The operator formalism in the theory of fermionic strings on Riemann surfaces

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(Submitted 27 December 1989)
Zh. Eksp. Teor. Fiz. **97**, 1429–1452 (May 1990)

A bosonic operator construction of $bc\beta\gamma$ theories (reparametrization and superconformal ghosts in superstring theories) on Riemann surfaces of arbitrary genus is proposed. The $\beta\gamma$ theory is considered in the phase II, in which differences from previously known results for the phase I on Riemann surfaces are uncovered. A global operator formalism is developed which takes into account the global holomorphic geometry in the structure of composite operators built from free fields (currents) on a Riemann surface. In particular, the "global" operators which break the supersymmetry of the world sheet are identified. An operator realization of "sewing" of two Riemann surfaces with $bc\beta\gamma$ theories defined on them is constructed. An explicit operator mechanism which leads to a rearrangement of the set of unphysical poles under sewing is constructed.

1. INTRODUCTION

The recent progress in multiloop calculations for superstrings (Refs. 1–7) and the attempts to derive explicit results for physical meaningful quantities (in particular, the cosmological constant) have demonstrated the significant complexity of a physical system like the superstring. It remains a hard problem to carry calculations to completion; the methods applicable to Riemann surfaces of lowest genera, i.e., to the first loop diagrams of string theory will, as a rule stop working for surfaces of higher genus.

It therefore seems desirable to carry out an analysis of multiloop amplitudes from the viewpoint of the structure of the more elementary objects which make it up. Such a "fundamentalist" approach is possible within the operator formalism for superstrings on Riemann surfaces of arbitrary genus (Ref. 8–20). In the sequel we shall mainly investigate the most nontrivial ingredient of the superstring: the superconformal $\beta\gamma$ ghosts (Ref. 21), which form a system of bosonic fields of the first order. At first sight the existence of an operator formalism for the $\beta\gamma$ system on a Riemann surface seems problematical since such systems exhibit so-called "unphysical" (spurious) poles (Refs. 1–3): these poles arise from the position of any of the field insertions into the many-point amplitude not on account of a confluence of the given point and another insertion, but are a global effect, which is sensitive to the position of all the other insertions and to the global structure of the Riemann surface.

An appropriate operator description of a $\beta\gamma$ system on Riemann surfaces nevertheless turns out to be possible within the framework of the formalism of global bosonization.[18] This approach is a consequence of the development of a global operator formalism for the bc system (reparametrization ghosts) on Riemann surfaces (Refs. 16 and 17). We propose to treat the objects related to the $\beta\gamma$ system as composite operators in a free bosonic theory, in such a manner that only operators are used which have controllable global features on the Riemann surface. The fundamental idea, applied first in the simpler bc-case (Refs. 16,17), consists in the use of the Baker-Akhiezer operator functions (Refs. 22–24), i.e., expressions of the form

$$\left\{\exp\int_{z_0}^{z} j\right\}\frac{\theta\left(z-z_0+\oint_b j\right)}{\theta(z-z_0)},\qquad(1.1)$$

where j is any (meromorphic) 1-differential (current) on the Riemann surface and $\oint_b j$ is the vector of its b-periods.[1)] The expression (1.1) generalizes to the case of Riemann surfaces the exponential

$$\exp\int_{z_0}^{z} j,$$

which locally represents the normal-ordered product of the well-known operator exponentials: $:\exp\phi(z)\exp(-\phi(z_0)):$, obtained from the local representation of the current in the form $j=\partial\phi$, where ϕ is a scalar field. In contrast to the "bare" exponential

$$\exp\int_{z_0}^{z} j,$$

which depends on how the integration path winds itself onto the "homologies" (the basis cycles a_i, b_i, $i=1,\dots,g$ of the homology), the expression (1.1) does not depend on the selection of the path from z_0 to z if in the Abel map and the θ functions one always chooses the same path as in the exponential.

Thus, in place of the scalar field

$$\phi(z)=\int^{z} j,$$

which is only locally defined and which effects "jumps around the homologies" (Refs. 25–27), we propose to consider the current j as a globally defined fundamental object. More precisely, in order to describe the $\beta\gamma$-theory one needs two currents, since the local bosonization formulas include two scalar fields ϕ and χ (Ref. 21):

$$\beta=\partial\xi e^{-\phi},\quad \gamma=\eta e^{\phi},\quad \xi=e^{-\chi},\quad \eta=e^{\chi}.\qquad(1.2)$$

 0038-5646/90/050807-13$03.00

The presence of the "auxiliary" fermionic $\xi\eta$ system in the equations (1.2) represents a remarkable distinction between the $\beta\gamma$- and bc-system (i.e., a distinction between commuting first-order theories and anticommuting theories). The fields ξ and η are anticommuting 0- and 1-differentials on the Riemann surface, but they do not reduce to the ordinary bc-system with conformal spin 1, since bc-systems do not exhibit unphysical poles on Riemann surfaces, whereas the $\xi\eta$-system is exactly the carrier of unphysical poles in the $\beta\gamma$-theory. It was noted recently (Refs. 28,29), that since the fields $\partial\xi$ and η are both 1-differentials with a (local) operator product $\partial\xi(x)\eta(y) = (x-y)^{-2}$, the substitution $\partial\xi\leftrightarrow\eta$ does not change (up to signs) the operator products, including the β and γ fields, and therefore an alternative bosonization is possible, with permuted $\partial\xi$ and η. Following Reference 30 we shall designate below this bosonization as the phase II of the $\beta\gamma$ theory. The phase II provides another splitting of the $bc\beta\gamma$ degrees of freedom into independent fields $\varphi,\bar{\varphi},\psi,\bar{\psi}$, compared to $\beta\gamma\to\xi\eta\phi$, of which the first two are scalars and the other two fermions of spin $\frac{1}{2}$.

The analysis of the $bc\beta\gamma$ theories in these new terms deserves attention in view of the importance of $bc\beta\gamma$-systems both in applications to superstrings, and in a completely different context (see Ref. 31). On Riemann surfaces of higher genera it is also not irrelevant which of the two 1-differentials $\partial\xi$ and η is exact, since the permutation $\partial\xi\leftrightarrow\eta$ leads to a change of the global properties, and in particular of the position of the unphysical poles. The correlation function in the phase II are also different from the corresponding ones in phase I. However, the theory preserves its invariant meaning: indeed, multiloop calculations for superstrings include an analysis of the position of unphysical poles in the presence of picture-changing operators (Refs. 1–4, 7). The structure of the picture-changing operators (and of the $BRST$-operators) in the phase II differs from that in the phase I (Ref. 30), and it is this difference which needs to be compensated by a permutation of the unphysical poles.

In Sec. 2 the $bc\beta\gamma$ theory is constructed in the phase II. The global operator bosonization is developed, corresponding to a splitting of the $\beta\gamma$ system into the set of independent fields $\varphi,\bar{\varphi},\psi,\bar{\psi}$, and the correlation functions are determined.

The phase II is related to an explicitly supersymmetric superfield bosonization of the combined $bg\beta\gamma$ system consisting of the reparametrization and superconformal ghosts, when the fields b,c,β,γ are considered as the component fields of two chiral superfields (Ref. 21):

$$B(x,\vartheta)=\beta(x)+\vartheta b(x),\quad C(x,\vartheta)=c(x)+\vartheta\gamma(x),\qquad(1.3)$$

where ϑ is the supercoordinate. The expression for the superfields in terms of two chiral scalar superfields leads, in terms of the component fields, exactly to the bosonization in the phase II.

On Riemann surfaces of higher genus one should not expect a realization of the global superfield formalism since the appropriate global Killing vectors are missing there. Therefore one does not succeed in giving a global meaning to the superfields (1.3) constructed in terms of the free fields $\varphi,\bar{\varphi},\psi,\ \bar{\psi}$: the monodromies of the putative superpartners around homologically nontrivial cycles are essentially different, preventing one from attributing a definite monodromy to the superfields B and C.

We shall also see that the nearest global analogs of the component fields in Eqs. (1.3) turn out to be nonfree on account of global effects (properly those which spoil the monodromy). These fields will be expressed, as are all objects in the $bc\beta\gamma$ theory, in terms of composite objects depending on the elementary fields $\varphi,\bar{\varphi},\psi$, and $\bar{\psi}$ (more precisely, their analogs in the global operator bosonization). The dressing of the fields $\varphi,\bar{\varphi},\psi,\bar{\psi}$ occurring on account of the global effects (i.e., the transformation of fields into nonfree fields) is another manifestation of the presence of unphysical poles in the $\beta\gamma$ system (and therefore in the $\varphi,\bar{\varphi},\psi,\bar{\psi}$ system).[2)]

One expects more from the operator formalism in string theory than a simple recovery of the correlation functions. In particular, one assumes that an operator formalism will turn out to be able to describe processes with a change of topology in string amplitudes, i.e., processes related to a change of genus of the Riemann surface. The two fundamental operations which change the genus are the gluing on of a handle to a Riemann surface of genus g, leading to a surface of genus $g+1$, and the sewing together of two Riemann surfaces of genera g_1 and g_2 into a single surface of genus g_1+g_2. The latter process is realized in the global operator formalism of Sec. 3 (the theory is again described in the phase II).

The global operator formalism turns out to be well adapted to the description of the sewing process. The composite operators constructed from currents are subject to two kinds of changes in the transition from the Riemann surfaces C_1 and C_2 to the sewn Riemann surface $C_1\propto C_2$: first of all the currents themselves change (which is similar to the variation of the argument in a tensor transformation law); secondly, there occurs a form-variation of the composite operators. The latter is realized by means of multiplication by some composite operator, also expressed in terms of currents (an additional normal-ordering is required, see Sec. 3).

A similar program was carried through for the anticommuting bc-theories in Ref. 17: there the transformation of the currents was practically trivial and the sewing reduced in essence to a fusion of the operators (i.e., multiplication and normal reordering in the product). The complication which arises in the $\beta\gamma$ system is again related to the unphysical poles: the necessity to restructure the system of unphysical poles requires a nontrivial transformation law for the currents. As a result of this the whole construction turns out to be less explicit than the bc-system. This represents, however, a manifestation of the nature of the $\beta\gamma$ theory rather than a deficiency of the formalism; working with the $bc\beta\gamma$ theory we construct a unified formalism which generates not only a simpler sewing in the bc system, but also a more complex construction for the $\beta\gamma$ system.

To conclude the Introduction we stress the fact that the global operator formalism is "invariant in the differential-geometric sense"; the formalism does not use any special parametrization of the Riemann surfaces and does not use distinguished coordinate systems in the neighborhood of distinguished points (none of these exist). We do not appeal in general to a metric on the Riemann surfaces and deal exclusively only with analytic (holomorphic or meromorphic) objects. However, in the sewing process certain supplementary information about each of the Riemann surfaces becomes necessary. It consists of the coordinate systems fixed near those points between which the sewing cylinder gets glued in (Refs. 32, 33). This is a minimal set of requirements since the geometric process of sewing itself depends on these coordinates. We note that in Refs. 33 and 34 a systematic program is developed for removing the metric in favor of a coordinating system, including its application to sewing of Riemann surfaces.

2. THE PHASE II OF THE βγ THEORY AND THE "SUPER" BOSONIZATION OF *bc*βγ SYSTEMS ON RIEMANN SURFACES

Thus, let us consider a $\beta\gamma$ system in which β and γ are respectively $\frac{3}{2}$ and $-\frac{1}{2}$ differentials on a Riemann surface C of genus $g\leqslant 2$. The standard bosonization formulas[21] have the form (1.2). As already noted in the Introduction, it is interesting to use an alternative bosonization of the $\beta\gamma$ theory[28,29] with the 1-differentials $\partial\xi$ and η permuted relative to the formulas of Ref. 21. In this bosonization, which we call the phase II,[30] it is found to be possible to further split the fields ξ and η into a pair of spin $\frac{1}{2}$ fermions and an isotropic scalar field. As a result of this the fields β and γ are (locally) expressed in terms of the system of fermions ψ and $\bar\psi$ and the scalars φ and $\bar\varphi$ with the operator products:[3)]

$$\psi(x)\bar\psi(y)=\frac{1}{x-y},\quad \varphi(x)\bar\varphi(y)=\ln(x-y) \tag{2.1}$$

(and $\psi\psi,\bar\psi\bar\psi,\varphi\varphi,\bar\varphi\bar\varphi\sim 0$) by means of

$$\beta=e^{\varphi},\ \gamma=e^{-\varphi}(-\psi\bar\psi+\partial\bar\varphi). \tag{2.2a,b}$$

In addition, the b- and c-fields (anticommuting reparametrization ghosts in string theory) are expressed in terms of the same $\psi\bar\psi\varphi\bar\varphi$:

$$b=\psi e^{\varphi},\ c=\bar\psi e^{-\varphi}. \tag{2.2c,d}$$

The fields ξ and η are now "composite" and have the expressions

$$\xi=\psi e^{-\bar\varphi},\ \eta=\bar\psi e^{\bar\varphi}. \tag{2.2e,f}$$

Then the formulas $\beta=e^{-\phi}\eta$, $\gamma=e^{\phi}\partial\xi$ (see Footnote 3) reproduce the equalities (2.2a,b) and are properly speaking the defining relations of the phase II.

The complete bosonization of the $bc\beta\gamma$ system requires the introduction of three bosonic objects. As noted in the Introduction, the scalar fields lose their single-valuedness on Riemann surfaces, and in their place the intrinsically defined objects are the currents. These currents, J, $\bar J$, and H correspond heuristically to the expressions $\partial\varphi$, $\partial\bar\varphi$, and $\psi\bar\psi$. Thus the fermions ψ and $\bar\psi$ bosonize together with the other fields. This allows for an exhaustive description of the $bc\beta\gamma$-system in operator language, and as a consequence, for a construction of all the correlation functions in phase II. Then, when we return to the discussion of supersymmetry and of superfields, we represent the fields β, γ in terms of $J\bar J$, ψ and $\bar\psi$.

Being guided by the analogy with Eqs. (2.1) we define the following nonvanishing correlation functions in the system formed by the three currents J, $\bar J$ and H:

$$\langle 0|H(x)H(y)|0\rangle=\langle 0|J(x)\bar J(y)|0\rangle=\omega(x,y). \tag{2.3}$$

Here $\langle 0|0\rangle=1$ and $\omega(\cdot,\cdot)$ denotes the symmetric meromorphic bidifferential (see Eq. (A4) of the Appendix). In addition, we normalize the currents by the conditions

$$\oint_{a_i} H=\oint_{a_i} J=\oint_{a_i} \bar J=0,\quad i=1,\dots,g, \tag{2.4}$$

and set $\langle 0|H|0\rangle=\langle 0|J|0\rangle=\langle 0|\bar J|0\rangle=0$.

The theory of the currents, J, $\bar J$ and H is considered as free, which is expressed through the validity of the Wick rule for the calculation of correlation functions of higher monomials (see Appendix B) and the definition of the normal ordering of composite operators as a result of subtraction of all possible contractions which are simply defined by means of the correlation functions (2.3). Thus, in particular,

$$H(x)H(y)={:}H(x)H(y){:}+\omega(x,y).$$

There are some delicate points related to this and other similar formulas: even on the Riemann sphere $\mathbf{CP}^1$ the left-hand side assumes not the ordinary product but the radial (time-ordered) product (this is almost never explicit in the notation). Similarly, in order to give a meaning to the left-hand side on a Riemann surface one must apply the procedure developed in Ref. 14, picking two points P_+ and P_- on C and the dipole differential ω_{P_+-P} which defines "equal-time lines" on C.

We first consider the background operators which bosonizes the minimal set of insertions on the given Riemann surface C. The number of these insertions is determined by the Riemann-Roch theorem in the presence of unphysical poles (see Refs. 1 and 36 for bosonization in the phase I). As will be seen below, the background operator describes the effects of the interactions of all other insertions with the global geometry of the surface C. We will not attempt to list the geometric motivation for the selection of a minimal set to be made below. The justification of the whole construction is the self-consistent correlation functions on the Riemann surface.

The minimal set includes:

1. g ψ-insertions, situated at the points $x_0,N_1,\dots,N_{g-1}$ on C;
2. $g-1$ exp φ-insertions at $M_1,\dots,M_{g-1}$;
3. $2g-2$ exp $\bar\varphi$-insertions at the points $P_1,\dots,P_{g-1}$ and one more exp φ-insertion at x_0.

For notational convenience we introduce the appropriate divisors $\mathcal{N}=\Sigma N_i$, $\mathcal{M}=\Sigma M_i$, $\mathcal{P}=\Sigma P_a$. Thus, we have the heuristic equality

$$\mathcal{B}=\prod_{M\in\mathcal{M}} e^{\varphi(M)}\prod_{P\in\mathcal{P}} e^{\bar\varphi(P)}\prod_{N\in\mathcal{N}}\psi(N)\psi(x_0)e^{-\bar\varphi(x_0)}. \tag{2.5}$$

To give it a precise meaning in the bosonized theory one uses a construction whose basic ingredient, in addition to the operator θ functions, is the operator insertions of the divisors [see Eq. (B1) in the Appendix]. With their help we define

$$\mathcal{B}=C:D_1(\mathcal{N},x_0|H)D_1(\mathcal{M},x_0|J)D_2(\mathcal{P},x_0|\bar J):\frac{\theta\left[\begin{smallmatrix}\beta\\ \alpha\end{smallmatrix}\right]\left(\oint_b (h+j)\right)}{\theta\left[\begin{smallmatrix}\delta\\ \varepsilon\end{smallmatrix}\right]\left(\oint_b j\right)}, \tag{2.6}$$

where C is a normalized constant [it is explicitly given below in Eq. (2.8b)], and where we have introduced the currents

$$h(u)=H(u)-d_u \ln E(x_0, u), \qquad j(u)=J(u)-2d_u \ln E(x_0, u), \tag{2.7a,b}$$

which together with the current

$$\bar{j}(u)=\bar{J}(u)-d_u \ln E(x_0, u) \tag{2.7c}$$

will be used below for the construction of the ghost currents. For generality we have also allowed the bc- and $\beta\gamma$-theories to have different characteristics (respectively $[{\beta \atop \alpha}]$ and $[{\delta \atop \varepsilon}]$).

In the expression (2.6) normal ordering of the right-hand side as a whole has not yet been carried out. Reducing it to normal-ordered form according to Eq. (B5) of the Appendix we obtain

$$\mathscr{B}(\mathscr{M},\mathscr{N},\mathscr{P},x_0) =: \mathscr{A}(\mathscr{M},\mathscr{N},\mathscr{P},x_0)\frac{\theta\left[{\beta \atop \alpha}\right]\left(\mathscr{M}+\mathscr{P}-3\Delta+\oint_b (H+J)\right)}{\theta\left[{\delta \atop \varepsilon}\right]\left(\mathscr{P}-2\Delta+\oint_b J\right)}. \tag{2.8a}$$

Making use of the explicit form of the D factors and choosing the normalization C we have

$$\mathscr{A}(\mathscr{M},\mathscr{N},\mathscr{P},x_0)=\exp\left\{\int_{(g-1)x_0}^{\mathscr{N}} H+\int_{(g-1)x_0}^{\mathscr{M}} J+\int_{2(g-1)x_0}^{\mathscr{P}} \bar{J}\right\}$$

$$\times\exp\frac{-1}{\pi i}\sum_{i=1}^{g}\oint_{a_i}\omega_i(u)\int_{x_0}^{u}(H+J+2\bar{J})$$

$$\times\prod_{\substack{M\in\mathscr{M}\\ P\in\mathscr{P}}}E(M,P)\prod_{i<p}E(N_i,N_j)\prod_{P\in\mathscr{P}}\sigma(P)$$

$$\times\prod_{M\in\mathscr{M}}\frac{\sigma(M)^2}{E(x_0,M)}\prod_{N\in\mathscr{N}}\sigma(N)E(x_0,N). \tag{2.8b}$$

The normalizing c-number multipliers have been selected so that the structure of the zeros and poles should satisfy the formal correspondence with the naive expression (2.5). In addition, all the expressions are $\frac{3}{2}$-differentials with respect to each of the points M_i, $\frac{1}{2}$-differentials with respect to the P_α and N_i, and $\frac{1}{2}-\frac{1}{2}=0$-differentials with respect to x_0.

In Eqs. (2.8) Δ denotes the Riemann class and σ is the Fay[32] $g/2$-differential:

$$\sigma(y)=\exp\frac{-1}{2\pi i}\sum_{i=1}^{g}\oint_{a_i}\omega_i(y)\ln E(u,y); \tag{2.9}$$

$(x-y)$ in the argument of the θ-function denotes the Abel map

$$\left[\int_y^x \omega_i\right]_{i=1,\dots,g}.$$

Comparing Eq. (2.8) with the equation (2.5) we see that at the heuristic level we have $\psi(x)\sim\exp\int^x H$. In terms of the neutral $\psi\bar{\psi}$-insertion this can be formulated precisely in the form[4)]

$$\overbrace{\psi(x)\bar{\psi}(y)}=\frac{1}{E(x,y)}\exp\int_y^x H. \tag{2.10}$$

The brace on top of the operator denotes that we do not insist on the existence of the individual factors in it. The anomaly of the ghost numbers is already saturated by the background operator, and for the construction of arbitrary correlation functions it now suffices to use operators which are neutral with respect to the ghost number, which can be thought of as insertions into the operator background (2.8).

The expression (2.10) is a $\frac{1}{2}$-differential in each variable. Similarly we set

$$\overbrace{\exp\varphi(x)\exp(-\varphi(y))}=\left[\frac{E(y,x_0)}{E(x,x_0)}\right]^3\exp\int_x^y J, \tag{2.11}$$

$$\overbrace{\exp(-\bar{\varphi}(x))\exp\bar{\varphi}(y)}=\frac{E(x,x_0)}{E(y,x_0)}\exp\int_x^y \bar{J}. \tag{2.12}$$

Here, in distinction from Eq. (2.10) there exist zeros and poles at $x=y$ in agreement with the local equalities $\langle\varphi\varphi\rangle=\langle\bar{\varphi}\bar{\varphi}\rangle=0$.

Now we can form the simplest of the composite neutral insertions, $b(x)c(y)$. The local formulas (2.2c,d) which express b and c are applied to the neutral insertions listed above. As a result we obtain

$$\overbrace{b(x)c(y)}=\frac{1}{E(x,y)}\frac{E(y,x_0)^3}{E(x,x_0)^3}:\exp\int_y^x (J+H):. \tag{2.13}$$

However, the expressions of the fields β and γ undergo changes compared to the local case, as soon as we have chosen the local ψ, $\bar{\psi}$ and $\exp(\pm\varphi)$, and not the $\psi^*,\bar{\psi}^*$, $\exp(\pm\varphi^*)$ (defined below; see footnote 4). We obtain these by first constructing the bosonized representation for the $\xi\eta$ insertion. One can give a meaning to the "global" modifications of the fields ξ and η separately (and not only the "synthetic" expression $\xi(x)\eta(y)$). Operator θ functions accompany each of the fields ξ and η on the Riemann surface[5)]

$$\xi(x)=\psi(x)\theta\left[{\delta \atop \varepsilon}\right]\left(y-x_0+\oint_b j\right)^{-1}e^{-\bar{\varphi}(x)}, \tag{2.14a}$$

$$\eta(y)=\bar{\psi}(y)\theta\left[{\delta \atop \varepsilon}\right]\left(y-x_0+\oint_b j\right)e^{\bar{\varphi}(x)}. \tag{2.14b}$$

These products are not assumed to be normal-ordered as a whole. Of, course, taken separately the fields ψ and $\bar{\psi}$ (and similarly $e^{-\bar{\varphi}}$ and $e^{\bar{\varphi}}$) are not neutral insertions, and the field $\bar{\varphi}$ has no unique definition on the Riemann surface. Equations (2.14) mean that every time one forms the neutral combination $\xi(x)\eta(y)$, the ψ and $\bar{\psi}$ entering into them, as well as $e^{-\bar{\varphi}}$ and $e^{\bar{\varphi}}$, must be combined pairwise into neutral combinations, and expressed in terms of currents according to Eqs. (2.10) and (2.12), after which the expression has to be normal-ordered as a whole.

Following this prescription we calculate the neutral insertion

$$\overbrace{\xi(x)\eta(y)} = :\psi(x)\bar\psi(y): \frac{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(y - x_0 + \oint_b j\right)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x - x_0 + \oint_b j\right)} \overbrace{:\exp(-\bar\varphi(x))\exp\bar\varphi(y):}$$

$$= \frac{1}{E(x,y)}\frac{E(x,x_0)}{E(y,x_0)}\cdot :\exp\int_y^x H::\exp\int_x^y \bar J: \frac{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(y - x_0 + \oint_b j\right)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x - x_0 + \oint_b j\right)}$$

$$= \frac{1}{E(x,y)}\frac{E(x,x_0)}{E(y,x_0)}:\exp\int_y^x (H-\bar J)\frac{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(2y - x - x_0 + \oint_b j\right)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(y - x_0 + \oint_b j\right)}:. \tag{2.15}$$

For future reference we write out the $\xi\eta$-current obtained from this (the square brackets $[\cdot]$ denote the current, and the letters inside are the name of the current):

$$[\xi\eta](x) = H(x) - \bar j(x) - d_x \ln\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x - x_0 + \oint_b j\right). \tag{2.16}$$

A similar procedure allows one to express the β- and γ-insertions in terms of ξ and η, namely:

$$\gamma(y) = \lim_{x\to y} :\bar\psi(x)\exp[\bar\varphi(x) - \varphi(x)]:\ :d_y\xi(y):$$

$$= -\lim_{x\to y} d_y\left\{\frac{1}{E(x,y)}:\left[\exp\int_x^y H\right]\frac{E(y,x_0)}{E(x,x_0)}\left[\exp\int_y^x \bar J\right]e^{-\varphi(x)}: \times\frac{E(x,y)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(y - x_0 + \oint_b j\right)}\right\}.$$

The prime-form $E(x,y)$ at power $+1$ appeared from the fusion of $\exp - \varphi(x)$ with $\exp - \bar\varphi(x)$. Here a more serious attitude toward the scalar fields on a Riemann surface is required than that adopted earlier: the fields φ and $\bar\varphi$ must be assumed to live on the universal cover of our Riemann surface and the correlation functions of $\varphi(x)$ and $\bar\varphi(y)$ must be set equal to $\ln E(x,y)$. (We remind the reader that the prime form is also defined on the universal cover!)

The overall sign is not completely determined as long as we have not established conventions about the cocycle multipliers in front of the exponentials in the scalar fields. We shall not do this, however. The minus sign was chosen in the last formula, following Ref. 29. One still needs to normal-order the expression we have obtained, i.e., introduce a θ function under the sign of the Wick product. This will shift the argument of the θ function by $x - y$, after which it becomes independent of y and can be taken out of the differentiation sign with respect to y. Finally, we obtain

$$\gamma(y) = [\bar j(y) - H(y)]e^{-\varphi(y)}\frac{E(x,y)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(y - x_0 + \oint_b j\right)}. \tag{2.17}$$

Similarly,

$$\beta(x) = \lim_{y\to x} :\psi(y)\exp[\varphi(y) - \bar\varphi(y)]:\ :\bar\psi(x)\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} \times\left(x - x_0 + \oint_b j\right):$$

$$= :[\exp\varphi(x)]\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x - x_0 + \oint_b j\right):. \tag{2.18}$$

The meaning of the representations of the fields β and γ separately is the same as for the fields ξ and η (see above). It is, however, easy to obtain the neutral insertion $\beta(x)\gamma(y)$:

$$\overbrace{\beta(x)\gamma(y)} = :[\exp\varphi(x)]\,\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left[x - x_0 + \oint_b j\right]::(\bar J(y) - H(y) - d_y\ln E(y,x_0))[\exp(-\varphi(y))]\frac{1}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(y - x_0 + \oint_b j\right)}:, \tag{2.19}$$

where we have explicitly indicated the contractions required to bring the expressions into normal-ordered form. Thus,

$$\overbrace{\beta(x)\gamma(y)} = :[\omega_{x-x_0}(y) + \bar J(y) - H(y) + \omega_i(y)\partial_i\ln\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x - x_0 + \oint_b j\right)] \times\left[\frac{E(y,x_0)}{E(x,x_0)}\right]^3 \frac{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x - x_0 - \oint_b j\right)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(y - x_0 + \oint_b j\right)}\left[\exp\int_y^x J\right]:. \tag{2.20}$$

Here $\omega_{a-b}(y) = d_y \ln E(a,y)/E(b,y)$ is a dipole differential (A7), and ∂_i differentiates the θ function with respect to its ith argument. The $\beta\gamma$ current obtained from (2.20) has the form

$$[\beta\gamma] = \bar j - j - h. \tag{2.21}$$

We also indicate the expression of the third [in addition to (2.21) and (2.13)] independent current—the ghost current of the bc-system:

$$[bc] = j + h. \tag{2.22}$$

All the global bosonic representations we have obtained acquire a "physical" meaning when superposed on the operator $\mathscr{B}$ of Eq. (2.8). We show how the fusion of each of the neutral pairs with the operator background occurs (the case of arbitrary, i.e., multiple and/or mixed fusions does not differ except in being more tedious).

Applying the formulas of Appendix B, we easily find

$$\overbrace{\psi(x)\overline{\psi}(y)}\mathscr{B}(\mathscr{M},\mathscr{N},\mathscr{P},x_0)$$

$$=\frac{1}{E(x,y)}\frac{E(x,\mathscr{N})}{E(y,\mathscr{N})}\frac{E(x,x_0)}{E(y,x_0)}\frac{\sigma(x)^1}{\sigma(y)}$$

$$\times:\left[\exp\int_y^x H\right]$$

$$\times\mathscr{A}\frac{\theta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}\left(\mathscr{N}+x-y+\mathscr{P}-3\Delta+\oint_b(H-J)\right)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(\mathscr{P}-2\Delta+\oint_b J\right)}:, \qquad (2.23)$$

where we have utilized the notation (B4). We note that in Eq. (2.23) the antisymmetry between $\psi(x)$ and the insertions $\psi(N)$, $N\in\mathscr{N}$, was asserted earlier (x_0 enters asymmetrically, as before, since in the background operator $\exp[-\bar{\varphi}(x_0)]$ still occurs. The poles and zeroes of the prime-forms in Eq. (2.23) coincide exactly with those of the operator monomial

$$\prod_{N\in\mathscr{N}}\psi(N)\psi(x_0)\psi(x)\overline{\psi}(y).$$

Similarly we find

$$\overbrace{\exp\varphi(x)\exp(-\varphi(y))}\mathscr{B}$$

$$=\frac{E(x,\mathscr{P})}{E(y,\mathscr{P})}\frac{E(y,x_0)}{E(x,x_0)}\left[\frac{\sigma(x)}{\sigma(y)}\right]^2:\left[\exp\int_x^y J\right]\mathscr{B}:,$$

$$\overbrace{\exp(-\overline{\varphi}(x))\exp\overline{\varphi}(y)}\mathscr{B}=\frac{E(y,\mathscr{M})}{E(x,\mathscr{M})}\frac{\sigma(x)}{\sigma(y)} \qquad (2.24)$$

$$\times:\mathscr{A}\left[\exp\int_x^y J\right]$$

$$\times\frac{\theta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}\left(\mathscr{N}+y-x+\mathscr{P}-3\Delta+\oint_b(H+J)\right)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(\mathscr{P}+y-x-2\Delta+\oint_b J\right)}:, \qquad (2.25)$$

$$b(x)c(y)\mathscr{B}=\frac{1}{E(x,y)}\frac{E(x,\mathscr{N})}{E(y,\mathscr{N})}\frac{E(x,\mathscr{P})}{E(y,\mathscr{P})}$$

$$\times\left[\frac{\sigma(x)}{\sigma(y)}\right]^3:\left[\exp\int_y^x(J+H)\right]\mathscr{A}$$

$$\times\frac{\theta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}\left(\mathscr{N}+x-y+\mathscr{P}-3\Delta+\oint_b(H+J)\right)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(\mathscr{P}-2\Delta+\oint_b J\right)}:. \qquad (2.26)$$

Although we have not worried especially about including in the background operator the $3g-3$ b-insertions which provide the $3g-3$ zeroes of the b field (and as many poles for the field c), on the Riemann surface, nevertheless the required zeros and poles have correctly appeared in Eq. (2.26) (the b-insertions are now effectively distributed over the points of the divisor $\mathscr{N}+\mathscr{P}$).

Slightly more care is required in introducing the $\xi\eta$-insertions into the operator back-ground. Procedures similar to the ones carried out above lead to the following result:

$$:\overbrace{\xi(x_1)\eta(y_1)}:\ldots:\overbrace{\xi(x_n)\eta(y_n)}::\mathscr{B}:$$

$$=\frac{\prod_{\mu<\nu}E(x_\mu,x_\nu)\prod_{i<j}E(y_i,y_j)}{\prod_{\mu,i}E(x_\mu,y_i)}$$

$$\times\prod_\mu\frac{E(x_\mu,\mathscr{N})}{E(y_\mu,\mathscr{N})}\prod_i\frac{E(y_i,\mathscr{M})}{E(x_i,\mathscr{M})}$$

$$\times\frac{\prod_{i=1}^n\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(y_i+\Sigma y_j-\Sigma x_\nu+\mathscr{P}-2\Delta+\oint_b J\right)}{\prod_{\mu=0}^n\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x_i+\Sigma y_j-\Sigma x_\nu+\mathscr{P}-2\Delta+\oint_b J\right)}\cdot$$

$$\theta\cdot\exp\int_{\Sigma_i y_i}^{\Sigma_i x_i}(H-\bar{J})D_1(\mathscr{N},x_0|H)D_1(\mathscr{M},x_0|J)D_2(\mathscr{P},x_0|\bar{J})$$

$$\times\theta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}\left(\mathscr{N}+\mathscr{P}-3\Delta+\oint_b(H+J)\right): \qquad (2.27)$$

This expression is still to be multiplied by normalization factors which do not depend on x_μ and y_i, and follow from Eq. (2.8). Here

$$(x_\mu)=x_0,x_1,\ldots,x_n;\ (y_i)=y_1,\ldots,y_n.$$

The formal vanishing of all operator currents in the normal-ordered expression (2.27) corresponds to taking the $\langle 0|\ldots|0\rangle$ expectation value and yields the correlation function

$$\Big\langle\xi(x_1)\eta(y_1)\ldots\xi(x_n)\eta(y_n)\xi(x_0)$$

$$\times\prod_{M\in\mathscr{M}}e^{\varphi(M)}\prod_{P\in\mathscr{P}}e^{\overline{\varphi}(P)}\prod_{N\in\mathscr{N}}\psi(N)\Big\rangle. \qquad (2.28)$$

It is immediately obvious that the monodromy of the $\xi\eta$-insertions around the cycles b_k is expressed by

$$\exp\left\{\pm\int_{\mathscr{N}}^{\mathscr{M}}\omega_k\right\},$$

so that single-valuedness in the $\xi\eta$ sector requires the choice $\mathscr{M}=\mathscr{N}$. As a consequence $g-1$ insertions of fields $b=\psi\exp\varphi$ appear in the background operator. If one does not insist on considering the $\xi\eta$ sector, and constructs the $bc\beta\gamma$ system directly in terms of the $\varphi\bar{\varphi}\psi\bar{\psi}$ system, then the choice $\mathscr{M}=\mathscr{N}$ is not compulsory, as soon as the $\beta\gamma$ correlation functions will have the correct monodromy. The fields $\beta\gamma$ do indeed exhibit good behavior on the operator background (2.8), as follows from the fusion

$$\overline{\beta(x)\gamma(y)}\mathscr{B}$$

$$= \frac{E(y,x_0)}{E(x,x_0)}\frac{E(x,\mathscr{P})}{E(y,\mathscr{P})}\left[\frac{\sigma(x)}{\sigma(y)}\right]^2\Bigg[\omega_{x-x_0}(y)+\bar{J}(y)-H(y)$$

$$+\omega_{\mathscr{M}-\mathscr{N}}(y)+\omega_i(y)\partial_i$$

$$\times\ln\frac{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x-x_0+\mathscr{P}-2\Delta+\oint_b J\right)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(\mathscr{P}-2\Delta+\oint_b J\right)}\Bigg]$$

$$\times\frac{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x-x_0+\mathscr{P}-2\Delta+\oint_b J\right)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(y-x_0+\mathscr{P}-2\Delta+\oint_b J\right)}\left[\exp\int_y^x J\right]\mathscr{B}, \qquad (2.29)$$

where overall normal ordering is implied in the right-hand side. Now the correlation function

$$\left\langle\beta(x)\gamma(y)\xi(x_0)\prod_{M\in\mathscr{M}}e^{\varphi(M)}\prod_{P\in\mathscr{P}}e^{\bar{\varphi}(P)}\prod_{N\in\mathscr{N}}\psi(N)\right\rangle$$

normalized to unit residue at the pole $(x-y)^{-1}$ follows from Eq. (2.29) in the form

$$\frac{\langle 0|\beta(x)\gamma(y)\mathscr{B}|0\rangle}{\langle 0|\mathscr{B}|0\rangle}=\frac{E(y,x_0)}{E(x,x_0)}\frac{E(x,\mathscr{P})}{E(y,\mathscr{P})}\left[\frac{\sigma(x)}{\sigma(y)}\right]^2$$

$$\times\Bigg[\omega_{x-x_0}(y)+\omega_{\mathscr{M}-\mathscr{N}}(y)$$

$$+\omega_i(y)\partial_i\ln\frac{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(x-x_0+\mathscr{P}-3\Delta)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(\mathscr{P}-2\Delta)}\Bigg]$$

$$\times\frac{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(x-x_0+\mathscr{P}-2\Delta)}{\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(y-x_0+\mathscr{P}-2\Delta)}. \qquad (2.30)$$

The many-point $\beta\gamma$ correlation functions are obtained analogous by fusion of the appropriate number of insertions (2.20) with the background operators (2.22).

The results differ from those initially obtained on Ref. 1 for the phase I. The structure of the background insertions in phase II is different (and turns out to be more delicate) than those in phase I. But the main distinction appears owing to the fact that in the phase II the field γ (and not β, as in the phase I) is expressed in terms of the differential $\partial\xi$; this implies the differential structure of the square bracket in Eq. (2.30) [or, more generally, in Eq. (2.20)]—the expression in the square brackets is a 1-differential with respect to y rather than x. A similar "partial" [i.e., one which affects the extra θ operator which originally come from Eq. (2.14)] permutation $x\leftrightarrow y$ in (2.20) and (2.29) could have been expected, since it compensates the differences between the two phases (which are also caused by the permutation $\partial\xi\leftrightarrow\eta$) in the picture-changing operators and the *BRST*-operators. This "compensation," i.e., the reconstitution of identical results in the two phases for the measure on the moduli space, occurs possibly only up to exact derivatives on the moduli space, a fact which needs additional investigation.

Thus, the presence of in the neutral insertions (2.15), (2.20) of θ functions which depend on the operator periods $\oint_b J$ makes the composite β-, γ-, ξ- and η-operators nonfree. Correspondingly, the operator θ functions lead to the appearance of terms in the operator products, e.g., of the form

$$\omega_i\partial_i\ln\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(\ldots+\oint_b J\right),$$

which have no local poles, but are sensitive to the global structure of the Riemann surface. Also, during fusion with other operator insertions, namely those containing $\exp\int\bar{J}$, they react to them by changes of the arguments of the θ function. We stress the fact that in themselves the currents H, J, and $\bar{J}$ remain free during our whole consideration.

Alternatively, the "global non-freedom" of the composite operators β, γ, ξ and η can be described as a result of a "dressing" of the initial $\varphi\bar{\varphi}\psi\bar{\psi}$ fields. Then the operators β, γ, ξ and η will be expressed in terms of the dressed fields by the same formulas (2.2) as in the local case. Indeed, as follows from Eq. (2.20), the field operator β can be tentatively identified with the exponential of the scalar field

$$\varphi^*(x)=\varphi(x)+\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x-x_0+\oint_b j\right)-3\ln E(x_0,x). \qquad (2.31a)$$

The c-number term $3\ln E(x_0,x)$ is not essential here, the most nontrivial part being the logarithm of the θ function. The operator periods $\oint_{b_i} J$ are identified with the jumps $\Delta_i\varphi$ of the field φ around the b-cycles, and similarly for $\bar{\varphi}$. The monodromies of the field φ^* are quite interesting:

$$\Delta_k\varphi^*(x)=2\int_{u_k'}^{x}\omega_k-2\pi i\varepsilon_k, \qquad (2.32)$$

where u_k' is the initial point of the cycle b_k and ε_k is the lower characteristic of the θ-function. The operator terms in Eq. (2.32) have thus canceled out in (2.32), leaving behind a prescribed system of jumps dependent on the point x.

Similarly we introduce

$$\psi^*(x)=\psi(x)\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x-x_0+\oint_b\right)^{-1}, \qquad (2.31b)$$

$$\bar{\psi}^*(x)=\bar{\psi}(x)\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x-x_0+\oint_b\right), \qquad (2.31c)$$

$$\bar{\varphi}^*(x)=\bar{\varphi}(x)-\ln E(x_0,x) \qquad (2.31d)$$

and formally define the superfields

$$\Phi^*=\varphi^*+\vartheta\psi^*,\quad \bar{\Phi}^*=\bar{\varphi}^*+\vartheta\bar{\psi}^*, \qquad (2.33)$$

where ϑ is the supercoordinate. Then the bosonization formulas (2.20), (2.30) can be formally rewritten in terms of the superfields (1.3) as follows

$$B(x,\vartheta)\equiv\beta(x)+\vartheta b(x)=\exp\Phi^*(x,\vartheta), \qquad (2.34a)$$

$$C(x,\vartheta)=c(x)+\vartheta\gamma(x)=D\Phi^{\bullet}(x,\vartheta)\exp(-\Phi^{\bullet}(x,\vartheta)),\tag{2.34b}$$

where $D=\partial/\partial\vartheta+\vartheta\partial$ and $\partial=\partial/\partial x$. However, the fields Φ^* and $\bar{\Phi}^*$ are not globally defined on account of the different monodromy properties of their superpartners. Thus, for instance, if the point x winds around the cycle b_k, the operator $\psi^*(x)$ acquires the factor

$$\exp\Big\{{}^{3}/_{2}\tau_{kk}+\oint_{b_K}(H+J)+\int_{3x_0}^{x+2u_K'}\omega_k+2\pi i\varepsilon_k\Big\}.$$

Even if all the c-number factors which make this expression different from $\exp\Delta_k\varphi^*(x)$ [see Eq. (2.32)] are attributed to the behavior of the supercoordinate ϑ around the cycle b_k, there still remains a nonremovable difference on account of the operator terms. These nonremovable differences in monodromy are caused by the circumstance that the most natural candidates for the role of component fields in the superfields turn out not to be free owing to global effects on the Riemann surface.

We have thus obtained what one might call a violation of global supersymmetry on the world sheet on account of the boundary conditions. We note that the whole $N=2$ algebra of the supersymmetry of the $bc\beta\gamma$ system is broken on the Riemann surface in a similar manner.[21] The operators which generate the $N=2$ supersymmetry in a local theory (the ghost current BC, the $U(1)$-current $DB\cdot C+3/2B\cdot DC$ and the super-energy momentum tensor) acquire "global additions" of the type $d\ln\theta[{}^{\delta}_{\varepsilon}](...+\oint_b j)$ in the operator products. This can be seen from the fundamental operator product

$$\begin{aligned}B(1)C(2)=&\frac{\vartheta_{12}}{z_{12}}+D\Phi^{\bullet}(2)+z_{12}\,\partial\Phi^{\bullet}(2)D\Phi^{\bullet}(2)\\&+\vartheta_{12}\{D\Phi^{\bullet}(2)D\bar{\Phi}^{\bullet}(2)+\partial\Phi^{\bullet}(2)-d\ln\theta(2)\}\\&+z_{12}[{}^{1}/_{12}\vartheta_1+{}^{1}/_{6}\vartheta_2]S+{}^{1}/_{2}\vartheta_1 z_{12}[(d\ln\theta(2))^2-d^2\ln\theta(2)]\\&+\vartheta_1\vartheta_2 D\Phi^{\bullet}(2)\,d\ln\theta(2)+\vartheta_2 z_{12}\omega_i\omega_j\partial_i\partial_j\ln\theta(2),\end{aligned}$$

where, as usual, $z_{12}=z_1-z_2-\vartheta_1\vartheta_2$, $\vartheta_{12}=\vartheta_1-\vartheta_2$, $\Phi^*(2)=\Phi^*(z_2,\vartheta_2)$ and $d\ln\theta(2)=d_{z2}\ln\theta[{}^{\delta}_{\varepsilon}]\times(z_2-x_0+\oint_b j)$. The current which we already know from Eq. (2.16) is present in the last equation, which in fact violates the global supersymmetry on the world-sheet.

3. SEWING OF $bc\beta\gamma$-THEORIES ON RIEMANN SURFACES IN THE GLOBAL OPERATOR FORMALISM

We first consider the geometric process of sewing together two Riemann surfaces C_1 and C_2 of genera g_1 and g_2, respectively (Refs. 32, 33). For this we choose on each of the surfaces a point $p_i\in C_i$. Identification of p_1 and p_2 leads to a surface C_0 with a necking-down. The opening of the neck is governed by a complex parameter t, as explained in Refs. 32, 33. This process requires the introduction of local coordinates near these points, the removal of small disks around the points and subsequent gluing in of a cylinder). Thus one obtains a family $\{\mathscr{C}_t\}$ of Riemann surfaces of genus g_1+g_2. For a given sewing procedure $(C_1,C_2)\to C_1\propto C_2=\mathscr{C}_t$ we assume known the expressions for the holomorphic and meromorphic differentials on $\mathscr{C}_t$ in terms of the corresponding objects on C_1 and C_2. Assuming an identification of the points on C_1 and C_2 (not too close to p_1 and p_2) with their images on the sewn surface, we obtain for the memorphic bidifferential on the "large" surface the expression

$$\Omega(x_1,y_1)=\overset{1}{\omega}(x_1,y_1)+{}^{1}/_{4}t\overset{1}{\omega}(x_1,p_1)\overset{1}{\omega}(y_1,p_1)+\mathcal{O}(t^2),\tag{3.1a}$$

$$\Omega(x_1,y_2)={}^{1}/_{4}t\overset{1}{\omega}(x_1,p_1)\overset{2}{\omega}(y_2,p_2)+\mathcal{O}(t^2),\tag{3.1b}$$

$$\Omega(x_2,y_2)=\overset{2}{\omega}(x_2,y_2)+{}^{1}/_{4}t\overset{2}{\omega}(x_2,p_2)\overset{2}{\omega}(y_2,p_2)+\mathcal{O}(t^2),\tag{3.1c}$$

where $x_i\in C_i$. The right-hand sides of these equalities depend on the choice of the local coordinates in the neighborhood of the points p_1 and p_2.

A canonical homology basis on $\mathscr{C}_t$ is chosen in the form

$$\mathbf{A}=(A_\mu)_{\mu=1,\dots,g_1+g_2}=(a_{i_1},a_{i_2}),\ \mathbf{B}=(B_\mu)_{\mu=1,\dots,g_1+g_2}=(b_{i_1},b_{i_2}),$$

where $i_1=1,\dots,g_1$, $i_2=1,\dots,g_2$. Integrating $\Omega(x,\cdot)$ around the $\mathbf{B}$-cycles we obtain the holomorphic differentials $\Omega_\mu(x)$. Thus, for example,

$$\Omega_{i_1}(x)$$

$$=\begin{cases}\omega_{i_1}(x_1)+{}^{1}/_{4}t\omega_{i_1}(p_1)\overset{1}{\omega}(x_1,p_1)+\mathcal{O}(t^2), & x=x_1\in C_1, \quad (3.2a)\\ {}^{1}/_{4}t\omega_{i_1}(p_1)\overset{2}{\omega}(x_2,p_2)+\mathcal{O}(t^2), & x=x_2\in C_2, \quad (3.2b)\end{cases}$$

and similarly for Ω_{i2}. Yet another integration yields the matrix of periods for the surface $\mathscr{C}_t$:

$$\Upsilon=\begin{pmatrix}\overset{1}{\tau}_{i_1j_1}+\sigma_{i_1j_1}(t), & v_{i_1j_2}(t)\\ u_{i_2j_1}(t) & \overset{2}{\tau}_{i_2j_2}+\sigma_{i_2j_2}(t),\end{pmatrix},\tag{3.3}$$

where all the quantities which depend on t have order of at least t (and are written explicitly mod t^2 in Ref. 32).

With the period matrix (3.3) is associated a "large" θ function $\Theta[{}^{\beta_i}_{\alpha_i}\ {}^{\beta_2}_{\alpha_2}]$. In order to distinguish the Abel maps for different Riemann surfaces, we introduce the notation

$$X-X_0=\Big[\int_{x_0}^{x}\Omega_\mu\Big]_{\mu=1,\dots,g_1+g_2},$$

where $x-x_0$, as before denotes

$$\Big[\int_{x_0}^{x}\omega_{i_1}\Big]_{i_1=1,\dots,g_1},$$

and a similar expression for C_2.

For the restrictions to C_1 and C_2 of the prime form $\mathscr{E}(\cdot,\cdot)$ on $\mathscr{C}_t$ we have

$$\ln\mathscr{E}(x_1,y_1)=\ln\overset{1}{E}(x_1,y_1)+\ln e(x_1,y_1),\tag{3.4}$$

$$\ln e(x_1,y_1)=-{}^{1}/_{4}t[\overset{1}{\omega}_{x_1-y_1}(p_1)]^2+\mathcal{O}(t^2).\tag{3.5}$$

We note that $\ln e(x,x)=0$. Finally, $\overset{1}{\sigma}(\cdot)$, $\overset{2}{\sigma}(\cdot)$ and $\Sigma(\cdot)$

denote the Fay differentials (2.9) respectively on C_1, C_2, and $\mathscr{C}_t$.

We now ask how to carry out at the operator level the sewing process we just described. Imagine that the equations (2.6)–(2.20) are written down three times, for the surfaces C_1, C_2, and $\mathscr{C}_t$. How can one relate these three sets of operators by means of operator manipulations?

First of all, on the big surface there is its own background operator, constructed according to the same rules as the operator $\mathscr{B}$ in Eqs. (2.5), (2.8), but in terms intrinsic to the surface $\mathscr{C}_t$:

$$\mathscr{B}[\mathscr{C}_t] = ⁑\mathscr{D}_1(\mathfrak{N}\,|\,\mathscr{H}+\mathscr{Y})\mathscr{D}_2(\mathfrak{P}\,|\,\overline{\mathscr{Y}})⁑ \times \frac{⁑\Theta\begin{bmatrix}\beta_1 & \beta_2\\ \alpha_1 & \alpha_2\end{bmatrix}\left(\oint\limits_{B}(\mathscr{H}+\mathscr{Y}-3d\ln\mathscr{E}(x_0,\cdot))\right)⁑}{\Theta\begin{bmatrix}\delta_1 & \delta_2\\ \varepsilon_1 & \varepsilon_2\end{bmatrix}\left(\oint\limits_{B}(\mathscr{Y}-2d\ln\mathscr{E}(x_0,\cdot))\right)}$$
$$\times\prod_{\mu=1}^{g-1}\prod_{\alpha=1}^{2g-2}\mathscr{E}(N_\mu,P_\alpha)\prod_{\mu<\nu}\mathscr{E}(N_\mu,N_\nu)\prod_\alpha\Sigma(P_\alpha)\prod_\mu\Sigma(N_\mu)^3, \tag{3.6}$$

$$\mathscr{D}_Q(\mathfrak{K}\,|\,\mathscr{J}) = \exp\left\{\int\limits_{Q(g-1)x_0}^{\mathfrak{K}}\mathscr{J} - \frac{Q}{2\pi i}\sum_{\mu=1}^{g}\oint\limits_{a_\mu}\Omega_\mu(u)\int\limits_{x_0}^{u}\mathscr{Y}\right\},$$

$$g = g_1 + g_2, \tag{3.7}$$

where $\mathfrak{K}$ is a positive divisor, this time of degree $Q(g_1+g_2-1)$. The currents $\mathscr{H}, \mathscr{J}, \overline{\mathscr{J}}$ and the normal product ⁑ can be introduced on the surface $\mathscr{C}_t$ without any reference to the sewing process, in complete analogy with the currents $H, J, \overline{J}$ and the normal ordering ::, introduced on the surfaces C_1 and C_2. Thus, [cf. Eq. (B2)],

$$\mathscr{H}(x)\mathscr{H}(y) = ⁑\mathscr{H}(x)\mathscr{H}(y)⁑ + \Omega(x,y),$$
$$\mathscr{Y}(x)\overline{\mathscr{Y}}(y) = ⁑\mathscr{Y}(x)\overline{\mathscr{Y}}(y)⁑ + \Omega(x,y), \tag{3.8a,b}$$

and in the other cases

$$\mathscr{K}_1(x)\mathscr{K}_2(y) = ⁑\mathscr{K}_1(x)\mathscr{K}_2(y)⁑. \tag{3.8*}$$

These equations assume the existence of a new "free vacuum" $|t\rangle$ associated with the surface $\mathscr{C}_t$ such that $\langle t|\mathscr{H}(x)\mathscr{H}(y)|t\rangle = \Omega(x,y)$, etc. We note that $|t=0\rangle$ coincides with the vacuum $|0\rangle$ of Sec. 2. We also assume that [cf. Eq. (2.4)]

$$\oint\limits_{A_\mu}\mathscr{H} = \oint\limits_{A_\mu}\mathscr{J} = \oint\limits_{A_\mu}\overline{\mathscr{J}} = 0.$$

If, however, the surface $\mathscr{C}_t$ is really sewn together from two others C_1 and C_2, then the currents denoted in script letters must somehow be related with those currents which we had earlier on C_1 and C_2. In order to establish this correspondence we recall the expressions for the ghost currents derived in Sec. 2:

$$[bc] = J - H - 3d\ln E(x_0,\cdot),\quad [\beta\gamma] = \overline{J} - H - J + 2d\ln E(x_0,\cdot), \tag{3.9a,b}$$

$$[\xi\eta](x) = H(x) - J(x) + d_x\ln E(x_0,x) - d_x\ln\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(x - x_0 + \oint\limits_{b} j\right). \tag{3.9c}$$

Similar expressions must hold on each of the Riemann surfaces C_1, C_2 and $\mathscr{C}_t$. Thus, for the sewn surface the last equation takes the form

$$[\xi\eta](x) = \mathscr{H}(x) - \overline{\mathscr{J}}(x) + d_x\ln\mathscr{E}(x_0,x) - d_x\ln\Theta\begin{bmatrix}\delta_1 & \delta_2\\ \varepsilon_1 & \varepsilon_2\end{bmatrix}\left(X - X_0 + \oint\limits_{B}\hat{j}\right), \tag{3.9c'}$$

where, by analogy to Eq. (2.7b)

$$\hat{j}(y) = \mathscr{J}(y) - 2d_y\ln\mathscr{E}(x_0,y). \tag{3.10}$$

By analogy with the construction of gluing on a handle and sewing together with Riemann surfaces in the bc-case (Refs. 16,17) we require that the ghost points should remain "the same" in the transition to the big Riemann surface. Thus, the left-hand side of the equation (3.9c) at $x\in C_1$ (respectively C_2) must coincide with the restriction of (3.9c') to the image of C_1 (respectively C_2 in the surface $\mathscr{C}_t$, and similarly for all the other currents. We are thus led to the following equations for the substitution of the currents:

$$\overset{1}{J}(x_1) = \mathscr{Y}(x_1) + d_{x_1}\ln\frac{\Theta\begin{bmatrix}\delta_1 & \delta_2\\ \varepsilon_1 & \varepsilon_2\end{bmatrix}\left(X_1 - X_0 + \oint\limits_{B}\hat{j}\right)}{\theta\begin{bmatrix}\delta_1\\ \varepsilon_1\end{bmatrix}\left(x_1 - x_0 + \oint\limits_{b^1} j^1\right)} - 3d_{x_1}\ln e(x_0,x_1), \tag{3.11a}$$

$$\overset{1}{\overline{J}}(x_1) = \overline{\mathscr{Y}}(x_1) - d_{x_1}\ln e(x_0,x_1), \tag{3.11b}$$

$$\overset{1}{H}(x_1) = \mathscr{H}(x_1) - d_{x_1}\ln\frac{\Theta\begin{bmatrix}\delta_1 & \delta_2\\ \varepsilon_1 & \varepsilon_2\end{bmatrix}\left(X_1 - X_0 + \oint\limits_{B}\hat{j}\right)}{\overset{1}{\theta}\begin{bmatrix}\delta_1\\ \varepsilon_1\end{bmatrix}\left(x_1 - x_0 + \oint\limits_{b^1} j^1\right)}. \tag{3.11c}$$

We note that in the right-hand sides of the equations (3.11a,c) neither the ::-, nor the ⁑ normal ordering is required, since the θ functions entering these equations depend on isotropic currents and are therefore insensitive to normal ordering. Similar relations hold on the "C_2-half" of the surface $\mathscr{C}_t$. For example, the corresponding analog of Eq. (3.11a) is

$$\overset{2}{J}(x_2) = \mathscr{Y}(x_2) + d_{x_2}\ln\frac{\Theta\begin{bmatrix}\delta_1 & \delta_2\\ \varepsilon_1 & \varepsilon_2\end{bmatrix}\left(X_2 - X_0 + \oint\limits_{B}\hat{j}\right)}{\overset{2}{\theta}\begin{bmatrix}\delta_2\\ \varepsilon_2\end{bmatrix}\left(x_2 - x_* + \oint\limits_{b^2} j^1\right)} - 3d_{x_2}\ln\mathscr{E}(x_0,x_2) + 3d_{x_2}\ln\overset{2}{E}(x_*,x_2), \tag{3.11a'}$$

where x_* plays the role of the point x_0 on the surface C_2. The two Riemann surfaces are considered somewhat asymmetrically from here onwards, since we intend to preserve x_0 as the position of the "redundant" ξ-insertion into $\mathscr{C}_t$; at the same time x_* will play a different role on the sewn surface, as will be seen below.

The transformations for the currents J and H are quite nontrivial, owing to the presence of the operator θ functions.[6] We note that the B-periods of the current $\mathscr{J}$ turn out to be independent of $\oint_{b^1} J^1$ and $\oint_{b^2} J^2$, since the integration of the equalities (3.11a,a') around the $b^1_{k_1}$-cycles yields $0 = 0$. The periods of $\oint_B \mathscr{J}$ must therefore be defined axiomatically, by giving the fusion rules for them, similar to the following

$$\bar{\mathscr{J}}(x) \oint_{B_\mu} \mathscr{J} = \Omega_\mu(x) + {}^{*}_{*}\bar{\mathscr{J}}(x) \oint_{B_\mu} \mathscr{J}{}^{*}_{*}.$$

One should remember that the new currents generally do not have simple contractions relative to the ::-ordering. Only some combinations of these currents have simple ::-contractions: from the above formulas it follows that

$$\mathscr{H}(x)\mathscr{H}(y) = :\mathscr{H}(x)\mathscr{H}(y): + \omega(x, y), \tag{3.12a}$$

$$(\mathscr{H}(x)+\mathscr{J}(x))\bar{\mathscr{J}}(y) = :(\mathscr{H}(x)+\mathscr{J}(x))\bar{\mathscr{J}}(y): + \omega(x, y), \tag{3.12b}$$

$$\mathscr{J}(x)\mathscr{J}(y) = :\mathscr{J}(x)\mathscr{J}(y):, \quad \bar{\mathscr{J}}(x)\bar{\mathscr{J}}(y) = :\bar{\mathscr{J}}(x)\bar{\mathscr{J}}(y):, \tag{3.12c}$$

$$\mathscr{H}(x)\mathscr{J}(y) = :\mathscr{H}(x)\mathscr{J}(y):, \tag{3.12d}$$

where x and y are assumed to be situated in the same "half" of the surface $\mathscr{C}_t$ and $\omega = \omega^1$ or $\omega = \omega^2$, respectively.

These relations will be quite important below for the normal reordering of the ingredients of the bosonized theories on C_1 and C_2 from the corresponding $(::)_1$ and $(::)_2$ orderings to the ${}^{*}_{*}\,{}^{*}_{*}$ ordering. The fact that the normal ordering leads to the appearance of only c-number ω- and Ω-pairings means that we are dealing with free fields. Then the ${}^{*}_{*}\,{}^{*}_{*}$ ordering reduces to a subtraction of all possible contractions, but since the ::-contractions have already been taken into account by the ::-ordering of the expression, there remains only to take into account the $\mathscr{O}(t)$ part in each contraction [see Eq. (3.1)].

In order to see how the normal reordering really takes place we first consider the neutral insertions (2.13), (2.15), (2.20). The transformation of Eq. (2.13) turns out to be the simplest and it follows easily from Eqs. (3.11) and (3.12) that

$$\frac{1}{E^1(x_1, y_1)} \frac{E^1(y_1, x_0)^3}{E^1(x_1, x_0)^3} : \exp \int_{y_1}^{x_1} (\overset{1}{J} + \overset{1}{H}):$$
$$= \frac{1}{\mathscr{E}(x_1, y_1)} \left[\frac{\mathscr{E}(y_1, x_0)}{\mathscr{E}(x_1, x_0)}\right]^3 {}^{*}_{*} \exp \int_{y_1}^{x_1} (\mathscr{J} + \mathscr{H}) {}^{*}_{*}. \tag{3.13}$$

The right-hand side is exactly the bc-insertion on the C_1-half of the surface $\mathscr{C}_t$, expressed in terms intrinsic to $\mathscr{C}_t$. Similarly the substitution of (3.11) for the currents on C_2 into the expression for $b(x_2)c(y_2)$ yields identities similar to (3.13) [but with the same x_0 in the right-hand side, see Eq. (3.11a')].

Finally, before the sewing the neutral insertion $b(x_1)c(y_2)$ and did not exist on $C_1 \sqcup C_2$. It is reconstructed by considering on $\mathscr{C}_t$ insertions of the form $b(x_1)c(y_1)b(x_2)c(y_2)$ $C_1 \text{II} C_2$; from these the C_1 C_2-insertion is extracted self-consistently, since both $b(x_1)c(y_1)$ and $b(x_2)c(y_2)$ were expressed in terms intrinsic to the surface $\mathscr{C}_t$.

Further, it is convenient to split the $\xi\eta$-insertion (2.15) into two normal-ordered factors:

$$\xi(x_1)\eta(y_1) = \frac{1}{E^1(x_1, y_1)} \cdot$$
$$\times \frac{E^1(x_1, x_0)}{E^1(y_1, x_0)} : \left[\exp \int_{y_1}^{x_1} \overset{1}{H}\right] \frac{\overset{1}{\theta}\left(y_1 - x_0 + \oint_{b^1} j^1\right)}{\overset{1}{\theta}\left(x_1 - x_0 + \oint_{b^1} j^1\right)} :: \exp \int_{x_1}^{y_1} \overset{1}{J} : \tag{3.14}$$

(the characteristics of the θ functions have not been indicated). Making use of Eq. (3.11), we obtain

$$\xi(x_1)\eta(y_1) = \frac{1}{E^1(x_1, y_1)} \frac{\mathscr{E}(x_1, x_0)}{\mathscr{E}(y_1, x_0)} : \left[\exp \int_{y_1}^{x_1} \mathscr{H}\right]$$
$$\times \frac{\Theta\left(Y_1 - X_0 + \oint_B \jmath\right)}{\Theta\left(X_1 - X_0 + \oint_B \jmath\right)} :: \exp \int_{x_1}^{y_1} \bar{\mathscr{J}} :.$$

Now the application of the appropriate formulas (3.12) to each of the ::-ordered factors yields

$$\xi(x_1)\eta(y_1) = \frac{1}{\mathscr{E}(x_1, y_1)} \frac{\mathscr{E}(x_1, x_0)}{\mathscr{E}(y_1, x_0)}$$
$$\times {}^{*}_{*} \left[\exp \int_{y_1}^{x_1} \mathscr{H}\right] \frac{\Theta\left(Y_1 - X_0 + \oint_B \jmath\right)}{\Theta\left(X_1 - X_0 + \oint_B \jmath\right)} {}^{*}_{*}\,{}^{*}_{*} \exp \int_{x_1}^{y_1} \bar{\mathscr{J}} {}^{*}_{*}, \tag{3.15}$$

which is the $\mathscr{C}_t$-analog of the second line in eq. (2.15). It is now obvious that the ${}^{*}_{*}\,{}^{*}_{*}$-ordering in Eq. (3.15) yields exactly the $\mathscr{C}_t$-analog of Eq. (2.15) in its definitive form. Similar considerations are applicable to the C_2-part of the surface $\mathscr{C}_t$ and to the "mixed" insertions.

Finally, the analysis of the $\beta\gamma$-insertion (2.20) requires a representation in terms of its limit for $y_1' \to y_1$ of the following expression with separated points and "split" normal ordering:

$$: \left[\exp \int_{y_1'}^{x_1} J\right] \frac{\overset{1}{\theta}\left(x_1 - x_0 + \oint_{b^1} j^1\right)}{\overset{1}{\theta}\left(y_1' - x_0 + \oint_{b^1} j^1\right)} : \frac{E^1(y_1', x_0)^3}{E^1(x_1, x_0)^3}$$
$$\times : \left[\overset{1}{\omega}_{y_1' - x_0}(y_1) + \overset{1}{J}(y_1)\right.$$
$$\left. - \overset{1}{H}(y_1) + \omega_{i_1}(y_1)\partial_{i_1} \ln \overset{1}{\theta}\left(y_1 - x_0 + \oint_{b^1} j^1\right)\right] :. \tag{3.16}$$

As a result of the substitution (3.11) and the normal reordering in each factor, the equation (3.16) takes the form

$$\ast\!\!\ast \left[\exp \int_{y_1'}^{x_1} \mathcal{J}\right] \frac{\Theta\left(X_1 - X_0 + \oint_{\mathbf{B}} \jmath\right)}{\Theta\left(Y_1' - X_0 + \oint_{\mathbf{B}} \jmath\right)} \ast\!\!\ast \frac{E^1(y_1', x_0)^3}{E^1(x_1, x_0)^3}$$

$$\times \ast\!\!\ast \left[\bar{\mathcal{J}}(y_1) - \mathcal{H}(y_1) + \Omega_\mu(y_1)\,\partial_\mu \ln \Theta\left(Y_1 - X_0 + \oint_{\mathbf{B}} \jmath\right) + d_{y_1} \ln E^1(y_1', y_1) - d_{y_1} \ln \mathcal{E}(x_0, y_1)\right] \ast\!\!\ast . \tag{3.17}$$

The total $\ast\!\!\ast$ ordering leads to an expression in which one can set y_1' equal to y_1 (recall that $\ln e(y_1,y_2) = 0$). As a result one obtains an expression for the $\beta\gamma$-insertion in terms intrinsic to the surface $\mathscr{C}_t$.

Now the problem of transformation of the background operator for the surface C_1 into the background operator for the surface $\mathscr{C}_t$ [see Eqs. (3.6), (3.7)] can be reformulated in more symmetric manner: we describe the sewing of $bc\beta\gamma$ theories as a bilinear map $\{\mathscr{B}[\cdot]\}\times\{\mathscr{B}[\cdot]\}\to\{\mathscr{B}[\cdot]\}$, realized by operator multiplication (together with normal reordering) of the product $\mathscr{B}[C_1]\cdot\mathscr{B}[C_2]$ by some composite operator. The structure of the product of the background operators $\mathscr{B}[C_1]$ and $\mathscr{B}[C_2]$ which underlies the transformation in $\mathscr{B}[\mathscr{C}_t]$, is the following:

$$\mathscr{B}[C_1]\cdot\mathscr{B}[C_2] = :\overset{1}{D}_1\overset{1}{D}_2::\overset{2}{D}_1\overset{2}{D}_2::\overset{1}{\theta}::\overset{2}{\theta}:\frac{1}{\bar{\theta}^1\bar{\theta}^2} \tag{3.18}$$

(here $\bar{\theta}$ denotes the isotropic θ function and the normal ordering on C_1 and C_2 are denoted in the same manner).

Thus our target is now the transformation of (3.18) into (3.6), the background operator for the surface $\mathscr{C}_t$. We first consider the θ function in the numerator of Eq. (3.18). It turns out to be sufficient simply to fuse them into one $\ast\!\!\ast$ normal-ordered expression. As was shown in Ref. 17, this fusion occurs in the following manner

$$:\overset{1}{\theta}\begin{bmatrix}\beta_1\\ \alpha_1\end{bmatrix}\left(\oint_{b^1}[\overset{1}{H} + \overset{1}{J} - 3d\ln E^1(x_0,\cdot)]\,|\,\overset{1}{\tau}\right):$$

$$:\overset{2}{\theta}\begin{bmatrix}\beta_2\\ \alpha_2\end{bmatrix}\cdot\left(\oint_{b^2}[\overset{2}{H} + \overset{2}{J} - 3d\ln E^2(x_\star,\cdot)]\,|\,\overset{2}{\tau}\right):$$

$$= \ast\!\!\ast\, \Theta\begin{bmatrix}\beta_1 & \beta_2\\ \alpha_1 & \alpha_2\end{bmatrix}\left(\oint_{\mathbf{B}}[\mathcal{H} + \mathcal{J} - 3d\ln\mathcal{E}(x_0,\cdot)]\,|\,\Upsilon\right)\ast\!\!\ast . \tag{3.19}$$

Here the period matrices have been explicitly indicated [see Eq. (3.3)]. The right-hand side of (3.19) is indeed the required ingredient of the operator (3.6).

We further consider the product of D-factors in (3.18). In distinction from the preceding case, here it does not suffice to do a simple normal reordering and two types of corrections become necessary.

First, the deformation of the holomorphic differentials ω_{i_1} and ω_{i_2} into the differentials Ω_μ on the big surface requires the following factors:

$$\mathfrak{S}_D = \exp\frac{-1}{2\pi i}\left\{\sum_{i_1=1}^{g_1}\oint_{a_{i_1}}\delta\omega_{i_1}(u)\int^u(\mathcal{H} + \mathcal{J} + 2\bar{\mathcal{J}}) + \sum_{i_2=1}^{g_2}\oint_{a_{i_2}}\delta\omega_{i_2}(u)\int^u(\mathcal{H} + \mathcal{J} + 2\bar{\mathcal{J}})\right\} \tag{3.20}$$

where the integrals do not depend on the lower limits. Here $\delta\omega = \Omega - \omega$ are the variations of the holomorphic differentials ω_{i_1} and ω_{i_2} [see Eq. (3.2)].

The different c-number factors which come from various sources—the transformation of the D-factors under the substitution (3.11), the normal ordering and the fusion with (3.20)—are successfully absorbed almost completely into a redefinition of the c-number normalization factors. Thus, for instance

$$\ast\!\!\ast\exp\frac{-1}{2\pi i}\sum_{i_1=1}^{g_1}\oint_{a_{i_1}}\delta\omega_{i_1}(u)\left[\int^u(\mathcal{H} + \mathcal{J} + 2\bar{\mathcal{J}}) - 5\ln\mathcal{E}(u,x_0)\right]\ast\!\!\ast$$

$$\times :\overset{1}{D}_1(\mathcal{N}^1\,|\,\overset{1}{H} + \overset{1}{J})\,\overset{1}{D}_2(\mathcal{P}^1\,|\,\overset{1}{J}): \overset{1}{E}(\mathcal{N}^1,\mathcal{P}^1)\,\overset{1}{\sigma}(\mathcal{N}^1)^3\prod_{i<j}\overset{1}{E}(N_i^1,N_j^1)$$

$$= \ast\!\!\ast\,\tilde{D}_1(\mathcal{N}^1\,|\,\mathcal{H} + \mathcal{J})\,\tilde{D}_2(\mathcal{P}^1\,|\,\bar{\mathcal{J}})\ast\!\!\ast\,\mathcal{E}(\mathcal{N}^1,\mathcal{P}^1)$$

$$\times\,\Sigma^{(1)}(\mathcal{P}^1)\,\Sigma^{(1)}(\mathcal{N}^1)^3\prod_{i<j}\mathcal{E}(N_i^1,N_j^1)$$

up to a numerical factor which does not depend on x_0, $\mathcal{N}^1$ and $\mathcal{P}^1$. Here $\tilde{D}$ is given by Eq. (B1) with ω_{i_1} replaced by Ω_{i_1}, $i_1 = 1\ldots g_1$. The notations $\sigma(\mathcal{N})$ and $\mathcal{E}(\mathcal{N},x)$ are analogous to (B4); further

$$\Sigma^{(1)}(x) = \exp\frac{-1}{2\pi i}\sum_{i_1=1}^{g_1}\oint_{a_{i_1}}\Omega_{i_1}(u)\ln\mathcal{E}(u,x)$$

is the "C_1-contribution" to the $(g_1 + g_2)/2$ Fay differential Σ on $\mathscr{C}_t$. Similar relations hold for the quantities defined on C_2.

Secondly, in summing the powers of the divisors of the insertions in C_1 and C_2 we do not obtain the required power of the divisor on $\mathscr{C}_t$. Indeed, similar to Eq. (2.5), the "big" background operator effectively bosonizes the following insertions

$$\mathscr{B}[\mathscr{C}_t] \propto \prod^{g-1}[e^{\varphi}\psi]\prod^{2g-2}[e^{\bar{\varphi}}]\,\psi(x_0)\,e^{-\bar{\varphi}(x_0)}, \tag{3.21}$$

where $g = g_1 + g_2$. In addition to the presence of the naively obtained two $\xi = \psi e^{-\varphi}$-insertions, the operator $\mathscr{B}[C_1]\mathscr{B}[C_2]$ misses the insertions $e^{\varphi}\psi$ and $e^{\bar{\varphi}}e^{\bar{\varphi}}$ at some points. Fortunately, the bosonizing background operator is not literally the expression (3.21), so that there does not arise a problem with the two ξ. One must just complete the divisors $\mathcal{N}^1 + \mathcal{N}^2$ and $\mathcal{P}^1 + \mathcal{P}^2$ with some points N', P' and P''. The choice of the latter is arbitrary, but the simplest formalism results when all three points coincide with $x_\star$—the position of the ξ-insertion on C_2. In this case no further operator multipliers are required because the product of the expression (3.20) with the D-factors in (3.18) already depends on $x_\star$, since

$$\exp\int^{x_\star}(\mathcal{H} + \mathcal{J} + 2\bar{\mathcal{J}}) \sim \psi(x_\star)\,e^{-\varphi(x_\star)}\,e^{2\bar{\varphi}(x_\star)}.$$

Finally, when the "isotropic" denominator in (3.18) is replaced by the one which is defined on $\mathscr{C}_t$ no complications arise in the normal ordering and the desired changes can be realized simply by multiplication.

Thus, the expression (3.20), multiplied by the appropriate isotropic θ functions, represents the sewing operator up to normalization. Collecting all the numerical factors and taking into account the difference in normalization on $C_1 \sqcup C_2$ and $\mathscr{C}_t$, we introduce the sewing operator $\mathfrak{S}$

$$\mathfrak{S} = {}^*_* \exp \frac{-1}{2\pi i} \sum_{\mu=1}^{g} \oint_{a_\mu} \delta\omega_\mu(u) \times \int^{u} (\mathscr{H} + \mathscr{Y} + 2\overline{\mathscr{Y}} - 5d \ln \mathscr{E}(x_0, \cdot)) {}^*_* \times \frac{c}{\mathscr{E}(x, x_0)^5} \frac{\overset{1}{\theta}\begin{bmatrix}\delta_1\\ \varepsilon_1\end{bmatrix}\left(\oint_{b^1} j^1\right)\overset{2}{\theta}\begin{bmatrix}\delta_2\\ \varepsilon_2\end{bmatrix}\left(\oint_{b^2} j^2\right)}{\Theta\begin{bmatrix}\delta_1 & \delta_2\\ \varepsilon_1 & \varepsilon_2\end{bmatrix}\left(\oint_{B} j\right)}, \tag{3.22}$$

where the constant c depends only on the three surfaces C_1, C_2 and $\mathscr{C}_t$, but not on the chosen divisors, etc. In the right-hand side of Eq. (3.22) an overall ${}^*_*\,{}^*_*$ normal ordering is not assumed (otherwise the arguments of the θ functions would change).

In the presence of the relation (3.11) between the three sets of currents (on C_1, C_2 and $\mathscr{C}_t$) the sewing operator satisfies the equality

$$\mathfrak{S}\mathscr{B}[C_1]\mathscr{B}[C_2] = \mathscr{B}[C_1 \infty C_2]. \tag{3.23}$$

The background operators in the left-hand-side are given by Eqs. (3.11), rewritten respectively for C_1 and C_2. Similarly, in the right-hand side, the expression $\mathscr{B}[C_1 \propto C_2]$ is defined in Eqs. (3.6) and (3.7), where one must set

$$\mathfrak{N} = \mathscr{N}^1 + \mathscr{N}^2 + x_\star, \quad \mathfrak{P} = \mathscr{P}^1 + \mathscr{P}^2 + 2x_\star \tag{3.24}$$

[the prime forms $E(x_\star, x_\star)$ for coinciding arguments are removed from the normalization factors].

Thus the global operator formalism for the $bc\beta\gamma$ theories on the sewn surface are reproduced as some bilinear combination of analogous operator constructs on the surfaces being sewn together. It should be noted that we have never really used either the explicit expressions for the variations of the differentials on C_1 and C_2 into such on $C_1 \propto C_2 = \mathscr{C}_t$, or any expansions in powers of t. The equation (3.23) is thus exact, including the agreement of normalizations.

The operator $(\mathfrak{S}\mathscr{B}[C_2])$ in Eq. (3.23) can also be interpreted asymmetrically, as the gluing of C_2 to C_1. In this manner one can in principle achieve arbitrary Riemann surfaces, starting from a given one $C = C_1$, for which one many choose the Riemann sphere (the previously assumed restriction to $g_1, g_2 \leqslant 2$ was technical and is easily removed). However, the effectiveness of the whole construction suffers from the essential nonlinearity in the transformations of the currents; see Eq. (3.11). This nonlinearity arises on account of the $d \ln \theta$ term in the bosonized $\xi\eta$ current (3.9c), which in turn comes from the θ operators in Eqs. (2.14)–(2.15).

APPENDIX A

The prime form and the formulas related to it

The prime form $E(x,y)$ is a 1/2-differential in each of the variables x and y belonging to the universal cover $\hat{C}$ of the Riemann surface C. The prime-form $E(x,y)$ is skew-symmetric under a permutation of the arguments and has the local expansion

$$\ln \frac{E(x,y)}{x-y} = \sum_{m,n \geqslant 1} Q_{mn} \frac{x^m y^n}{mn}, \tag{A1}$$

where $Q_{mn} = Q_{nm}$. It can be seen from (A1), in particular, that the prime form has a first-order zero at $x = y$. When the point y traverses the cycles a_i and b_i, the prime-form behaves respectively as

$$E(x,y) \to E(x,y), \quad E(x,y) \to E(x,y) \exp\left\{\frac{\tau_{ii}}{2} - \int_x^y \omega_i\right\}, \tag{A2a,b}$$

where ω_i is the ith holomorphic differential.

The meromorphic differentials on C with a pole at the single point a are defined as

$$\omega_a^{(r)}(z) = \frac{1}{r!} d_z \frac{\partial^r}{\partial a^r} \ln E(a,z), \quad r \geqslant 1. \tag{A3}$$

These are objects globally defined on C; the multivaluedness in (A2) disappears from (A3). Most important is the differential $\omega_a^{(1)}(x)$ with a second-order pole; in the main text it manifests itself as the bidifferential $\omega(x,a)$ which is symmetric with respect to x and a:

$$\omega(x,a) = \omega_a^{(1)}(x)\,da = d_x d_a \ln E(x,a) = da\,dx\left\{\frac{1}{(x-y)^2} + \sum_{m,n \geqslant 1} Q_{mn} a^{m-1} x^{n-1}\right\}. \tag{A4}$$

We now obtain from Eqs. (A2) that

$$\oint_{a_i} \omega(\cdot, a) = 0, \quad \oint_{b_i} \omega(\cdot, a) = \omega_i(a). \tag{A5a,b}$$

The "diagonal" of the regular part of the expansion (A4) defines a projective connection S on C (Ref. 32):

$$S(z) = 6 \sum_{m,n \geqslant 1} Q_{mn} z^{m+n-2}. \tag{A6}$$

One more important object is the so-called dipole meromorphic differential with simple poles at two selected points a and b with residues respectively equal to -1 and $+1$:

$$\omega_{b-a}(x) = d_x \ln \frac{E(b,x)}{E(a,x)}. \tag{A7}$$

Obviously

$$\int_c^d \omega_{b-a} = \int_a^b \omega_{d-c} = \ln \frac{E(b,d)E(a,c)}{E(a,d)E(b,c)} \tag{A8}$$

and similar to Eqs. (A5),

$$\oint_{a_i} \omega_{b-a} = 0, \quad \oint_{b_i} \omega_{b-a} = \int_a^b \omega_i. \tag{A9}$$

APPENDIX B

The operator insertions of the divisors

We consider the Riemann surface C of genus g. For an additional divisor $\mathscr{K}$ of degree $Q(g-1)$ we define the following composite operator, depending on the operator current I with zero a-periods:

$$\mathscr{D}_Q(\mathscr{K}, x_0|I) = \exp\left\{ \int_{Q(g-1)x_0}^{\mathscr{K}} I - \frac{Q}{2\pi i}\sum_{i=1}^{g} \oint_{a_i} \omega_i(u) \int_{x_0}^{u} I \right\}. \tag{B1}$$

Then if the current I^* is conjugate in the sense

$$I(x)I^*(y) = \omega(x, y) + :I(x)I^*(y): \tag{B2}$$

(and the II^* theory is free, i.e., the Wick rule holds), then

$$:\exp\int_x^y I^*::D_Q(\mathscr{K}, x_0|I):=\frac{E(y,\mathscr{K})}{E(x,\mathscr{K})}\left[\frac{E(y,x_0)}{E(x,x_0)}\frac{\sigma(y)}{\sigma(x)}\right]^Q$$

$$\times:\left[\exp\int_x^y I^*\right]\mathscr{D}_Q(\mathscr{K}, x_0|I):, \tag{B3}$$

where we have introduced the notation

$$E(x,\mathscr{K}) = \prod_{K\in\mathscr{K}} E(x,K) \tag{B4}$$

(we remind the reader that the divisor $\mathscr{K}$ is positive). Similarly,

$$:\mathscr{D}_Q(\mathscr{K}, x_0|I^*)::\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(\oint_b I\right):=:\mathscr{D}_Q(\mathscr{K}, x_0|I^*)$$

$$\times\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}\left(\mathscr{K} - Q\Delta + 2\pi Q + \oint_b [I + Qd_u \ln E(x_0, u)]\right):, \tag{B5}$$

which is established by a direct calculation. The term $2\pi Q$ in (B4) is added to each component of the argument of the θ function. On account of the quasi-periodicity it can be taken out as the common factor

$$\exp\sum_{i=1}^{g} Q\delta_i,$$

which we neglected in the main text, in order to simplify the writing (it is not hard to include it, but is manifestly unnecessary for even Q).

[1)] We adhere to the invariant notation for the integrals: all integrands are considered 1-differentials, i.e., in one (and therefore in any) coordinate system they contain the differential of the integration variable.

[2)] The topological (more correctly, holomorphic) nonfreedom of the fields in the $\beta\gamma$ theory does not contradict the presence of the free Lagrangian $\beta\bar\partial\gamma$, and in the language of functional integrals it is related to nontrivial "boundary conditions," i.e., to the fact that the soliton sectors (Refs. 25–27) for each of the two scalars which bosonize the $\beta\gamma$ system are not independent.

[3)] In this section φ denotes a scalar field which is isotropic in the sense of operator products, whereas the field ϕ in Eq. (1.2) is expressed in terms of the new variables as follows:

$$e^{\Phi} = \bar\psi e^{\bar\varphi - \varphi}, \quad e^{-\Phi} = \psi e^{\varphi - \bar\varphi}.$$

[4)] Equations (2.10)–(2.12) involve the free fields ψ, $\bar\psi$, $\exp(\pm\varphi)$ and $\exp(\pm\bar\varphi)$. The difference between the β- and γ-fields and the free fields, occurring on account of global effects, is then associated with the additional factors which modify the local expressions (2.2a,b). Alternatively, one may preserve the expressions for β and γ in the form (22a,b), but in terms of modified nonfree fields ψ^*, $\bar\psi^*$, φ^* and $\bar\varphi^*$ (cf. below).

[5)] We note that Eq. (2.14a) agrees with the assertion that the background operator (2.8) describes, among other things, the insertion $\xi(x_0)$.

[6)] We note that in terms of the multivalued scalar fields of Sec. 2 the equation (3.11) can be interpreted as the conservation of the "superfields" $\Phi^* = \varphi^* + \vartheta\psi^*$ and $\bar\Phi = 2\bar\varphi + \vartheta\bar\psi^*$ in the sewing process.

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Translated by Meinhard E. Mayer