Structure of the electron mass operator in a homogeneous magnetic field close to the critical strength

V. N. Baĭer, V. M. Katkov, and V. M. Strakhovenko

Institute of Nuclear Physics, Siberian Division of the USSR Academy of Sciences
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The mass operator M of the electron in a homogeneous magnetic field H close to the critical field $H_0 = 4.41\times10^{13}$ Oe is investigated. The real part of M (containing information about the anomalous magnetic moment of the electron) and the imaginary part of M (related to the probability of emission) are calculated in the form of series in powers of $\eta\equiv H/H_0$ for all values of the principal quantum number n. For the emission intensity a general expression valid for arbitrary η and n is obtained, together with its asymptotic expansions. It is shown that the series for $\eta\ll1$ is asymptotic. The region $\eta\gtrsim1$ is analyzed, and, in particular, the radiative corrections to the ground state for $\eta\gg1$ are calculated to within algebraic corrections. Explicit expressions are obtained for the intensity and probability of emission in this region for $n\gg1$.

1. INTRODUCTION

It is generally accepted that the magnetic field of neutron stars (pulsars) attains values $H\sim4\times10^{12}$ Oe (see, e.g., Ref. 1), close to the critical field $H_0 = m^2/e = (m^2c^3/e\hbar) = 4.41\times10^{13}$ Oe. It has been established recently that in flaring γ-ray sources, which, evidently, are also neutron stars, the magnetic field has magnitude $H\sim2\times10^{12}$ Oe (Ref. 2). In view of this, it seems worthwhile to obtain an adequate description of radiative effects in such fields. These effects include the emission, and also corrections to the mass and to the anomalous magnetic moment. In other words, we are speaking of the determination of the corrections $\sim\eta\equiv H/H_0$ to the formulas of quasiclassical quantum electrodynamics in an external field,[3] which gives a simple and effective description of electromagnetic phenomena that is exact in the parameter $\chi=\gamma\eta\,(\gamma=\varepsilon/m)$ and ignores terms proportional to η.

Since the characteristic lengths of formation of the processes indicated above ($l_f\sim\lambda_c/\eta$) are much shorter than the scale of the gradient of the magnetic field of a neutron star, the calculation can be performed assuming a constant magnetic field H. An exact expression for the mass operator of the electron to order α in a constant magnetic field was obtained in Refs. 4 and 5. It is a double integral of a surprisingly compact expression, which, however, contains oscillating functions. For this reason, direct use of this expression for the calculation of radiative effects turns out to be extremely difficult.

In this paper, we perform the expansion of the mass operator and calculate corrections to the quasiclassical approximation for arbitrary values of the parameter γ. The results are expressed in terms of single integrals and make it easy to follow the structure of the mass operator as one passes from the nonrelativistic to the ultrarelativistic limit. For the emission intensity we obtain an exact expression that is a function of η and the quantum number n. Using this expression, we find asymptotic expansions of the intensity in various regions. The region $\eta\gtrsim1$ for $n\gg1$ is investigated separately, as is the region $\eta\gg1$, which is of undoubted theoretical interest. The correction to the electron mass in the ground state $n=0$ in an ultrastrong field ($\eta\gg1$) is calculated to a finite order in the power-series expansion.

2. THE MASS OPERATOR IN FIELDS BELOW THE CRITICAL FIELD

We use the expression of diagonal form obtained in Ref. 5 for the mass operator of the electron in a homogeneous magnetic field:

$$M=\frac{\alpha m}{2\pi}\int_0^\infty\frac{dx}{x}\int_0^1 du\, e^{-iux/2\eta}\left\{\frac{1}{\Delta}\exp\left[\frac{i\rho}{\eta}\left(a(x)-\frac{ux}{2}\right)\right]\right.$$
$$\times\Big[(\rho(1-u)-u)\,(c(x)+u(s(x)-c(x)))-us(x)$$
$$\left.+i\zeta\gamma_\perp u(1-u)x\left(\frac{c(x)}{x^2}-1+s(x)\right)+1+u\Big]-1-u\right\}. \tag{1}$$

Here

$$\rho=2n\eta=\gamma_\perp{}^2-1,\quad \eta=\frac{H}{H_0},\quad n=n_0+\frac{1-\zeta}{2},$$
$$a(x)=\operatorname{arctg}\frac{uc(x)}{x(1-us(x))},$$
$$\Delta=1-2u(1-u)s(x)+u^2\left[\frac{2c(x)}{x^2}-1\right], \tag{2}$$
$$c(x)=1-\cos x,\quad s(x)=1-\frac{\sin x}{x};$$
$$\zeta=\pm1\ (n\geqslant1),\ \zeta=1\ (n=0).$$

For $\eta\ll1$, values $ux\lesssim\eta\ll1$ make a contribution to the integral (1). Expanding the functions $a(x)$ and $\Delta^{-1}(x)$ in the quantity ux gives

$$a(x)\approx\frac{uc(x)}{x}[1+us(x)+u^2s^2(x)]-\frac{u^3c^3(x)}{3x^3},$$
$$\Delta^{-1}(x)\approx1+2u(1-u)s(x)$$
$$+4u^2(1-u)^2s^2(x)+u^2\left(1-\frac{2c(x)}{x^2}\right). \tag{3}$$

The possibility of further expansion of the exponential in (1) is connected with the magnitude of the parameter $\rho\eta^2\equiv\chi^2$.

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For $\rho \lesssim 1$ and $\eta \ll 1$ the parameter χ is always small. In the ultrarelativistic limit ($\rho \gg 1$) a contribution to the integral (1) is given by values $x \lesssim 1/\rho^{1/2} \ll 1$; then we have $\rho[a(x) - ux/2]/\eta \sim u^3x^3\rho/\eta \lesssim \rho\eta^2 = \chi^2$, and the expansion of the corresponding exponential can be performed only for $\chi \ll 1$. Since the limit $\rho \gg 1$ has been well studied for all values of the parameter χ (see, e.g., Ref. 3), we shall consider the case $\chi \ll 1$ for arbitrary values of the parameter ρ. In this case, keeping terms $\sim \eta^2$ inclusive, we obtain

$$\exp\left[\frac{i\rho}{\eta}a(x)\right]$$

$$\approx \exp\left(\frac{i\rho}{\eta x}uc\right)\left[1+\frac{i\rho u^2c}{\eta x}\left(s+us^2-\frac{uc^2}{3x^2}\right)-\frac{\rho^2u^4c^2s^2}{2\eta^2x^2}\right]. \tag{4}$$

As is well known, the imaginary part of the mass operator is connected with the probability of emission of a photon by the following relation:

$$W=-\frac{2}{\gamma}\operatorname{Im} M. \tag{5}$$

A contribution to the integral over u for the imaginary part of the mass operator is given by values $u \ll 1$, and the region $u \sim 1$ is exponentially suppressed. This makes it possible to extend the integral over u to ∞, which significantly simplifies the subsequent calculations. The physical meaning of the exponential suppression of the emission probability for $u \sim 1$ is as follows. Since the quantity M is invariant under Lorentz transformations, we go over to a reference frame in which the longitudinal (along the magnetic field) momentum is sufficiently large ($p_{\parallel}/m \to \infty$). Here, both the magnitude H of the magnetic field and the value of the parameter $\rho = p_\perp^2/m^2$ remain unchanged. However, in this frame we have $\gamma \gg 1$ and the variable u is uniquely related to the frequency of the emitted photon by $u = \omega/\varepsilon$. On the other hand, it is well known that in the classical region, for $\chi \ll 1$, the emission of frequencies $\omega \sim \varepsilon$ is exponentially suppressed. Performing the integration over u with allowance for what has been said above, we obtain for the probability of emission of a photon the following expression (see the Appendix):

$$W=\frac{2\alpha m^2}{3\varepsilon}\eta^2\left\{\frac{1}{\eta}f_0(\rho)-1-2\rho\right.$$
$$\left.+\zeta\gamma_\perp\left[f_1(\rho)-\frac{\eta}{2}(7+9\rho)\right]+\eta f_2(\rho)\right\}, \tag{6}$$

where

$$f_0(\rho)=\frac{3}{\pi}\int_0^\infty\frac{dx}{x^2}\left(\frac{1+\rho c}{\mu}-1\right),$$
$$\mu=1+\rho\left(1-\frac{2c}{x^2}\right),$$
$$f_1(\rho)=\frac{6}{\pi}\int_0^\infty\frac{dx}{x^2}\left(\frac{1}{\mu^2}\left(\frac{c}{x^2}+s-1\right)+\frac{1}{2}\right),$$
$$f_2(\rho)=\frac{24}{\pi}\int_0^\infty\frac{dx}{x^2}\left\{\frac{1}{\mu^3x^2}\left[s\left(1\right.\right.\right.$$
$$\left.-\frac{3(1+\rho)}{\mu}\right)(1+s(\rho-1)-(1+2\rho)c)$$
$$+\frac{1}{\mu}(1+\rho c)(1+\rho)\left(\frac{c^2}{x^2}\right.$$
$$\left.\left.\left.+3s^2\left(1-\frac{2(1+\rho)}{\mu}\right)\right)+(1+\rho)(s-c)\right]$$
$$\left.+\frac{7\rho+5}{12}\right\}. \tag{7}$$

Plots of the functions f_0, f_1, and f_2 are presented in Fig. 1. The calculation of Re M is a more complicated problem, since a contribution to the real part of M comes from the entire region of integration over the variable u. After the elementary integrals over u have been taken, the remaining integral over x in the region $\eta \ll x \ll 1$ contains terms proportional to dx/x, which lead to the appearance in Re M of the factor $\ln(1/\eta)$. To identify explicitly the terms with the logarithm it is convenient to divide the range of integration over x into two parts, choosing as their boundary a value x_0 satisfying the condition $1 \gg x_0 \gg \eta$. As a result, we obtain (see the Appendix)

$$\operatorname{Re} M=\frac{\alpha m}{2\pi}\left\{-\frac{1}{2}\zeta\gamma_\perp\eta+\eta^2\left[\varphi_1(\rho)+\frac{4}{3}(1+2\rho)\ln\frac{1}{2\eta}\right]\right.$$
$$\left.-\zeta\gamma_\perp\eta^3\left[\varphi_2(\rho)-\left(\frac{14}{3}+6\rho\right)\ln\frac{1}{2\eta}\right]\right\}, \tag{8}$$

where

$$\varphi_1(\rho)=4\int_0^\infty\frac{dx}{x^3}\left\{1-2(1+2\rho)s+\frac{1}{\mu^2}[(1-\rho)s+(1+2\rho)c-1]\right.$$
$$\left.-2s(1+\rho)(1+\rho c)/\mu^3\right\}-\frac{1}{18}+\frac{20}{9}\rho,$$

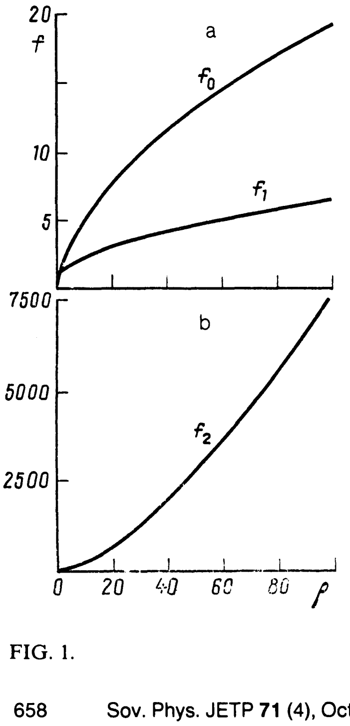

FIG. 1.

$$\varphi_2(\rho)=4\int_0^\infty \frac{dx}{x^3}\left\{\frac{4}{\mu^3}\left(\frac{c}{x^2}+s-1\right)\left[\left(\frac{3(1+\rho)}{\mu}-1\right)s-1\right]\right.$$
$$\left.-2+(7+9\rho)s\right\}+\frac{1}{9}+3\rho.$$

We find the mass operator in the nonrelativistic limit $\rho \ll 1$, keeping terms $\sim \eta^3$. The corresponding expansions of the functions f and φ in this case have the form

$$f_0(\rho)\approx\rho\left(1-\frac{1}{5}\rho\right),\quad f_1(\rho)\approx1+\frac{1}{10}\rho,\quad f_2\approx\frac{7}{2};$$
$$\varphi_1(\rho)\approx-\frac{13}{18}+\left(\frac{293}{90}-\frac{32}{5}\ln 2\right)\rho,\quad \varphi_2(\rho)\approx\frac{32}{5}\ln 2-\frac{83}{90}. \tag{9}$$

Taking into account also that $\gamma_\perp \approx 1+\rho/2$, we obtain

$$W=\frac{2\alpha m^2}{3\varepsilon}\eta^2\left[2n_0-\frac{4}{5}n^2\eta-4n\eta+\frac{6}{5}\zeta n\eta+\frac{7}{2}(1-\zeta)\eta\right]. \tag{10}$$

We note that in the leading order in η the emission probability is determined entirely by the value of the orbital quantum number n_0 and does not depend on the electron spin. For Re M, which determines the corrections to the mass and also the anomalous magnetic moment of the electron, we have

$$\operatorname{Re} M=\frac{\alpha m}{2\pi}\left\{-\frac{1}{2}\zeta\gamma_\perp\eta\left[1-\eta^2\left(4\left(\frac{7}{3}+3\rho\right)\right.\right.\right.$$
$$\left.\left.\times\ln\frac{1}{2\eta}-\frac{64}{5}\ln 2+\frac{83}{45}\right)\right]$$
$$\left.+\eta^2\left[\frac{4}{3}\ln\frac{1}{2\eta}-\frac{13}{18}+n\eta\left(\frac{16}{3}\ln\frac{1}{2\eta}-\frac{64}{5}\ln 2+\frac{293}{45}\right)\right]\right\}. \tag{11}$$

The expansions (10) and (11) coincide with the asymptotic forms obtained in Ref. 4.

The asymptotic forms of the functions f for $\rho \gg 1$ are as follows:

$$f_0\approx\frac{5}{4}(3\rho)^{1/3},\quad f_1\approx\frac{3}{8}(3\rho)^{1/3},\quad f_2\approx\frac{35}{8}\rho(3\rho)^{1/3}. \tag{12}$$

Substituting these values into Eq. (6), we obtain

$$W\approx\frac{\alpha m^2\chi}{\varepsilon}\left[\frac{5\sqrt{3}}{6}-\frac{4}{3}\chi+\frac{35}{4\sqrt{3}}\chi^2+\zeta\chi\left(\frac{\sqrt{3}}{4}-3\chi\right)\right]. \tag{13}$$

We give the corresponding asymptotic forms of the functions $\varphi(\rho)$:

$$\varphi_1\approx\frac{8}{3}\rho\left[\ln\left(\frac{12}{\rho}\right)^{1/2}+C-\frac{33}{16}\right],$$
$$\varphi_2\approx-6\rho\left[\ln\left(\frac{12}{\rho}\right)^{1/2}+C-\frac{37}{12}\right],$$
$$C=0{,}577215\ldots \tag{14}$$

We than have the following expression for Re M for $\rho \gg 1$:

$$\operatorname{Re} M\approx\frac{\alpha m}{2\pi}\left\{-\frac{1}{2}\zeta\gamma_\perp\eta\left[1-12\chi^2\right.\right.$$
$$\left.\times\left(\ln\frac{1}{\chi}+C+\frac{1}{2}\ln 3-\frac{37}{12}\right)\right]$$
$$\left.+\frac{8}{3}\chi^2\left(\ln\frac{1}{\chi}+C+\frac{1}{2}\ln 3-\frac{33}{16}\right)\right\}. \tag{15}$$

The asymptotic forms of Re M for $n\sim 1$ [see (11)] and for $\rho \gg 1$ [see (15)] have been calculated previously by different methods and were independent expressions. Now, however, Eq. (8) gives a unified description of the entire region $\eta \ll 1$, and, in particular, describes the behavior of the anomalous magnetic moment of the electron in a field.

3. THE EMISSION INTENSITY

Important characteristics of the emission are its spectral composition and intensity. In the operator technique, the differential probabilities of processes are usually obtained with the aid of projection operators onto the corresponding states. We distinguish states with a definite photon frequency ω and with a definite projection of the photon momentum onto the axis $\mathbf{e}_z \| H$:

$$1=\int d^2q\,\delta(q-k)=\frac{1}{(2\pi)^2}\int d^2q\int d^2\xi\, e^{i(q-k)\xi},$$
$$q=(q_0,0,0,q_z),\quad \xi=(\xi_0,0,0,\xi_z). \tag{16}$$

The calculation of the mass operator in Ref. 5 was performed by means of an exponential parametrization of the squared electron propagator and the photon propagator. In the integrand in the integral over d^4k we shall be interested in the term $\hat{k}\exp(is\mathscr{H})$, where $\mathscr{H}=(\mathscr{P}^2-2\mathscr{P}k)u+k^2$. Its product with the factor $\exp(-ik\xi)$ can be represented in the form

$$e^{-ik\xi}\hat{k}\exp(is\mathscr{H})=\left(\hat{\varkappa}+\frac{\hat{\xi}}{2s}\right)\exp\left(is\mathscr{H}(\varkappa)-iu\mathscr{P}\xi-\frac{i\xi^2}{4s}\right), \tag{17}$$

where $\varkappa=k-\xi/2s$. After the integration over $d^4\varkappa$ the extra term in the matrix structure has the form $\hat{\xi}(c_1+c_2\sigma F)$; in a magnetic field, the matrices $\hat{\xi}$ and σF commute. The average value of the additional term in the mass operator on the mass shell can be calculated using the following anticommutators:

$$\{\hat{\xi},\hat{\mathscr{P}}\}=2\xi\mathscr{P},\quad \{\hat{\xi}\sigma F,R\}=\frac{1}{4}\{\hat{\xi}\sigma F,\{\sigma F,\hat{\mathscr{P}}\}\}$$
$$=\frac{1}{2}\xi\mathscr{P}(\sigma F)^2=2\xi\mathscr{P}H^2,\quad R=\gamma^5\mathscr{P}F^*\gamma,\quad R^2=\mathscr{P}F^{*2}\mathscr{P}=H^2\varepsilon_\perp^{\,2}. \tag{18}$$

Noting that the eigenvalue of the operator R is $\zeta\sqrt{R^2}$, we obtain

$$\langle\hat{\xi}\rangle=\frac{1}{m}\xi\mathscr{P},\quad \langle\hat{\xi}\sigma F\rangle=\frac{2}{\varepsilon_\perp}\zeta H\xi\mathscr{P}. \tag{19}$$

When (19) is taken into account the integral over ξ of the expression (17) is Gaussian:

$$\frac{1}{(2\pi)^2}\int d^2\xi\,(1,\xi\mathscr{P})\exp\left(iq\xi-iu\xi\mathscr{P}-\frac{i\xi^2}{4s}\right)$$

$$=\frac{s}{\pi}\left(1,i\frac{d}{du}\right)\exp[i(q-up)^2s],\quad p=(\varepsilon,0,0,p_z).\quad (20)$$

As a result, we obtain for the mass operator the following covariant expression:

$$dM=\frac{\alpha d^2q}{2\pi^2m}\int_0^\infty\frac{dx}{2\eta}\int_0^1\frac{du}{u\Delta}\exp\left[-\frac{iux}{2\eta}+\frac{ix(q-up)^2}{2\eta um^2}\right]$$
$$\times\left\{\exp\left[\frac{i\rho}{\eta}\left(a(x)-\frac{ux}{2}\right)\right]\left[\left(\rho-\frac{pq}{m^2}\right)[c+u(s-c)]\right.\right.$$
$$+i\zeta\gamma_\perp u\left(1-\frac{pq}{\varepsilon_\perp^2}\right)\left(\frac{c}{x}-\sin x\right)+i\zeta\left(u\gamma_\perp-\frac{pq}{m\varepsilon_\perp}\right)(1-s)$$
$$\left.\left.-us+1-u\rho+\frac{pq}{m^2}\right]-1+u\rho-\frac{pq}{m^2}\right\}.\quad (21)$$

The simplest form of the spectral distribution of the emission probability for arbitrary values of the parameter ρ is obtained in the infinite-momentum frame ($p_z\to\infty$). (We note that for $p_z/m\gg 1$ the classical motion of the particle in the magnetic field is the same as in a spiral undulator.) In this approximation we have $\gamma=\varepsilon/m\gg 1$ and a contribution to the integral over d^2q is given by the values $q_0-q_z\sim q_0/\gamma$. With relativistic accuracy, we obtain

$$q_0-q_z\approx\frac{q^2}{2q_0},\quad qp=q_0\varepsilon-q_zp_z\approx\frac{q^2\varepsilon}{2q_0}+\frac{q_0p^2}{2\varepsilon},$$
$$\frac{(q-up)^2}{u}\approx\left(\frac{1}{u}-\frac{\varepsilon}{q_0}\right)q^2+\left(u-\frac{q_0}{\varepsilon}\right)p^2.$$

We multiply the expression for dM by the factor i/γ and extend the integration over x from $-\infty$ to ∞. In the theory of the emission, in the integration over the angles of emission of the photon [or, as in (21), over d^2q], integrals of the following form arise:

$$\int_{-\infty}^{\infty}\int_0^{\infty}F(x)f(xy)\,dx\,dy=\lim_{a\to\infty}\int_{-\infty}^{\infty}F(x)\,dx\int_0^a f(xy)\,dy$$
$$=\int_{-\infty}^{\infty}\frac{dx}{x}[F(x)-F(0)]\int_0^{\infty}f(y)\,dy.$$

In the expression for the emission probability the subtraction that is contained in the mass operator (1), and that also arises in the integration over d^2q, can be ensured by displacing the contour of integration a little below the real axis. Changing now in (21) to the variables $q_0-q_z\approx q^2/2q_0$, $q_0\equiv\omega$ and performing the integration over d^2q and the variable u, we obtain for the spectral distribution of the emission probability the following expression:

$$dW=\frac{i\alpha m^2d\omega}{2\pi\varepsilon^2}\int_{-\infty}^{\infty}\frac{dx}{x-i0}\left\{\left[\rho-(1+\rho)u+\frac{i\eta}{x}\frac{d}{du}u\right][c+u(s-c)]\right.$$
$$+i\zeta\gamma_\perp\left[1-u+\frac{i\eta}{(1+\rho)x}\frac{d}{du}u\right]u\left(\frac{c}{x}-\sin x\right)$$
$$\left.+\left(u-\frac{\zeta\eta}{\gamma_\perp}\frac{d}{du}u\right)(1-s)+1-\frac{i\eta}{x}\frac{d}{du}u\right\}$$
$$\times\frac{1}{\Delta}\exp\left\{-\frac{iux}{2\eta}+\frac{i\rho}{\eta}\left[a(x)-\frac{ux}{2}\right]\right\},\quad u=\frac{\omega}{\varepsilon}.$$
$$(22)$$

By multiplying dW by ω we obtain the spectral distribution of the emission intensity in the undulator limit. Performing the integration over $\omega(u)$, we obtain the following expression for the total emission intensity:

$$I=\frac{i\alpha m^2}{2\pi}\int_{-\infty}^{\infty}\frac{dx}{x-i0}\int_0^1\frac{udu}{\Delta}\exp\left\{-\frac{iux}{2\eta}+\frac{i\rho}{\eta}\left[a(x)-\frac{ux}{2}\right]\right\}$$
$$\times\left\{\left[\rho-(1+\rho)u-\frac{i\eta}{x}\right][c+u(s-c)]\right.$$
$$+i\zeta\gamma_\perp u\left[1-u-\frac{i\eta}{(1+\rho)x}\right]$$
$$\left.\times\left(\frac{c}{x}-\sin x\right)+\left(\frac{\zeta\eta}{\gamma_\perp}+u\right)(1-s)+1+\frac{i\eta}{x}\right\}.$$
$$(23)$$

We note that the expression (23) has an invariant form and does not depend on the reference frame in which the calculations were performed.

We now perform the expansion of the emission intensity in powers of η for $\eta\ll 1$, $\chi^2=\rho\eta^2\ll 1$, as was done above for the probability of the process. As a result, we have (see the Appendix

$$I=\alpha m^2\eta^2\left\{\frac{2}{3}\rho+\eta\left[f_4(\rho)+\zeta(1+\rho)^{1/2}\left(\rho+\frac{2}{3}\right)\right]\right.$$
$$\left.+\eta^2\left[8(4\rho+1)\left(\rho+\frac{2}{3}\right)+\zeta(1+\rho)^{1/2}f_5(\rho)\right]\right\},\quad (24)$$

where

$$f_4(\rho)=\frac{16}{\pi}\int_0^\infty\frac{dx}{x^4}\left\{\frac{1}{\mu^3}\left[(1+2\rho)c+s\right.\right.$$
$$\times\left(1+(1+\rho c)\left(1-\frac{3}{\mu}(1+\rho)\right)\right)$$
$$\left.\left.-\frac{1}{4}c\mu\right]-\frac{x^2}{24}(12\rho+5)\right\},$$

$$f_5(\rho)=\frac{16}{\pi}\int_0^\infty\frac{dx}{x^4}\left\{\frac{6}{\mu^4}\left(\frac{c}{x^2}+s-1\right)\left[1+2s\left(1-\frac{2}{\mu}(1+\rho)\right)\right]\right.$$
$$+3-\frac{x^2}{4}(12\rho+7)$$
$$+\frac{1}{1+\rho}\left[\frac{1}{\mu^3}\left(1-s-\frac{c}{x^2}+s(1-s)\left(1-\frac{3}{\mu}(1+\rho)\right)\right)\right.$$
$$\left.\left.-\frac{1}{2}+\frac{x^2}{24}(15\rho+11)\right]\right\}.\quad (25)$$

Plots of the functions f_4 and f_5 are presented in Fig. 2. In the limit of nonrelativistic transverse motion ($\rho\ll 1$), we have

$$f_4(\rho)\approx-\frac{2}{3}-\frac{11}{3}\rho,\quad f_5(0)=-\frac{16}{3}.\quad (26)$$

In this case, with allowance for corrections $\sim\eta$, we obtain for the emission intensity the following expression:

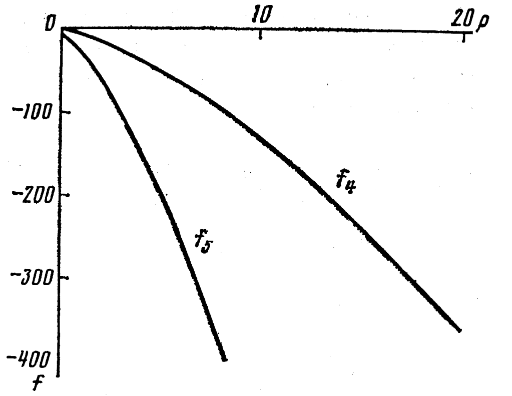

FIG. 2.

$$I \approx \frac{2}{3}\alpha m^2\eta^3\{2n-1+\zeta+\eta[(4\zeta-11)n+8(1-\zeta)]\}$$
$$=\frac{2}{3}\alpha m^2\eta^3\left\{[2+\eta(4\zeta-11)]n_0+\frac{1-\zeta}{2}\eta\right\}. \quad (27)$$

In the case of ultrarelativistic transverse motion ($\rho\gg 1$), but when the parameter $\chi=\eta p^{1/2}$ is nevertheless small ($\chi\ll 1$), we obtain

$$f_4(\rho)\approx-\frac{55}{8\sqrt{3}}\rho^{3/2},\quad f_5(\rho)\approx-\frac{35\sqrt{3}}{4}\rho^{3/2},$$

$$I\approx\alpha m^2\chi^2\left[\frac{2}{3}-\frac{55}{8\sqrt{3}}\chi+32\chi^2+\zeta\chi\left(1-\frac{35\sqrt{3}}{4}\chi\right)\right]. \quad (28)$$

We note that the expansions (24), (27), and (28) are asymptotic, and this leads to large magnitudes of the numerical coefficients in them and, correspondingly, decreases their ranges of applicability. For example, in the case $\rho\gg 1$, even for $\chi=0.1$ the emission of unpolarized particles is weaker by a factor of approximately 2/3 than the classical intensity $I_{cl}=2\alpha m^2\chi^2/3$ (Ref. 6). For the same reason, for $\eta\sim 0.1$ Eq. (27) [i.e., the expansion (24)] becomes inapplicable, even for the lowest levels (e.g., for $n_0=1$ and $\zeta=-1$).

The characteristics of the emission with a spin flip are of great interest, since this process can preferentially polarize the particles. The corresponding probability and intensity of the emission for nonrelativistic transverse motion ($\rho\ll 1$) can be obtained from the expressions (10) and (27) by taking into account that the desired quantities in this case do not depend on ρ and by setting $n_0=0$ in these expressions. As a result, we obtain, e.g., for the probability of emission with a spin flip,

$$W_{\zeta,-\zeta}=\frac{\alpha m^2}{3\varepsilon}\eta^3(1-\zeta),\quad \rho\ll 1. \quad (29)$$

The probability of emission with a spin flip for $\rho\gg 1$, $\chi\ll 1$ can be obtained in quasiclassical theory (see, e.g., Ref. 3):

$$W_{\zeta,-\zeta}=\frac{\alpha m^2}{2\varepsilon}\chi^3\left(\frac{5\sqrt{3}}{8}-\zeta\right). \quad (30)$$

Recently, the problem of the radiation from an electron in an arbitrary external electromagnetic field has been solved with allowance for the first quantum correction.[7] In the emission problem the quantum corrections appear as an expansion in powers of the parameters χ and η (if the electric field satisfies $E=0$). In this sense, a way of finding the first term of the expansion in χ or in η was formulated in Ref. 7. The quasiclassical expressions that we are using are exact[1)] in χ (but require $\eta=0$), while Eq. (1) is exact both in χ and in η, i.e., contains all quantum effects exactly. It should be borne in mind that in our earlier paper[5] the mass operator (and hence the emission probability) was found in a constant external field of arbitrary configuration (with arbitrary fields **H** and **E**), so that the generalization of (1) to this case follows directly from Ref. 5. As regards transitions with a spin flip, it should be taken into account that the spin-flip amplitude is proportional to $\hbar$ (more precisely, to $\hbar\omega/\varepsilon$), and, consequently, from the outset the expression for the probability contains the necessary power of $\hbar$. Therefore, to calculate the leading term in this case it is sufficient to perform the calculations on the classical trajectory. By virtue of what has been said, the last formula in Ref. 7 agrees with that obtained earlier in Ref. 9.

Above, we considered the region $\chi\ll 1$, for arbitrary values of the parameter ρ, including the lowest states $n\sim 1$. For $\chi\gtrsim 1$ in a weak field ($\eta\ll 1$) the transverse motion of the particle is certainly ultrarelativistic ($\rho=\chi^2/\eta^2\gg 1$). In this case, in the calculation of radiative effects the quasiclassical theory of emission first formulated by two of the present authors for the case of a (generally speaking) inhomogeneous magnetic field (see Ref. 3) is applicable. In the framework of this theory[3] the anomalous magnetic moment of the electron and corrections to the electron mass in an external field were considered. Taking into account that the quasiclassical theory has been described in detail in textbooks and monographs (see, e.g., Refs. 10 and 11), we shall not dwell on this region ($\eta\ll 1$, $\chi\gtrsim 1$), but give only the results. In the quasiclassical theory, an integral representation of the mass operator that is convenient for obtaining the series in χ for $\chi\ll 1$ and in inverse powers of χ for $\chi\gg 1$ has the form

$$M=\frac{\alpha m}{48\sqrt{3}}\frac{1}{2\pi i}\int_{-0-i\infty}^{-0+i\infty} ds\,(3\chi)^{s+1}\frac{e^{-i\pi s/2}}{\sin\pi s}\left[\frac{i}{\cos(\pi s/2)}\right.$$
$$\times(3s^2+3s+10)\Gamma\left(\frac{s}{2}+\frac{1}{6}\right)\Gamma\left(\frac{s}{2}+\frac{5}{6}\right)$$
$$\left.+\frac{3\zeta s(s+1)}{\sin(\pi s/2)}\Gamma\left(\frac{s}{2}+\frac{1}{3}\right)\Gamma\left(\frac{s}{2}+\frac{2}{3}\right)\right]. \quad (31)$$

For the emission intensity, correspondingly, we have

$$I=\frac{\alpha m^2}{24\sqrt{3}}\frac{1}{2\pi i}\int_{-0-i\infty}^{-0+i\infty} ds\,(3\chi)^{s+2}\frac{(s+1)}{\sin\pi s}\left[\zeta s(s+2)\Gamma\left(\frac{s}{2}+\frac{5}{6}\right)\right.$$
$$\left.\times\Gamma\left(\frac{s}{2}+\frac{7}{6}\right)-(s^2+2s+8)\Gamma\left(\frac{s}{2}+\frac{2}{3}\right)\Gamma\left(\frac{s}{2}+\frac{4}{3}\right)\right]. \quad (32)$$

For $\chi\ll 1$, closing the integration contour on the right in the expressions (31) and (32), we obtain asymptotic series in powers of χ. For $\chi\gg 1$ the contour of integration must be closed on the left, the pole singularities of the integrands lie at $s<0$, and series in inverse powers of χ are obtained.

Since the leading (in $\rho\gg 1$) terms of the expansion of the mass operator and of the emission intensity are functions of only the one invariant parameter χ, $\chi \ll 1$ series for the expressions (31) and (32). An analysis of the radiative effects in the constant-field approximation for $\chi \gtrsim 1$ can be found not only in the monographs Refs. 10 and 11 but also in Refs. 6 and 12. For example, in Ref. 12 it is shown that in the region $1 \lesssim \chi \lesssim 15$ the emission intensity is described by the simple formula $I \approx 2\alpha m^2 \chi/15$ with accuracy better than 10%. For such values of χ we have $\bar{\omega} \approx \varepsilon\chi/(2+5\chi)$, and the production of e^+e^- pairs by the emitted photon in the magnetic field becomes possible. However, the consideration of cascade processes lies outside the scope of the present paper.

4. RADIATIVE EFFECTS IN STRONG FIELDS *($H \gtrsim H_0$)*

Above, we considered radiative effects in a weak field ($\eta \ll 1$) for arbitrary values of the quantum number n (parameter ρ) from $n=0$ to $n \gg \eta^{-3}$. We now turn to the case when the magnetic field is of the order of the critical field ($\eta \sim 1$), or considerably greater ($\eta \gg 1$). For $\eta \sim 1$ the calculations of the radiative effects for particles in the lowest Landau levels must be carried out numerically. An example of such calculations is the computation of the anomalous magnetic moment of the electron carried out in Ref. 13. Simple analytical expressions can be obtained for $n=0$, $\eta \gg 1$, or for large values $n \gg 1$ but without restriction on the parameter η. Corrections to the electron mass in the ground state were considered in Refs. 14–16. The most advanced quantitative result was obtained in Ref. 16, in which the correction to the electron mass was calculated with logarithmic accuracy. In Ref. 16, the expression obtained in Ref. 14 for the mass operator in the ground state, which agrees with Eq. (1) for $n=0$, was used. Below, we perform calculations for this case with algebraic accuracy.

By making use of the relation

$$\Delta=\delta\delta^*,\quad \delta=1-u+\frac{iu}{x}(e^{-ix}-1), \tag{33}$$

we can represent the expression for M determined by Eq. (1), for $n=0$ ($\zeta=1$), in the form

$$M=\frac{\alpha m}{2\pi}\int_0^\infty \frac{dx}{x}\int_0^1 du\, e^{-iux/2\eta}\left(\frac{1+ue^{-ix}}{\delta}-1-u\right). \tag{34}$$

Taking into account that the function $\delta(x)$ does not have zeros in the fourth quadrant of the plane of the complex variable x, we expand the integration contour onto the lower semi-axis. As a result, we obtain

$$M=\frac{\alpha m}{2\pi}\int_0^\infty dz\int_0^1 du\, e^{-uz/2\eta}\left[\frac{1+ue^{-z}}{(1-u)z+u(1-e^{-z})}-\frac{1+u}{z}\right]. \tag{35}$$

In the case $\eta \gg 1$ the region of integration over z can be divided conveniently into two regions: $z \ll z_0$, and $z \gg z_0$, where $\eta \gg z_0 \gg 1$. In the first region we can replace the common exponential factor by unity, and in the second we can neglect terms proportional to e^{-z}. In addition, in the region $z \gg z_0$ we divide the range of integration over u into two, with a boundary u_0 satisfying the conditions $1-u_0 \ll 1$ and $(1-u_0)z_0 \gg 1$, and carry out the corresponding expansions. By separating out the logarithmic terms in explicit form and "matching" the integrals in the indicated regions, we obtain with algebraic accuracy the following expression for the correction to the electron mass in the ground state:

$$M=\frac{\alpha m}{4\pi}\left[\left(\ln 2\eta-C-\frac{3}{2}\right)^2+A+O\left(\frac{1}{\eta}\right)\right], \tag{36}$$

where

$$A=\frac{5\pi^2}{6}-\frac{19}{4}+2\left[\int_0^1 dz\left(g(z)-\frac{3}{2z}\right)+\int_1^\infty dz\left(g(z)-\frac{\ln z}{z-1}\right)\right]=4{,}02816\ldots,$$

$$g(z)=\frac{e^{-z}}{\xi(z)}+\left(\frac{1}{\xi(z)}-\frac{ze^{-z}}{\xi^2(z)}\right)\ln\left(1+\frac{\xi(z)}{z}\right),$$
$$\xi(z)=1-z-e^{-z}. \tag{37}$$

Let us consider the mass operator of the electron in a strong field ($\eta \gtrsim 1$) for high excited levels ($n=\rho/2\eta \gg 1$). In this case, the main contribution to the integral (1) is determined by the region that limits the magnitude of the term proportional to n in the argument of the exponential; we represent this term in the form

$$2n\left(\operatorname{arctg}\frac{u(1-\cos x)}{u\sin x+(1-u)x}-\frac{ux}{2}\right)$$
$$=-2n\left(\operatorname{arctg}\frac{zt}{1-z(1-t\operatorname{ctg}t)}-zt\right)\equiv-2n\varphi(t),$$
$$z=1-u,\quad t=x/2. \tag{38}$$

It can be seen from the expression (38) that for $n \gg 1$ the main contribution to the integral (1) is given by the region $zt \ll 1$, or $u \ll 1$. In the case $zt \ll 1$ the expansions of the functions $\varphi(t)$ and Δ^{-1} in series in this parameter have the form

$$\varphi(t)=z^2t\left[(1-z^2t^2)b(t)+zb^2(t)-\frac{zt^2}{3}+\frac{z^3t^4}{5}+\ldots\right],$$
$$b(t)=1-t\operatorname{ctg}t,\quad \Delta^{-1}=\frac{t^2}{\sin^2 t}(1+2zb-z^2t^2+\ldots). \tag{39}$$

It can be seen from this expansion that for $z \sim 1$ a contribution is given by values $t \sim n^{-1/3} \ll 1$, while for $t \sim 1$ we have $z \sim n^{-1/2} \ll 1$. In the integrand in (1) the leading term is the term proportional to $z\rho$, which for $t \ll 1$ is of order $\rho t^2 \sim \eta n^{1/3} \sim \chi^{2/3}$, while for $z \ll 1$, with allowance for the phase volume, it is of order $\rho z^2 \sim \eta$. A contribution $\sim 1/n$ to the mass operator (but not to the emission intensity) is given by the region $u \sim 1/n \ll 1$ as well. From this it can be seen that the leading term of the expansion in $n^{-1/3}$ in the mass operator should coincide with the corresponding quasiclassical asymptotic form for $\chi \gg 1$ (31), and the difference can have an effect only in the subsequent terms of the expansion.

Taking into account the analysis performed above, and making use of Eqs. (38) and (39), we obtain for the first three terms of the expansion of the mass operator the following expressions (see the Appendix):[2)]

$$W=-\frac{2}{\gamma}\,\mathrm{Im}\,M=\frac{\alpha m^2}{\varepsilon}\left\{\frac{14}{27}\Gamma\left(\frac{2}{3}\right)(3\chi)^{2/3}-\frac{5}{6}-k_0\eta\right.$$
$$+\frac{10}{9}\Gamma\left(\frac{1}{3}\right)(3\chi)^{-2/3}\left(1+\frac{7\eta^2}{25}\right)+\zeta'\left[\frac{1}{27}\Gamma\left(\frac{1}{3}\right)(3\chi)^{1/3}\right.$$
$$\left.\left.-\frac{2}{9}\Gamma\left(\frac{2}{3}\right)(3\chi)^{-1/3}+\frac{1}{4\chi}\left(1+\frac{22}{15}\eta^2\right)\right]\right\},$$
$$k_0=\frac{2}{\pi}\int\limits_0^\infty dt\left(\frac{3}{t^2}-\frac{\sin^2 t}{t^2-\sin^2 t}\right)=1{,}5213\ldots, \tag{40}$$

$$\Delta m=\mathrm{Re}\,M=\alpha m\left\{\frac{7\sqrt{3}}{81}\Gamma\left(\frac{2}{3}\right)(3\chi)^{2/3}\right.$$
$$-\frac{1}{\pi}\left[1+\frac{5}{6}l(\chi)-\frac{1}{4}\eta k_1(\eta)\right]$$
$$-\frac{5}{9\sqrt{3}}\Gamma\left(\frac{1}{3}\right)(3\chi)^{-2/3}\left(1+\frac{7}{25}\eta^2\right)-\zeta'\left[\frac{1}{54\sqrt{3}}\Gamma\left(\frac{1}{3}\right)(3\chi)^{1/3}\right.$$
$$+\frac{1}{9\sqrt{3}}\Gamma\left(\frac{2}{3}\right)(3\chi)^{-1/3}$$
$$\left.\left.-\frac{1}{4\pi\chi}\left(\left(1+\frac{22}{15}\eta^2\right)l(\chi)+\frac{3}{4}+\eta^2\left(k_2(\eta)-\frac{1}{5}\right)\right)\right]\right\}, \tag{41}$$

where

$$\chi^2=2n\eta^3,\quad \zeta'=\zeta(1+1/\rho)^{1/2},\quad l(\chi)=\ln\frac{\chi}{\sqrt{3}}-C,$$
$$k_1(\eta)=\int\limits_0^\infty dt\,e^{-t/\eta}\left(\frac{1}{t\,\mathrm{cth}\,t-1}-\frac{\mathrm{cth}\,t}{t}-\frac{2}{t^2}\right),$$
$$k_1(\eta\gg1)\approx-0{,}56902\ldots+(\ln\eta)/\eta,$$
$$k_2(\eta)=\int\limits_0^\infty\frac{dt}{t}e^{-t/\eta}\left(\frac{1}{t\,\mathrm{cth}\,t-1}-\frac{3}{t^2}-\frac{1}{5}\right), \tag{42}$$
$$k_2(\eta\gg1)\approx-\frac{1}{5}\ln\eta+0{,}5519.$$

Plots of the functions k_1 and k_2 are given in Fig. 3. For $\eta\ll1$ Eqs. (40) and (41) coincide with the corresponding expansion of the integral (31) in inverse powers of χ.

In the limit of an ultrastrong field ($\eta\gg1$) we obtain from Eqs. (40)–(42) the following expressions for the emission probability and the correction to the electron mass:[3)]

$$W=\frac{\alpha m^2}{\varepsilon}[\eta(a_1n^{1/3}-k_0+a_2n^{-1/3})$$
$$+\zeta\eta^{1/2}(a_3n^{1/6}+a_4n^{-1/6})], \tag{43}$$

where $a_1=1.8401$, $a_2=0.3180$, $a_3=0.1606$, and $a_4=0.2593$; and

$$\Delta m=\alpha m\left\{\eta(b_1n^{1/3}-\varkappa_0+b_2n^{-1/3})\right.$$
$$+\zeta\eta^{1/2}\left[-b_3n^{1/6}+\frac{1}{2\pi\sqrt{2n}}(\ln\eta\right.$$
$$\left.\left.+c_1\ln n-c_2)\right]\right\}, \tag{44}$$

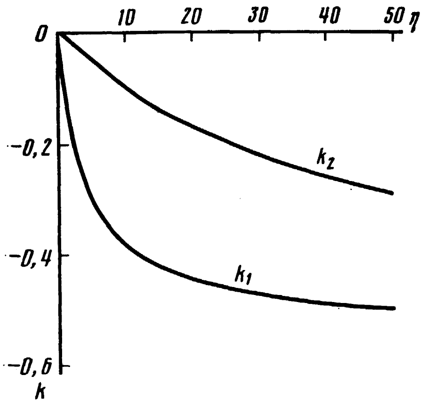

FIG. 3.

where

$$b_i=\frac{1}{2\sqrt{3}}a_i,\quad \varkappa_0=-\frac{1}{4\pi}k_1(\infty),$$
$$c_1=\frac{11}{30},\quad c_2=0{,}3964.$$

The calculation of the emission intensity from Eq. (43) is performed in the same way as for the mass operator, with the sole difference that the region $u\ll1$ does not make a contribution to the expansion terms kept. As a result, we have

$$I=\alpha m^2\left\{\frac{32}{243}\Gamma\left(\frac{2}{3}\right)(3\chi)^{2/3}\right.$$
$$-\frac{2}{3}+\frac{2}{81}\Gamma\left(\frac{1}{3}\right)(3\chi)^{-2/3}\left(55+\frac{4}{5}\eta^2\right)$$
$$+\zeta'\left[\frac{5}{243}\Gamma\left(\frac{1}{3}\right)(3\chi)^{1/3}\right.$$
$$\left.\left.-\frac{14}{81}\Gamma\left(\frac{2}{3}\right)(3\chi)^{-1/3}+\frac{1}{4\chi}\left(1+\frac{76}{45}\eta^2\right)\right]\right\}. \tag{45}$$

For $\eta\gg1$ we obtain

$$I=\alpha m^2[\eta(d_1n^{1/3}+d_2n^{-1/3})+\zeta\eta^{1/2}(d_3n^{1/6}+d_4n^{-1/2})], \tag{46}$$

where $d_1=0.4673$, $d_2=0.0202$, $d_3=0.0892$, and $d_4=0.2986$.

From the spin part of the correction to the electron mass we can obtain an expression for the anomalous magnetic moment. In units of $\alpha/2\pi$ this expression has the form

$$\mu(\chi,\eta)=\frac{2\pi}{9\sqrt{3}}\left[\Gamma\left(\frac{1}{3}\right)(3\chi)^{-2/3}+6\Gamma\left(\frac{2}{3}\right)(3\chi)^{-4/3}\right]$$
$$-\frac{1}{\chi^2}\left[\left(1+\frac{22}{15}\eta^2\right)l(\chi)+\frac{3}{4}+\eta^2\left(k_2(\eta)-\frac{1}{5}\right)\right]. \tag{47}$$

For $\eta\gg1$, using the asymptotic form (42) of $k_2(\eta)$, we obtain

$$\mu_n(\eta)=\frac{1}{\eta}\left[\frac{c_3}{n^{1/3}}-\frac{1}{n}(\ln\eta+c_1\ln n-c_2)\right], \tag{48}$$

where c_1 and c_2 are defined in Eq. (44) and $c_3=0.4120$.

We note that, with logarithmic accuracy, Eq. (48) is also valid for the lower excited levels $n \gg 1$ (see Ref. 13 and the literature cited therein). It can be seen from the expression (48) that for $n \gg 1$ the anomalous magnetic moment vanishes $[\psi_n(\eta_0) = 0]$ and has a minimum $\psi_n(n_m)$ at large values of the parameter η, justifying the use of the asymptotic form (48) in this region. Here,

$$\eta_0 = \exp(c_3 n^{2/3} - c_1 \ln n + c_2), \qquad \eta_m = e\eta_0. \tag{49}$$

Using Eq. (49) for $n = 1$ and $n = 2$, we obtain $n_m(n = 1) \approx 6.10$, $\mu_1(\eta_m) \approx -0.164$, $n_m(n = 2) \approx 6.03$, and $\mu_2(\eta_m) = -0.083$. These values and the behavior of the function (48) for $\eta \gtrsim n_m$ agree fairly well with the results of the numerical calculation of the anomalous magnetic moment performed in Ref. 13.

The expressions obtained above for the emission probability and intensity make it possible to estimate the characteristic frequencies of the emission. In a weak field ($\eta \ll 1$), in the case of nonrelativistic transverse motion ($\rho \ll 1$), from Eqs. (10) and (16) we obtain for the mean emission frequency ($p_z = 0$)

$$\overline{\omega} = \frac{I}{W} \approx \eta m = \omega_0, \tag{50}$$

which corresponds to dipole radiation in an oscillator with $\Delta n = 1$. If the longitudinal momentum is large ($p_z^2 \gtrsim m^2$), for the emitted frequencies it is necessary to take the Doppler shift into account. In the case when the parameter ρ is not small, but the parameter χ is small in magnitude, the mean photon energy is determined by the expression

$$\frac{\overline{\omega}}{\varepsilon} \approx \frac{\rho\eta}{f_0(\rho)}, \quad \frac{\overline{\omega}}{\varepsilon}(\rho \gg 1) \approx \frac{4\eta}{5}\left(\frac{\rho}{3}\right)^{1/2} \sim \frac{\chi}{2}. \tag{51}$$

For $\chi \gtrsim 1$ the energy of the emitted quanta becomes comparable to the electron energy, while for $\chi \gg 1$ we obtain $\overline{\omega} \approx \varepsilon a_1/d_1 \sim \varepsilon/4$. In the case of ultrastrong fields ($\eta \gg 1$), in the emission from the lowest excited levels the mean energy of the photons becomes even larger. Since the coefficients of the series in powers of $n^{-1/3}$ for the emission probability and intensity in this case fall off quite rapidly, to estimate $\overline{\omega}$ we can use Eqs. (43) and (46) up to $n \sim 1$. For unpolarized particles we obtain for $\eta \gg 1$ the following expression for $\overline{\omega}$:

$$\frac{\overline{\omega}}{\varepsilon} \approx \frac{1}{4}\left(1 - \frac{5}{6} n^{-1/3} + \frac{1}{6} n^{-2/3}\right)^{-1}. \tag{52}$$

APPENDIX

We illustrate the details of the calculation in the limit $\eta \ll 1$ for the example of the determination of the expansion of the mass operator to terms $\sim \eta^2$ inclusive. The next terms of the expansion in Eqs. (6) and (8) and the corresponding expression (24) for the intensity are calculated analogously. In accordance with what has been said in the text, we divide the range of integration over x in Eq. (1) into two: $(0,x_0)$ and (x_0,∞), where $\eta \ll x_0 \ll 1$. In the range $(0,x_0)$ we can use the expressions (3) and (4) and then expand the functions $c(x) = 1 - \cos x$ and $s(x) = 1 - (\sin x)/x$ in x. Then, to within the specified accuracy, after the change of variable $x \to 2\eta x$ the contribution of the region $(0,x_0)$ takes the form

$$M_1 = \frac{\alpha m}{2\pi} \int_0^1 du \int_0^{x_0/2\eta} dx\, e^{-iux} \Big\{ \eta^2 \frac{x}{3} \langle \rho(1-u)[2(3-2u) - iu(1-u)^2 x] - u(3u^2-5u+4) \rangle - i\eta\zeta\gamma_\perp u(1-u) \Big\}. \tag{A1}$$

The integrals over x in (A1) can be expressed in terms of the function $\varphi(u) = (1 - e^{iux_0/2\eta})/u$ and its derivatives with respect to u. In their turn, the terms containing derivatives of the function $\varphi(u)$ will be integrated by parts. In the integrated terms we can set $\varphi(0) = ix_0/2\eta$, $\varphi(1) = 1$, and $\varphi'(1) = -1$. The only nonelementary integral over u has the form

$$\int_0^1 du\, \varphi(u) \approx \ln\frac{x_0}{2\eta} + C + i\frac{\pi}{2}, \tag{A2}$$

where we have taken into account that $x_0/2\eta \gg 1$; C is the Euler constant. Performing the integration over u, we find for the contribution of the region $(0,x_0)$

$$M_1 = \frac{\alpha m}{2\pi} \Big\{ \eta^2 \Big[-\frac{5}{2} - \frac{8}{3}\rho + \frac{4}{3}(1+2\rho)\Big(\ln\frac{x_0}{2\eta} + C + i\frac{\pi}{2}\Big) - \frac{5}{6} i\rho \frac{x_0}{\eta} \Big] - \eta\zeta\gamma_\perp \Big(\frac{1}{2} + \frac{2i\eta}{x_0}\Big) \Big\}. \tag{A3}$$

In the region (x_0,∞), by making use of Eqs. (3) and (4), after the substitution $u = 2\eta z/x$ we have

$$M_2 = \frac{\alpha m\eta}{\pi} \Big\{ \int_{x_0}^{\infty} \frac{dx}{x^2} \int_0^{x/2\eta} dz \times \Big[e^{-i\mu z}\Big\langle 1 + \rho c(x) + \frac{2\eta z}{x}\Big[2s(x)(1+\rho c(x)) \times \Big(1 + iz\rho\frac{c(x)}{x^2}\Big) + 1 + (\rho-1)s(x) - (1+2\rho)c(x) + i\zeta\gamma_\perp x\Big(s(x) - 1 + \frac{c(x)}{x^2}\Big)\Big]\Big\rangle - e^{-iz}\Big(1 + \frac{2\eta z}{x}\Big)\Big]\Big\}. \tag{A4}$$

Since in (A4) $\mu = 1 + \rho(1 - 2c(x)/x^2) > 1$ and the upper limit of the integral over z is equal to $x/2\eta \gg x_0/2\eta \gg 1$, within exponentially small terms this integration can be extended to ∞, after which it becomes trivial. Thus, the term linear in η in M_2 has the form

$$M_{21} = -\frac{i\alpha m\eta}{\pi} \int_{x_0}^{\infty} \frac{dx}{x^2}\Big[\frac{1+\rho c(x)}{\mu} - 1\Big] \approx -\frac{i\alpha m\eta}{\pi}\Big\{ \int_0^{\infty} \frac{dx}{x^2}\Big[\frac{1+\rho c(x)}{\mu} - 1\Big] - \frac{5}{12}\rho x_0 \Big\}. \tag{A5}$$

In the term which is quadratic in η the integral over x diverges as $x_0 \to 0$. Subtracting and adding terms that cancel this divergence, we find to order η^2

$$M_{22}=-\frac{2\alpha m\eta^2}{\pi}\left\{\int_0^\infty\frac{dx}{x^3}\left\langle\left[2s(x)(1+\rho c(x))\right.\right.\right.$$

$$\times\left(1+\frac{2\rho c(x)}{\mu x^2}\right)+1$$

$$+s(x)(\rho-1)-(1+2\rho)c(x)$$

$$\left.+i\zeta\gamma_\perp x\left(s(x)-1+\frac{c(x)}{x^2}\right)\right]\mu^{-2}-1$$

$$\left.+i\frac{x}{2}\zeta\gamma_\perp+2(1+2\rho)s(x)\right\rangle$$

$$\left.-\int_{x_0}^\infty\frac{dx}{x^3}\left[2(1+2\rho)s(x)+i\frac{x}{2}\zeta\gamma_\perp\right]\right\}. \qquad \text{(A6)}$$

The last integral in (A6) is, to the required accuracy, equal to

$$\int_{x_0}^\infty\frac{dx}{x^3}\left[2(1+2\rho)s(x)+i\frac{x}{2}\zeta\gamma_\perp\right]$$

$$\approx\frac{(1+2\rho)}{3}\left[\frac{11}{6}+\ln\frac{1}{x_0}-C\right]+i\zeta\frac{\gamma_\perp}{2x_0}. \qquad \text{(A7)}$$

The sum of the expressions (A3), (A5), and (A6) [with (A7) taken into account], in which the quantity x_0 is cancelled, as it should be, gives, to within terms $\sim\eta^2$, the mass operator M, the imaginary part of which is related to the emission probability by (5).

A calculation of the asymptotic form of M for $n\gg1$ has also been performed by means of "matching." As an example, we shall find the spin-independent part of M. We denote this part by T. The terms in M proportional to ζ are calculated in exactly the same way. Starting from the analysis performed [see the discussion after Eqs. (38) and (39)], we divide the integration over u into three ranges: $(0,u_1)$, (u_1,u_2), and $(u_2,1)$. We subject the parameters u_1 and u_2 to the following conditions: $n^{-1}\ll u_1\ll1$, and $n^{-1/2}\ll 1-u_2\ll1$. In the first range we can perform the expansion in u, after which we obtain for the contribution of this region

$$T_1=\frac{\alpha m\rho}{2\pi}\int_0^\infty\frac{dx}{x}c(x)\int_0^{u_1}du\exp[-inux(1-2c(x)/x^2)]$$

$$=\frac{i\alpha m\eta}{\pi}\int_0^\infty\frac{dx}{x}c(x)\frac{\exp[-inu_1x(1-2c(x)/x^2)]-1}{1-2c(x)/x^2}$$

$$\approx\frac{\alpha m\eta}{\pi}\left\{i\int_0^\infty\frac{dx}{x^2}\left[6-\frac{c(x)}{1-2c(x)/x^2}\right]+(2nu_1)^{1/3}\Gamma\left(\frac{2}{3}\right)(-3i)^{2/3}\right.$$

$$\left.-\frac{7i\Gamma(1/3)}{45}\left(-\frac{3i}{2nu_1}\right)^{1/3}\right\}. \qquad \text{(A8)}$$

The integral over x in (A8) has also been calculated by "matching," with allowance for the condition $nu_1\gg1$. In the range (u_1,u_2) a contribution is given only by small values $x\sim n^{-1/3}$. Performing the corresponding expansion, we have

$$T_2=\frac{\alpha m}{2\pi}\int_{u_1}^{u_2}du\int_0^{x_1}\frac{dx}{x}e^{-iux/2\eta}\left\{\rho\frac{x^2}{2}(1-u)\left\langle\left(1-\frac{2}{3}u\right)\right.\right.$$

$$\times\left[1+ux^2\left(\frac{1}{3}-\frac{u}{4}\right)\right.$$

$$\left.\left.+\frac{inu(1-u)^2x^5}{16\cdot45}(9u^2-12u+2)\right]-\frac{x^2}{12}\left(1-\frac{4}{5}u\right)\right\rangle$$

$$\times\exp\left[-\frac{inu(1-u)^2x^3}{12}\right]+(1+u)$$

$$\left.\times\left(\exp\left[-\frac{inu(1-u)^2x^3}{12}\right]-1\right)\right\}. \qquad \text{(A9)}$$

The conditions imposed on u_1 and u_2 make it possible to choose a value $x_1\ll1$ such that $nu(1-u)^2x_1^3\gg1$, and then the integration over x in (A9) can be extended to ∞. Performing this integration [for compactness only, in (A9) the function $\exp(-iux/2\eta)$ has not been expanded], we find

$$T_2=\frac{\alpha m}{\pi}\int_{u_1}^{u_2}du\left\{\frac{(1-2u/3)}{1-u}\left[\frac{1}{3}\Gamma\left(\frac{2}{3}\right)\left(-\frac{3i}{z}\right)^{2/3}-1\right]\right.$$

$$+\frac{u(2/3-u)}{18(1-u)}\Gamma\left(\frac{1}{3}\right)(-3i)^{4/3}z^{2/3}$$

$$-\frac{\Gamma(1/3)(-3i/z)^{4/3}}{15\cdot27\rho(1-u)^3}\left(7-\frac{56}{3}u+17u^2-6u^3\right)$$

$$\left.+\frac{(1+u)}{3}\left[\frac{1}{2}\ln3+C-\ln\left(\frac{-i}{z}\right)\right]\right\}, \qquad \text{(A10)}$$

in which $z=u/(1-u)\eta\rho^{1/2}$. For the subsequent calculation of T_2, the terms in (A10) that have a singularity as $u\to0$ or $u\to1$ must be transformed by integration by parts, thereby isolating the explicit dependence on the matching parameters u_1 and u_2, after which the integrals reduce to tabulated integrals and for the quantity T_2 we obtain

$$T_2=\frac{\alpha m}{\pi}\left\{\Gamma\left(\frac{2}{3}\right)(-3i\eta\rho^{1/2})^{2/3}\left[\frac{14\pi}{27\sqrt3}-\frac{1}{6}\delta^{2/3}-u_1^{1/3}\right]\right.$$

$$+\frac{1}{3}(\ln\delta-2)$$

$$+\frac{i\Gamma(1/3)(-3i)^{1/3}}{(\eta\rho^{1/2})^{2/3}}\left(\frac{\delta^{-1/3}}{12}-\frac{10\pi}{27\sqrt3}\right)-\frac{i\Gamma(1/3)}{9\cdot15}\left(\frac{-3i\eta^4}{\rho}\right)^{1/3}$$

$$\times\left(\frac{14\eta}{\sqrt3}+\delta^{-2/3}-21u_1^{-1/3}\right)$$

$$\left.+\frac{1}{2}\left[\frac{1}{2}\ln3+C+\frac{1}{3}-\ln(-i\eta\rho^{1/2})\right]\right\}, \qquad \text{(A11)}$$

where $\delta=1-u_2$. In the third range we expand the integrand in the mass operator in the small quantity $y=1-u$:

$$T_3=\frac{\alpha m\rho}{2\pi}\int_0^\infty\frac{dx}{x}e^{-ix/2\eta}\left(\frac{x/2}{\sin(x/2)}\right)^2 s(x)\int_0^\delta dy\,y$$
$$\times\exp\left\{-inxy^2\left[1-\frac{\sin x}{x}\left(\frac{x/2}{\sin(x/2)}\right)^2\right]\right\}. \qquad (A12)$$

Performing the elementary integration over y in (A12) and replacing $x\to 2x$, we have

$$T_3=\frac{i\alpha m\eta}{4\pi}\int_0^\infty\frac{dx}{\sin^2 x}\frac{s(2x)e^{-ix/\eta}}{1-[(\sin 2x)/2x](x/\sin x)^2}F(x), \qquad (A13)$$

$$F(x)=\exp\left\{-irx\left[1-\frac{\sin 2x}{2x}\left(\frac{x}{\sin x}\right)^2\right]\right\},\quad r=2n\delta^2\gg 1.$$

Further analysis shows that the integration contour passes below the poles in (A13) that lie on the real axis. We choose a value x_2 that satisfies the condition $r^{-1/3}\ll x_2\ll 1$. In the range $(0,x_2)$ we can expand in $x\ll 1$, while in the range (x_2,∞) we can replace $F(x)\to -1$. We now add and subtract the integral, in which $F(x)$ is replaced by -1, over the contour $x=x_2e^{i\varphi}(-\pi/2\leqslant\varphi\leqslant 0)$ in the complex x plane. We can extend the resulting integral $(\int_{-ix_2}^{x_2}+\int_{x_2}^\infty)dx$ with $F=-1$ along the negative imaginary semi-axis and go over to the variable $-ix$, while the integrals in the region $|x|\ll 1$ can be calculated directly. As a result, we find for T_3

$$T_3=\frac{\alpha m\eta}{2\pi}\left\{\int_{x_2}^\infty\frac{dx\,e^{-x/\eta}}{2\,\mathrm{sh}^2x}\frac{1-(\mathrm{sh}\,2x)/2x}{1-[(\mathrm{sh}\,2x)/2x](x/\mathrm{sh}\,x)^2}\right.$$
$$+\frac{1}{\eta}\left[\ln\left(\frac{i}{x_2}\right)+\frac{1}{3}\ln\left(-\frac{3i}{r}\right)-\frac{C}{3}\right]$$
$$-\frac{1}{x_2}+\Gamma\left(\frac{2}{3}\right)(-i)^{2/3}\left(\frac{r}{3}\right)^{1/3}$$
$$\left.+\frac{i\Gamma(1/3)}{3}\left(-\frac{3i}{r}\right)^{1/3}\left(\frac{2}{45}-\frac{1}{2\eta^2}\right)\right\}. \qquad (A14)$$

Arranging in the usual way the expression that converges as $x\to 0$ in the remaining integral, and calculating the integrals of the subtraction terms in explicit form, after simple transformations we obtain from (A14)

$$T_3=\frac{\alpha m\eta}{2\pi}\left\{\frac{1}{2}\int_0^\infty\frac{dx}{x}e^{-x/\eta}\left[\frac{1}{\mathrm{cth}\,x-1/x}-\mathrm{cth}\,x-\frac{2}{x}\right]\right.$$
$$+\left(\frac{r}{3}\right)^{1/3}(-i)^{2/3}\Gamma\left(\frac{2}{3}\right)+\frac{i\Gamma(1/3)}{3}\left(-\frac{3i}{r}\right)^{1/3}\left(\frac{2}{45}-\frac{1}{2\eta^2}\right)$$
$$\left.+\frac{1}{\eta}\left[\ln\frac{1}{\eta}-1+\frac{1}{3}\ln\frac{3}{r}+\frac{2}{3}\left(C+i\frac{\pi}{2}\right)\right]\right\}. \qquad (A15)$$

It is not difficult to convince oneself that in the sum $T=T_1+T_2+T_3$ of the expressions (A8), (A11), and (A15) the parameters u_1 and u_2 cancel and the result given in the text is reproduced.

[1] In the case of large quantum numbers the quasiclassical approximation is applicable, naturally, in the nonrelativistic region as well. This situation has been encountered in the problem of the radiation in an undulator,[8] in which the transverse motion was nonrelativistic.

[2] An attempt to analyze the region $\eta\gtrsim 1$ for the emission probability was undertaken in Refs. 17 and 18.

[3] The correction to the electron mass in an ultrastrong electric field ($E\gg E_0$) has been discussed in Ref. 19.

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