 T. \tag{17}$$

The self-similar solution is a good approximation for $\tilde{\Omega}_{sc}(t) \gg \tilde{\Omega}_T$, or, in other words, when the energy of the phonon subsystem is primarily concentrated in phonons whose frequency exceeds the characteristic frequency of the equilibrium distribution with temperature T. The scaling approximation begins to break down as $\tilde{\Omega}_{sc}(t)$ approaches $\tilde{\Omega}_T$ in the process of relaxation; this is connected with the growth in importance of the role of amplification processes for these phonons. Setting Eqs. (16) and (17) equal, we can obtain an estimate for the time it takes this disruption to occur; we find $t \approx 6 \times 10^{-4}\ T^{-5}$. Actually, if we plot the function $S(t)$ for various temperatures (see Fig. 7) as a function of tT^5, then it is clear that these functions coincide almost completely in the right-hand portion of Fig. 8, which reflects the similarity in the nature of the disruption of the scaling solution at various temperatures. Thus, our calculations have confirmed that the time in which the scaling function exists is proportional to T^{-5}. The arrow in Fig. 8 points to the value of tT^5 obtained from Eqs. (16) and (17). It is clear that this estimate is in reasonable agreement with the results of the calculations.

Let us also note (and this is clear from comparing Figs. 1b, Fig. 3 and Fig. 6) that the role of the nonlinear terms in disrupting the self-similar approximation is almost independent of the form of the decay characteristics of the phonons.

CONCLUSIONS

Thus, we have arrived at the following picture of the evolution of a system of nonequilibrium phonons which at $t = 0$ is excited at a comparatively low level $T < 0.1$ such that high-frequency phonons are predominantly excited. In the initial stage of relaxation $(0 \leqslant t \leqslant 10)$ there is a transition of the phonon distribution to self-similar form. For time $t \geqslant 10$ the system achieves this self-similar distribution independent of the initial conditions and the form of the decay characteristics of the phonons. The time within which the phonon distribution remains self-similar depends on the level of excitation of the system and is proportional to T^{-5}. During this stage the relaxation of the phonon system is dominated by decay processes for those phonons with frequencies close to the maximum of its energy distribution. During later stages of the relaxation the scaling approximation is violated; this is associated with the growing role of those phonon amplification processes that lead to the establishment of an equilibrium distribution with a well-defined temperature in the system.

For higher levels of excitation $(T > 0.1)$ the role of amplification processes is important even in the initial stages of relaxation. In this case self-similar distribution of phonons in principle need not be established at all, and the relaxation process can lead directly to the establishment of an equilibrium temperature without a stage corresponding to the scaling approximation. This situation was discussed in Ref. 9.

The results of our calculations show that both the approach of the phonon distribution to self-similar form and the departure from this form depend only weakly on the specific characteristics of the phonon decay. This makes it possible to apply this model to describe phonon kinetics over a wide range of materials.

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