Upper critical fields and corresponding phases in superconductors with multicomponent order parameters

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Superconducting phases in a magnetic field are investigated in crystals of various symmetries using symmetry-based and analytic approaches to the Landau-Ginzburg expansion. The distinguishing property of uniaxial systems with threefold and sixfold axes is a competition between two superconducting phases with different symmetries to become the state of the system at $H = H_{c2}$. One phase arises for field orientations along the principal crystal axes, while the other arises for the perpendicular orientation. This leads to an unusual anisotropy in $H_{c2}(\theta)$ in planes passing through the principal axis. Also investigated are Abrikosov vortex lattices in the new phases.

1. INTRODUCTION

Ever since the appearance of the first experimental and theoretical papers devoted to unusual superconductivity in certain compounds, primarily in heavy-fermion systems (see the review Ref. 1), the attention of investigators has been drawn to the various magnetic properties of these superconductors.

A general symmetry analysis of the possible types of pairing which can arise in exotic superconductors by way of second-order phase transitions from the normal state gives rise to a selection of one of the representations of the full symmetry group G_0 corresponding to a transition to the superconducting phase at $T = T_c$ (Ref. 2). The group G_0 should include the crystal symmetry of the medium in which the Cooper pairs move as well as a strong spin-orbit coupling which arises in the pair due to the presence in the medium of atoms of the heavy elements (U and Ce for heavy fermions). In all, this gives

$$G_0 = G \times R \times U(1),$$

where G is the point group of the crystal symmetry, R is the time reversal operator, and $U(1)$ is the group of gauge transformations. Then the transition to a superconducting phase, which corresponds to some representation of G alone (representations of other symmetry elements from G_0 give rise to complex coefficients when expanded in terms of a basis of a representation of G) must be accompanied by breaking of various types of symmetries from G_0.

First of all, the very nature of the superconducting order parameter

$$\Delta_{\alpha\beta}(\mathbf{k}) = \langle a_{\mathbf{k}\alpha} a_{-\mathbf{k}\beta} \rangle$$

is bound up with the breaking of $U(1)$ symmetry, i.e., with the choice of a definite phase of the wave function of the electrons which form the Cooper pair. Therefore, the corresponding symmetry group for the superconductor cannot include a pure gauge transformation. It is also possible to have other kinds of symmetry breaking involving the spatial and magnetic (i.e., time-reversal) symmetries.

The symmetry of the superconducting phase plays an important role in the determination of properties such as the spin and orbital moments of the Cooper pairs and the types and locations of zeroes of the superconducting gap at the Fermi surface. The presence of zeros, in turn, determines the low-T temperature dependence of the thermodynamic quantities, e.g., the electron heat capacity, and also the various transport characteristics. Corresponding measurements for UPt_3, UBe_{13}, $CeCu_2Si_2$ and several other materials show that for these materials the thermodynamic quantities exhibit power-law dependence on T rather than the usual exponentials. However, it is impossible to identify unambiguously the symmetry classes of these superconductors on the basis of this data, because measurements of thermodynamic and transport properties do not allow us to discern easily the difference between experimental functions of the form T^2 (corresponding to lines of zeroes in the gap spectrum) and of the form T^3 (isolated point zeroes), let alone to identify the distribution of zeroes on the Fermi surfaces.

As was noted in Ref. 3, another and probably more effective method of determining the original representation corresponding to the transition at T_c would be a measurement of the anisotropy of the upper critical field H_{c2} in these materials. The authors of Ref. 3 and the later Ref. 4 treat the anisotropy of H_{c2} which appears in the basal plane of a crystal with a high-order axis of symmetry. In these papers, it is found that for non-one-dimensional representation of the point groups an additional anisotropy is possible which cannot be included in a simple model with a tensor effective mass.

Certain other directions taken by investigators of the upper critical field and the phases corresponding to them have stimulated a considerable number of experiments[5–7] based on the application of a variety of techniques, from measuring the absorption of ultrasound and jumps in the heat capacity to attenuation of torsional vibrations of pendulum. On the basis of these experiments, certain conclusions were arrived at regarding the complex phase diagram of the typical heavy-fermion superconductor UPt_3 ($G = D_{6h}$) in the $H - T$ plane. In this compound the symmetries of the phases for $H \sim 0$ and $H \sim H_{c2}$ are found to be different.[8] In Ref. 9 it was shown that when a magnetic field is applied to this compound along the sixfold symmetry axis, which thereby preserves the hexagonal symmetry of the system, a superconducting phase with broken translational symmetry can exist in addition to the usual hexagonal phase, in which

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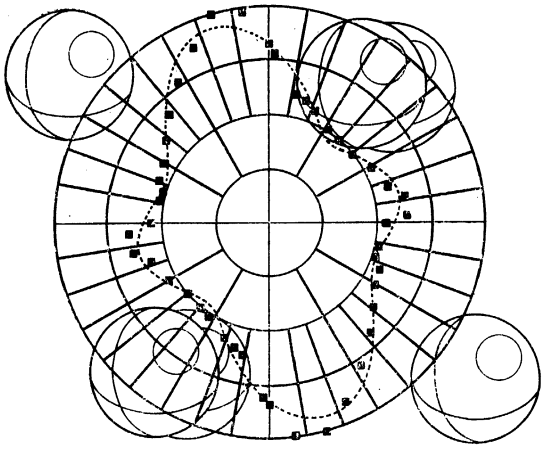

FIG. 1. Anisotropy of the upper critical field in the (210) plane plotted in polar coordinates. The data was obtained from Ref. 11 by measuring T_c in a constant field $H = 13.9$ kG and determining from it (using the known value of dH_{c2}/dT) the anisotropy of H_{c2}. Atoms of Mo are represented by small circles.

case the sixfold axis becomes a twofold axis. A further problem of interest is to trace the way this state changes as the magnetic field deviates from the crystallographic axis.

There is interest in this type of problem in connection with recent experiments in which the anisotropy of H_{c2} was measured in $Cu_{1.8}Mo_6S_8$ (a material which belongs to the class of compounds with a Chevrel phase[10,11]). In these papers, the dependence of H_{c2} on angle was measured in a plane passing through a threefold inversion axis (almost all compounds with a Chevrel phase have the rhombohedral structure). The form of this dependence (see Fig. 1), which looks like the envelope of two perpendicularly overlapping ellipses, cannot be reconciled with any description in terms of a tensor mass, which would give an ellipsoidal dependence of the upper critical field on angle in any crystal plane. The idea of connecting this type of angular dependence of the anisotropy with nontrivial pairing was proposed in Ref. 12, where, based on a numerical investigation of the Ginzburg-Landau (GL) functional for a two-component order parameters, parameter of this functional were found which led to a form of $H_{c2}(\theta)$ which agreed with experiment.

The goal of this paper will be to give a systematic analysis of the superconducting phases which arise in uniaxial crystals, and to show their relation to the various anisotropies of H_{c2}. In Sec. 2 we will carry out a symmetry analysis of the possible phases, some of which have already been obtained previously.[9] In Sec. 3 we will develop a method for solving certain linear homogeneous differences equations; this will allow us to construct a perturbation theory which will enable us to investigate the anisotropy of the upper critical field. In Sec. 4 we generalize expressions obtained earlier for vortex lattices in the new phases.

2. NONTRIVIAL SUPERCONDUCTIVITY IN A STRONG MAGNETIC FIELD (THE MOST SYMMETRIC PHASES)

The influence of a magnetic field on the symmetry of a superconducting state for systems with nontrivial pairing is well known from the example of the superfluid phase of ^{3}He. In this case the magnetic field, which reduces the full symmetry group of the system, stabilizes the A_1 phase over the entire range of the $p - T$ diagram near T_c compared to the A- and B-phases, which are energetically more favorable in the absence of a field. Two types of interaction are responsible for the change in the symmetry group of ^{3}He in a magnetic field—an interaction which orients the action of the field along the spontaneous local moments, and one which interacts with the induced paramagnetic moments in the system of atomic spins. As a result of these interactions, the critical temperature for the spin projection $s_z = +1$ is increased, i.e., the degeneracy in s_z is lifted. This implies that the lowering of the symmetry must be treated as a splitting of the original representation into several components.

A different picture applies to the exotic superconductors. In this case the particles which form the pairs have nonzero charge, so that at the first level of interactions of the system with a magnetic field we propose supercurrents generated by this very field, which are described by the gradient terms in the energy functional. On the other hand, both types of interaction mentioned above for ^{3}He become unimportant by virtue of the fact that the spins are "frozen" onto the lattice. Therefore the inclusion of a magnetic field (i.e., destruction of time-reversal symmetry) will not give rise to a splitting of the original representation into several components which transform into one another by complex conjugation.

Let us now turn to a direct description of the exotic superconductors in a magnetic field based on the GL functional. For this it is necessary to choose some irreducible representation of the point group of crystal rotations (different representations for space inversion and reflection will correspond to the cases of singlet and triplet pairing). The order parameter is decomposed according to the basis of this irreducible representation:

$$\Delta(\mathbf{k}) = \sum \eta_i \Phi_i(\mathbf{k}).$$

The GL functional is obtained, in turn, by expanding the energy near T_c in powers of η_i in the form of combinations of quantities which are invariant with respect to all the symmetry operators of the system. Thus, for hexagonal symmetry D_6, i.e., the case that is most popular in the literature (this corresponds to the compound UPt_3) there exist two two-dimensional representations E_1 and E_2; this case is also the primary subject of our investigation. The basis functions from E_1 transform as components of a two-dimensional vector perpendicular to the sixfold symmetry axis $\hat{z}$. The invariant form of the energy density, to within terms of second order, can be written in the following way:

$$F = \alpha(T - T_c)\eta_i^*\eta_i + K_1 p_i^* \eta_j^* p_i \eta_j + K_2 p_i^* \eta_i^* p_j \eta_j + K_3 p_i^* \eta_j^* p_j \eta_i + K_4 p_z^* \eta_i^* p_z \eta_i,$$

$$i = x, y, \quad \mathbf{p} = -i\hbar\partial + \frac{2e}{c}\mathbf{A}. \tag{1}$$

The basis functions of the E_2 representation [these are $(x+iy)^2$, $(x-iy)^2$] correspond to those projections of the moment of the pairs which equal ± 2. This should be con-

trasted with E_1, where the projections of the moment equal ± 1; the different projections determine the similarities and differences for these two cases. Because the results are found to be the same, we will omit the contributions for E_2.

Finding the exact form of the superconducting phases near H_{c2} requires direct minimization of (1). However, before doing this it is interesting to find all the possible remaining symmetries of the corresponding solutions independently, as was done, e.g., for ^{3}He in connection with the impossibility of an exact minimization of the terms of fourth order which determine the symmetry of the superfluid phases.[13] The symmetry group of the superconductor (i.e., of the gradient terms in (1)) including a field along the $\hat{z}$ axis has the form

$$G_0=[E,\ (C_{6z})^n,\ 3U_2R,\ 3U_2'R]\times U(1).$$

The remaining symmetries are determined by its subgroups (see also Ref. 2):

$$\begin{aligned} D_6(E)&=(E,\ (C_{6z}\varepsilon)^n,\ U_2\times R,\ U_2\varepsilon^2R,\ U_2\varepsilon^4R,\ U_2'\varepsilon R,\\ &\quad U_{2y}'\varepsilon^3R,\ U_2'\varepsilon^5R),\\ D_6(C_2)&=(E,\ (C_{6z}\varepsilon^2)^n,\ U_{2x}R,\ U_2\varepsilon^4R,\ U_2\varepsilon^2R,\\ &\quad U_2'\varepsilon^2R,\ U_{2y}'R,\ U_2'\varepsilon^4R),\\ &\quad \varepsilon=\exp(\pi i/3), \end{aligned}$$

and also $D_3(E)$, $D_2(E)$, $D_2(C_2)$, which are subgroups of those mentioned above. It is easy to convince oneself that by virtue of the properties of the basis functions, states with the symmetries $D_6(E)$ and $D_2(E)$ can appear which generate the representation E_1, while those with $D_6(C_2)$ and $D_2(C_2)$ symmetry generate E_2. The state $D_3(E)$ cannot appear in a hexagonal crystal.

Varying (1) with respect to η_i^* leads to the GL equations:

$$\begin{aligned} \alpha(T_c-T)\eta_1&=((K_1+K_2+K_3)p_x^2+K_1p_y^2+K_4p_z^2)\eta_1\\ &\quad+(K_2p_xp_y+K_3p_yp_x)\eta_2,\\ \alpha(T_c-T)\eta_2&=(K_2p_yp_x+K_3p_xp_y)\eta_1+(K_1p_x^2+(K_1+K_2+K_3)p_y^2\\ &\quad+K_4p_z^2)\eta_2. \end{aligned} \tag{2}$$

Introducing

$$\eta_\pm=\eta_x\pm i\eta_y,\quad a=\left(\frac{c}{4\hbar eH}\right)^{1/2}(p_x-ip_y),\quad [a,a^+]=1$$

and setting $p_z=0$ (because $H\|\hat{z}$), we are led to the eigenvalue problem

$$\lambda\Psi=\begin{pmatrix}(1+C)(2a^+a+1)-D & 2Ca^+a^+\\ 2Caa & (1+C)(2a^+a+1)+D\end{pmatrix}\Psi, \tag{3}$$

where Ψ is represented by the column

$$\Psi=\begin{pmatrix}\eta_+\\ \eta_-\end{pmatrix},\quad C=\frac{K_2+K_3}{2K_1},\quad D=\frac{K_2-K_3}{2K_1}.$$

In solving (3) we seek η_+, η_- in the form of a series based on eigenfunctions φ_n of the operator a^+a corresponding to the nth Landau level. Possible solutions to (1) will be: $(\varphi_0,0)$, $(\varphi_1,0)$, $(p_n\varphi_{n+2},q_n,\varphi_n)$. In determining the minimum eigenvalue λ, which is connected with H_{c2} by the relation

$$H_{c_2}=\frac{\phi_0}{2\pi\hbar^2}\frac{\alpha(T_c-T)}{K_1\lambda_{min}}$$

(here ϕ_0 is the flux quantum), we must take into account the requirement that the uniform state be stable; this is ensured by positive definiteness of the quadratic form made up of $p_x\eta_1$, $p_y\eta_1$, $p_x\eta_2$, and $p_y\eta_2$. It leads to the condition

$$K_1+K_2+K_3>|K_2|,\ K_1>|K_3|,\ K_4>0$$

or, if presented through C and D, it will look like

$$1+2C>|C+D|,\ 1>|C-D|. \tag{4}$$

Then for all the permissible regions of values of the parameters the minimum λ is attained for one of the two solutions:

$$\Psi\sim(\varphi_0,0),\ \lambda_1=1+C-D, \tag{5}$$
$$\Psi\sim\left(\varphi_2,\frac{\omega}{2^{1/2}}\varphi_0\right),$$

$$\lambda_2=3(1+C)-(8C^2+(2+2C-D)^2)^{1/2},\quad \omega=\frac{4C}{\lambda_2-(1+C+D)}. \tag{6}$$

Comparing λ_1 and λ_2, we see that for $D>C^2/(1+C)$ seeds of the phase (5) can appear near H_{c2} with symmetry $D_6(E)$. For the opposite sign of the inequality seeds of phase (6) can appear with the broken spatial symmetry $D_2(E)$ (see Fig. 2 of Ref. 9). Solutions analogous to (5) and (6) were found in Ref. 14, where superconductors were investigated with p-pairing in the weak-coupling approximation and in the absence of spin-orbit interaction. However, these do not play an important role, since the maximum H_{c2} corresponds in this case to a solution for the polar type of phase.

In addition to the completely general condition (4) on the coefficients of the GL functional, there exists still another limitation on them, connected with particle-hole symmetry near the Fermi surface. This symmetry relates the solutions of the Bogolyubov equations for quasiparticles with positive and negative energies, and must be treated in our case as a symmetry relative to the operation $\eta\to\eta^*$. If this symmetry is exact, then we obtain $D=0$. The breaking of this symmetry occurs as a consequence of the curvature of the Fermi surface, and the corresponding value of D is found to be of order $(T_c/\varepsilon_F)^2$ The physical meaning of this assertion is also explainable by the fact that terms with the coefficient D in (1) can be transformed to the form $DH_zi(\eta_1^*\eta_2-\eta_1\eta_2)$ and in this way can describe the interaction of the field with the dynamic orbital moment of the Cooper pair. As in the case of ^{3}He, this moment is small as a consequence of the large size of the pairs compared to the spacing between quasiparticles.[15] The majority of materials satisfy $T_c/\varepsilon_F<0.1$, and consequently it is possible to assume $D=0$ to good accuracy. This leads to the conclusion that a state with the broken spatial symmetry $D_2(E)$ will always arise in a magnetic field.

Let us now turn to the case where the magnetic field is applied perpendicular to the $\hat{z}$ axis. If the field is directed along one of the crystallographic axes which is simultaneously a second-order rotation axis, e.g., x, then the symmetry group reduces to $D_2(C_{2x})$ with elements E, C_{2x}, $C_{2y}R$, and $C_{2z}R$. For all the remaining directions of the field $D_6\times R$ reduces to $C_{2z}(E)$. Thus, the isotropy of the upper

critical field in the basal plane of the crystal is absent, generally speaking, and $H_{c2}(\theta)$ repeats the hexagonal symmetry of the sample. However, near T_c at $H=0$ the second-order terms in the free energy (both uniform and gradient terms) have an additional symmetry: $D_\infty \times R$ in place of $D_6 \times R$, i.e., the sixfold axis induces an additional continuous symmetry. As a result, we can obtain an equivalence of all the axis in the plane and isotropy of H_{c2}.[4] In this case the symmetry group for all $\mathbf{H} \perp \hat{z}$ becomes $D_2(C_{2h})$. The remaining possible symmetries of solutions which minimize the energy functional are also easy to find (the x-axis is chosen along the direction of $\mathbf{H}$):

$$D_2'(E)=(E, C_{2x}e^{i\pi}, C_{2y}R, C_{2z}e^{i\pi}R),$$
$$D_2'(C_{2x})=(E, C_{2x}, C_{2y}e^{i\pi}R, C_{2z}e^{i\pi}R).$$

In order to find the solution in analytic form, we set $p_x=0$ and rewrite (2) in the form

$$\alpha\frac{(T_c-T)}{K_1}\eta=\begin{pmatrix} p_y^2+Kp_z^2 & 0 \\ 0 & (1+2C)p_y^2+Kp_z^2 \end{pmatrix}\eta, \quad (7)$$

where $K=K_4/K_1$. The solutions we require will be

$$\lambda_{min}=K^{1/2},\ \eta_x\neq 0,\ \eta_y=0,\ C>0, \quad (8a)$$
$$\lambda_{min}=[K(1+2C)]^{1/2},\ \eta_x=0,\ \eta_y\neq 0,\ C<0. \quad (8b)$$

Nonzero values of η_x and η_y are described by wave functions for anisotropic Landau levels (p_x and p_z enter into (7) asymmetrically). The solution (8a) corresponds to the symmetry $D_2'(C_{2x})$, while (8b) corresponds to $D_2'(E)$. As for the condition $p_x=0$, which we imposed in order to reach the minimum λ, we now can no longer prove analytically, by finding $\lambda(p_x)$, that this should be the case, as we did in finding H_{c2} for superconductors with single-component order parameters or in the case (3). In Ref. 4 this supposition was verified numerically. Nevertheless, there are considerations which lead to just this condition on $\lambda_{\min}$. They are presented in the Appendix.

Including the pure continuous symmetry of the energy functional near T_c also changes the symmetry of the state (5). It becomes $D_\infty(E)$ and is continuous; in addition, the form of this phase does not depend on the specific values of the parameters in the GL expansion (1). In the terms used to describe the superfluid phase of ^{3}He, phase (5) is "inert." The symmetry of phase (6), as before, remains discrete—$D_2(E)$—and its form depends directly on these parameters. It is not "inert," and in contrast to ^{3}He it is just this noninert phase which apparently corresponds to the physically realized situation.

In concluding this section, we should pause briefly to discuss other types of crystal symmetries. In the case of rhombohedral symmetry D_3 there is one two-dimensional vector representation of interest to us. It is fully analogous to the representation E_1 of the group D_6 we discussed earlier. In particular, the GL expansion up to terms of second order will be the same as in (1), and all the conclusions arrived at earlier remain exactly valid for this representation as well. In the presence of a fourfold axis for the tetragonal symmetry D_4, the energy functional near T_c ceases to possess an additional continuous symmetry for its two-dimensional representation E. Thus, an invariant $K_5(p_x^*\eta_1^* p_x\eta_1+p_y^*\eta_2^* p_y\eta_2)$, is added to Eq. (1) which is independent of the fourfold axis, and which is preserved only for rotations around the z axis by an angle of $\pi/2$. From this we can immediately draw the conclusion that the only possible symmetry of the solutions in this case is $D_2(E)(\mathbf{H}\|\hat{z})$. The symmetry $D_4(E)$ could correspond only to a single-component solution of type (5); however, its full symmetry is $D_\infty(E)$, and therefore it cannot arise for arbitrary values of K_5 which breaks this symmetry.

3. ANISOTROPIES OF THE UPPER CRITICAL FIELD IN EXOTIC SUPERCONDUCTORS

For terms up to second order in the GL functional it follows from the same symmetry D_4 that H_{c2} must necessarily be anisotropic in the basal plane of a tetragonal crystal. The situation is analogous for a cubic crystal. For the two-dimensional and any of the three-dimensional representations of the group O, there already appear invariants in the second-order gradient terms which break the symmetry relative to arbitrary rotations. The invariant of this type for the three-dimensional representation F_1 has the form

$$p_x^*\eta_1^* p_x\eta_1+p_y^*\eta_2^* p_y\eta_2+p_z^*\eta_3^* p_z\eta_3$$
$$-{}^1/_2(p_x^*\eta_1^* p_y\eta_2+p_x^*\eta_1^* p_z\eta_3+p_y^*\eta_2^* p_z\eta_3+\text{c.c.}),$$

which leads to an unusual angular dependence of $H_{c2}(\theta)$ in the form of a rosetts.[3,4] For the exotic superconductors made from hexagonal and rhombohedral systems, this characteristic feature is absent, and near T_c the isotropy of H_{c2} is retained in a plane perpendicular to the principal axis. However, the continuous symmetry of the energy functional can lead to other interesting features. In the previous section we found that for symmetric orientations of the field along a sixfold (threefold) axis in a hexagonal (rhombohedral) crystal, the superconducting state arising from the two-dimensional representation is determined by one of the phases (5) and (6). These two phases differ from each other by the spatial symmetries both of the Copper pairs [this follows from (5) and (6)], and the vortex lattices which correspond to them (see Sec. 4 below). As the field deviates from the special direction, the symmetry of the system changes discontinuously from $D_6\times R$ to $(E, C_{2x}R)$, where the axis $\hat{x}\perp\mathbf{H}$ and lies on the basal plane. Correspondingly, the symmetries of the superconducting phases also become different. The behavior of these phases as the magnetic field goes off-axis will also be a goal of our investigations.

Let us discuss the energy functional (1) in the case where the field $\mathbf{H}$ is rotated by a definite angle θ in the (z,y) plane: $\mathbf{H}=(0, H\sin\theta, H\cos\theta)$. Let us pass to a new coordinate system, retaining the $\hat{x}$ axis as before, directing the $\hat{z}_1$ axis parallel to $\mathbf{H}$, and letting the y_1 axis be perpendicular to it:

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix},$$
$$\begin{pmatrix} p_y \\ p_z \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} p_{y_1} \\ p_{z_1} \end{pmatrix}.$$

If we now express p_y and p_z in terms of p_{y_1} and p_{z_1} and take into account that in order to determine H_{c2} it is necessary to set $p_{z_1}=0$ (see Appendix), then we can obtain a GL equation of the following form:

$$\frac{\alpha(T_c-T)}{K_1}\eta=[(1+C)p_x^2+((1+C)\cos^2\theta+K\sin^2\theta)p_y^2$$
$$+C\sigma_3(p_x^2-p_y^2\cos^2\theta)$$
$$+C\sigma_1\cos\theta(p_xp_y+p_yp_x)+Di\sigma_2\cos\theta(p_xp_y-p_yp_x)]\eta. \quad (9)$$

Here and in what follows, p_{y_1} will be denoted by p_y for brevity, $[p_x,p_y]=-2ie\hbar H/c$, and σ_i are the Pauli matrices, chosen as a basis for the space of 2×2 dimensional matrices. Let us introduce

$$a=\left(\frac{c}{4e\hbar H}\right)^{1/2}\left\{\left[\frac{1+C}{(1+C)\cos^2\theta+K\sin^2\theta}\right]^{1/4}p_x-i\right.$$
$$\left.\times\left[\frac{(1+C)\cos^2\theta+K\sin^2\theta}{1+C}\right]^{1/4}p_y\right\},$$
$$[a,a^+]=1$$

and use a unitary transformation to pass from η to $\Psi=(\eta_+,\eta_-)$; in this case $\sigma_3\to\sigma_1$, $\sigma_1\to-\sigma_2$, $\sigma_2\to-\sigma_3$, and Eq. (9) can be written in the form

$$\lambda\left[\cos^2\theta+\frac{K}{1+C}\sin^2\theta\right]^{1/2}\Psi=\Lambda\Psi$$
$$=\begin{pmatrix}(1+C)(2a^+a+1)-Df & R\\ R^+ & (1+C)(2a^+a+1)+Df\end{pmatrix}\Psi,$$
$$R=\tfrac{1}{2}Caa(1-f)^2+\tfrac{1}{2}Ca^+a^+(1+f)^2 \quad (10)$$
$$+\tfrac{1}{2}C(1-f^2)(a^+a+aa^+),$$

where

$$f=\frac{\cos\theta}{\left(\cos^2\theta+\frac{K}{1+C}\sin^2\theta\right)^{1/2}}.$$

For $\theta=0, f=1$, Eq. (10) goes to (3). In contrast to all the solutions found earlier, there is no change in the operators a and a^+ which will reduce the eigenfunctions of Eq. (10) to a form such that the components of the order parameter are expressible in terms of a finite number of wave functions of new Landau levels. The reason for this is the same as in the case of tetragonal symmetry D_4, where solutions with symmetry $D_2(E)$ (see Sec. 2) are also obtained only in the form of infinite series. If the corresponding solutions could be expressed in terms of a finite number of Landau levels, then there would necessarily exist a solution with a still smaller number of levels, and ultimately we would be left with a solution consisting of a single zero-order Landau level, i.e., to a solution of type (5) with symmetry which does not admit the overall group of identity transformations of the system. Analogous considerations are verified in this case.

Substituting

$$\eta_+=\sum x_n\varphi_n,\quad \eta_-=\sum y_n\varphi_n$$

(taking into account $a^+a\varphi_n=n\varphi_n$) into (10) and comparing coefficients of φ_n on the left and right sides, we obtain a system of two second-order difference equations. The upper critical field, as in the usual case, will be that field for which there first appear normalizable solutions to Eq. (10), i.e., superconducting seeds appear. This implies that the series formed from the sequences $\{x_n\}$ and $\{y_n\}$, $\Sigma|x_n|^2$ and $\Sigma|y_n|^2$, must converge. To decide this, it is necessary for us to know the behavior of x_n and y_n for large n. To do this, let us use the first equation to express y_n in terms of the elements of $\{x_n\}$; then by substituting the latter into the second equation, we obtain a linear homogeneous difference of fourth order:

$$b_{n,n-4}x_{n-4}+b_{n,n-2}x_{n-2}+b_{n,n}x_n+b_{n,n+2}x_{n+2}+b_{n,n+4}x_{n+4}=0,$$
$$b_{n+4,n}=b_{n,n+4}=-\frac{C^2(1-f^2)^2[(n+1)(n+2)(n+3)(n+4)]^{1/2}}{4[(1+C)(2n+5)+Df-\Lambda]}$$
$$b_{n+2,n}=b_{n,n+2}=-\frac{C^2(1+f)^2(1-f^2)(2n+1)[(n+1)(n+2)]^{1/2}}{4[(1+C)(2n+1)+Df-\Lambda]}$$
$$-\frac{C^2(1+f)^2(1-f^2)(2n+1)[(n+1)(n+2)]^{1/2}}{4[(1+C)(2n+5)+Df-\Lambda]}, \quad (11)$$
$$b_{n,n}=(1+C)(2n+1)-Df-\Lambda-\frac{C^2(1+f)^4n(n-1)}{4[(1+C)(2n-3)+Df-\Lambda]}$$
$$-\frac{C^2(1-f^2)^2(2n+1)^2}{4[(1+C)(2n+1)+Df-\Lambda]}-\frac{C^2(1-f)^4(n+1)(n+2)}{4[(1+C)(2n+5)+Df-\Lambda]}.$$

The upper critical field is connected with Λ by the following relation:

$$H_{c_2}=\frac{\phi_0}{2\pi\hbar^2}\frac{\alpha(T_c-T)}{K_1\Lambda_{min}}\left[\cos^2\theta+\frac{K}{1+C}\sin^2\theta\right]^{1/2}.$$

We can find the value of Λ for which there exists a decreasing sequence $\{x_n\}$ in the following way. The general theory of difference equations (see, e.g., Ref. 16) establishes the following facts which we need:

1. Any linear combination of solutions of a linear homogeneous difference equation is also a solution to this equation. Any solution to an n-th order equation is uniquely determined by specifying its initial conditions $-x_0,\ldots,x_{n-1}$. Correspondingly, all solutions form an n-dimensional linear space. Note that, because x_n appears in (11) only with the specific parity of n, the order of the equation is decreased by a factor of two. The sought-after $\Lambda_{\min}$ corresponds to the sequence $x_0,x_2,\ldots$.

2. The asymptotic form of the solutions of an arbitrary linear difference equations with variable coefficients coincides with the solution of an equation with constant coefficients which is obtained from the original equation by replacing all the coefficients $b_{n,n-k}$ by their limits as $n\to\infty$ (the Poincaré theorem). The solutions to the latter can be found exactly—they are geometric progressions and linear combinations of the latter.

3. If the coefficient $b_{n,n-k}$ with the largest k does not go to zero as n varies, then by choosing the initial conditions we can obtain a solution to the difference equation having any asymptotic form admitted by the limiting equation (the generalized Poincaré theorem).

Because a geometric progression is easily summed, our normalizable solutions must correspond to any decreasing solution to the difference equation. In the case of Eq. (11), the initial conditions are specified in a somewhat different fashion. For arbitrary x_0 and x_2 we find x_4 and x_6 from the first two recursion relations, and then all the subsequent x_n will be determined in terms of the four previous x_n, since $b_{n,n-k}\neq0$ for all $n\geqslant4$. This implies that our decreasing solution does not exist for all Λ.

The characteristic equation which describes the geometric progressions which satisfy the limiting equation for

(11) has, as is easy to verify, four real and distinct roots, and because

$$\lim_{n\to\infty}\frac{b_{n,n-4}}{b_{n,n+4}}=1,$$

two of them will have absolute value smaller, two larger than one. Corresponding to this, we can choose four independent solutions to (11). Let $\mathbf{l}_0=(x_0,x_2,x_4,x_6)$ be an initial condition such that $\mathbf{l}_0^k$ are the initial conditions for the four independent solutions; then $\mathbf{l}_0=\Sigma C_k\mathbf{l}_0^k$ and C_k are certain constants. If we now start with the third equation and iterate up to the equation with subscript $N/2+3$ and obtain a new quadruple $\mathbf{l}_N(x_N,x_{N+2},x_{N+4},x_{N+6})$, then we find $\mathbf{l}_N=\Sigma C_k\mathbf{l}_N^k$, where $\mathbf{l}_N^k$ is the same quadruple that was obtained from the initial conditions $\mathbf{l}_0^k$. For large N we have $\mathbf{l}_N^k\sim q_k^N\mathbf{l}^k$, where q_k is the base of the geometric progression we are looking for. Then, as an initial condition for finding the rest of the infinite sequence, the contribution to $\mathbf{l}_N$ from the solution with maximum $|q_k|$ is much larger than the contribution to the initial $\mathbf{l}_0$: $\mathbf{l}_N=\Sigma\tilde{C}_k\mathbf{l}^k$, where $\mathbf{l}^k$ are the initial conditions for the four independent solutions for large N, and $\tilde{C}_k=C_kq_k^N$; $|\tilde{C}_k|$ for the largest q_k is much larger than $|\tilde{C}_i|$ for the remaining q_i.

Conversely, if we specify any $\mathbf{l}_N$ assuming $\mathbf{l}_p=0$ for $p\geqslant N+8$, and from the equation with subscript $N/2+3$ we iterate backward in the direction of smaller subscripts, then, once we reach $\mathbf{l}_0$, we obtain a larger contribution to it from $\mathbf{l}_0^k$ with $|q_k|<1$ than with $|q_k|>1$. The condition that such a $\mathbf{l}_0$ is possible for a certain Λ is that the determinant of the matrix of the first $N/2+3$ homogeneous linear equations reduced to zero. From this it is obvious how the exact condition must appear for the existence of a decreasing series for a given Λ. It will consist of the vanishing of the following infinite determinant:

$$\begin{vmatrix} b_{00} & b_{02} & b_{04} & 0 & 0 & \dots \\ b_{20} & b_{22} & b_{24} & b_{26} & 0 & \dots \\ b_{40} & b_{42} & b_{44} & b_{46} & b_{48} & \dots \\ 0 & b_{62} & b_{64} & b_{66} & b_{68} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}=0. \tag{12}$$

In this notation, this determinant diverges in general. However, by dividing each row by the nonzero coefficient $b_{n,n-4}$ which is of no interest to us, we can achieve convergence of the entire infinite determinant. This approach is suitable for difference equations of any order. When the order of the difference equation equals two, the determinant of the matrix, which will have nonzero terms on the diagonal and two adjacent subdiagonals, reduces to a continued fraction:

$$b_{00}-\cfrac{b_{02}b_{20}}{b_{22}-\cfrac{b_{24}b_{42}}{b_{44}-\dots}}=0.$$

This equation, which is well-known in the theory of continued fractions, was first used in Ref. 14. It usually arises in problems where the transition temperature of the original representation is subject to a small splitting either as a result of weak spin-orbit interaction or because of interactions with an antiferromagnetic order parameter.

Equation (12) cannot be solved in the general case; however, it allows the perturbation theory we require to be developed rather simply. In what follows, we will set $D=0$. Correspondingly, it follows from (4) that $-1/3<C<1$. This allows us to consider C as a certain small parameter in the problem, and to attempt to write the required expressions in the form of a series in powers of C.

According to (11), all the off-diagonal elements (12) disappear for $\theta=0$. Then λ_1 and λ_2, which coincide in this case with Λ_1 and Λ_2, are the roots of the two equations $b_{00}=0$ and $b_{22}=0$. For $\theta\neq0$ the nondiagonal elements are nonzero; however, if we take into account that they appear in (12) only in the form $b_{n,n-k}b_{n-k,n}\sim C^4\sin^4\theta$, then they can be neglected to first approximation in C. After this, the root (12) as before will be found from the equations $b_{00}=0$, $b_{22}=0$, etc. This allows us to find the angular dependence:

$$\begin{aligned}\Lambda_1(\theta)&=1+C-\frac{|C|}{2}(1-f^2),\\ \Lambda_2(\theta)&=1+C-\frac{C^2(1+f)^2}{8(1+C)}.\end{aligned} \tag{13}$$

Thus, for $\theta=0$ (corresponding to $f=1$) we have established a direct correspondence between the solutions of (13) and phases (5) and (6) we found earlier: $\Lambda_2(0)<\Lambda_1(0)$. For $\theta=\pi/2$ (i.e., $f=0$), which corresponds to the case where the magnetic field is perpendicular to the principle axis, $\Lambda_1(\pi/2)$ corresponds to the smallest solution to (8), depending on the sign of C (again taking into account the smallness of C). In addition, $\Lambda_1(\pi/2)$ is much larger than $\Lambda_2(\pi/2)$. Consequently, in the range of angles $0<\theta<\pi/2$ there must be an intersection of the curves $\Lambda_1(\theta)$ and $\Lambda_2(\theta)$. Then the curve $\Lambda_{\min}(\theta)$ will have a kink for

$$(1-f^2)=\frac{C(1+f)^4}{4(1+C)}$$

However, this kink disappears if we "include" the nondiagonal elements. Limiting ourselves in first approximation only by the condition

$$\begin{vmatrix} b_{00} & b_{02} \\ b_{20} & b_{22} \end{vmatrix}=0,$$

we obtain

$$\Lambda_{min}(\theta)=\frac{1}{2}\Big\{\Lambda_1(\theta)+\Lambda_2(\theta)-\Big[(\Lambda_1(\theta)-\Lambda_2(\theta))^2+\frac{C^3(1+f)^4(1-f^2)}{32(1+C)}\Big]^{1/2}\Big\} \tag{14}$$

From this we see that the kink is smoothed out into a point of inflection. The considerations applied here are very similar to those used to deal with intersection of electronic terms in diatomic molecules, Superconducting states with initially different symmetries acquire the same symmetry as the field deviates from the symmetry axis. Therefore the corresponding curves $\Lambda_1(\theta)$ and $\Lambda_2(\theta)$ cannot intersect, i.e., they repel each other.

The perturbation theory we are using is automatically obtained from Eq. (12). In contrast, an attempt to obtain it directly from (10) would be very inconvenient, because we cannot isolate the perturbation operator we require in explicit form.

For values of C which are not small compared to unity, it is necessary to include higher-order terms of the expansion

in powers of C. However, if we keep in mind that Eq. (13) in the form we have written it gives an error in determining the eigenvalues (even for $C = 1$) of less than 10%, we can use it qualitatively together with (14) over the entire region $-1/3 < C < 1$.

The picture developed here accords well with the experimental shape of the curves given in Refs. 10, 11 (see Fig. 1). Furthermore, while the weak rhombohedral distortion in the Chevrel phases (the cluster Mo_6S_8 forms an easily-deformed cube) should introduce an anisotropy of only 1% into the effective mass and correspondingly into H_{c2}, the anisotropy for the two-dimensional representation which selects the axis perpendicular to the order parameter also appears even if the mass is isotropic [i.e., equal K_1 and K_4 in (1)] because C is nonzero. This can lead to the anisotropy of about 30% which is observed in experiment.

4. LATTICES OF VORTICES IN EXOTIC SUPERCONDUCTORS

In the previous sections we sought the upper critical fields in the new Type II superconductors as those magnetic fields for which the appearance of superconducting seeds in a thin normal sample becomes energetically favorable. The superconductivity in this case is found in a suppressed state which finds expression in the linearity of the of the equations which describe the superconducting seeds. The concrete definition of the gauge, which we have not been interested in up to now in general, leads to seeds which are localized in space. By virtue of the linearly of the equations from which these seeds are obtained, at H_{c2} they are formed independently throughout the entire volume. By shifting to the region of smaller fields and thereby causing the superconducting order parameter to develop up to some nonzero value, and by taking into account nonlinear terms in the GL equations, we obtain an interaction (specifically a repulsion) between the seeds, which packs them into a regular structure.

In contrast to the situation which arises when the superconducting sample is permeated with vortices for $H \gtrsim H_{c1}$, in which case there is a physical quantity—the magnetic field $\mathbf{B}$—which fixes the spacing between vortices, in our case it is meaningless to discuss two superconducting seeds. For any $H < H_{c2}$ it is necessary to construct a planar lattice out of them at the outset. In our treatment of this question for multicomponent superconductors we will base ourselves on the standard approach discussed in Refs. 17, 18.

The inclusion of nonlinear terms proceeds as follows: in Eq. (4) for the energy, which is quadratic in the absolute value of $\boldsymbol{\eta}$, we include terms of fourth order in $\boldsymbol{\eta}$, and also the energy of screening currents. In a hexagonal (rhombohedral) system only two fourth-order invariants are possible for the vector invariant: $\beta_1(|\eta_1|^2 + |\eta_2|^2)^2$ and $\beta_2(\eta_1^2 + \eta_2^2)(\eta_1^{*2} + \eta_2^{*2})$. The requirement of positive definiteness gives $\beta_1 > 0$ and $\beta_2 > -\beta_1$. For further calculations it will be convenient to pass to dimensionless units by setting $\alpha(T_c - T) = 1$, $\beta_1 = 1/2$, and $K_1 = 1$. Then

$$\mathbf{p} = -\frac{i}{\varkappa}\nabla + \mathbf{A},$$

where $\varkappa$ is the Landau-Ginzburg parameter for the material in question, while the dimensionless parameter λ is connected with the upper critical field by the relation $\lambda = \varkappa/H$. In these dimensionless units the energy functional for a superconductor in a magnetic field has the form

$$F = -\eta_i^*\eta_i + {}^1/_2(\eta_i^*\eta_i)^2 + {}^1/_2\beta_2|\eta_i\eta_i|^2 + \boldsymbol{\eta}^*\hat{H}_{\text{eff}}\boldsymbol{\eta} + \mathbf{h}^2 - 2\mathbf{hH}. \quad (15)$$

Here $\hat{H}_{\text{eff}}$ is the effective Hamiltonian in Eq. (2) [the passage from (1) to (15) is mediated by our neglect of the specific form of the boundary conditions], $\mathbf{h}$ is the magnetic field in the sample averaged over the atomic structure but not over the system of supercurrents, and $\mathbf{H}$ is the magnetic field intensity induced by an external source.

Out of all these phases, the most interesting is a lattice of Abrikosov vortices for phase (6). Lattices of vortices of the remaining phases, for which the components of the order parameter depend in the same way on coordinates, are analogous to lattices of the usual superconductors. The lattice of phase (5) is hexagonal, while that of (8) is distorted because of the anisotropy of the mass tensor.

According to Ref. 17, the first stage in determining the structure of the mixed state near H_{c2} is the determination of the field $\mathbf{h}_s$ caused by the supercurrents, $\mathbf{h} = \mathbf{H} + \mathbf{h}_s$. This is necessary not only for a correct determination of the energy density F obtained by using the order parameter equation (the angular brackets denote averaging over the inhomogeneities of the superconducting state, while η_i is expressed in terms of η_+ and η_-):

$$F = -H^2 + \langle -{}^1/_2({}^1/_4(|\eta_+|^2 + |\eta_-|^2) + \beta_2|\eta_+|^2|\eta_-|^2) + h_s^2\rangle, \quad (16)$$

but also for determining the absolute value of the order parameter. The simplest way to find it is to require that the energy functional be stationary with respect to the transformation $|\eta| \to (1 + \varepsilon)|\eta|$, where ε is a small number.[17] This leads to the following condition:

$$\langle {}^1/_4(|\eta_+|^2 + |\eta_-|^2)^2 + \beta_2|\eta_+|^2|\eta_-|^2 - 2h_s^2\rangle = (H_{c_2} - H)\langle -h_s\rangle. \quad (17)$$

The difficulty in finding h_s in this problem is due to the fact that the coordinate dependence of the order parameter involves more than the wave function of the zero-order Landau level. By using the equations for the components of the order parameter which follows from (6),

$$p_-\eta_- = 0, \quad {}^1/_2\omega p_-\eta_+ - p_+\eta_- = 0, \quad p_\pm = p_x \pm ip_y,$$

substituting them into the expression for the supercurrent, and integrating the equation $\operatorname{rot}\mathbf{h}_s = \mathbf{j}_s$, after a series of calculations we obtain

$$h_s = -\frac{1+C+D}{4\varkappa}|\eta_-|^2 - \frac{1+C-D}{4\varkappa}|\eta_+|^2 - \frac{C}{\varkappa\omega}|\eta_-|^2 + h_1,$$

$$\partial_y h_1 = -\frac{1+C+{}^1/_2C\omega}{\omega\varkappa}(\eta_+^*\partial_y\eta_- + \eta_+\partial_y\eta_-^*).$$

Here h_1 is the solution to the second equation and cannot be expressed in explicit form in terms of η_+ and η_-. Introducing the quantity δ according to (17) such that

$$\delta = \frac{\langle {}^1/_4(|\eta_+|^2 + |\eta_-|^2)^2 + \beta_2|\eta_+|^2|\eta_-|^2 - 2h_s^2\rangle}{\langle h_s\rangle^2 H_{c_2}^2}, \quad (18)$$

allows us to write (16) in the form

$$F = -\frac{2}{\delta}\left(\frac{H_{c_2} - H}{H_{c_2}}\right)^2 - H^2$$

and thereby reduce the problem to the minimization of a function of δ, which is a generalization of the usual relation to be minimized, i.e., $\langle|\psi|^4\rangle/\langle|\psi|^2\rangle|^2$. The choice of gauge

$\mathbf{A} = (-Hy,0,0)$ allows us to write in explicit form a solution which has the form of a lattice of superconducting seeds with periods a, b, and an angle α between them:

$$\eta_+ \propto \sum_n \exp\left[\pi i \frac{b}{a}\cos\alpha n(n-1)\right] \times\exp\left(\frac{2\pi}{a} inx\right)[2\varkappa H(y-nb\sin^2\alpha)-1] \times\exp\left[-\frac{\varkappa H}{2}(y-nb\sin\alpha)^2\right],$$

$$\eta_- \propto \omega\sum_n \exp\left[\pi i \frac{b}{a}\cos\alpha n(n-1)\right] \times\exp\left(\frac{2\pi}{a} inx\right)\exp\left[-\frac{\varkappa H}{2}(y-nb\sin\alpha)^2\right].$$

The condition for periodicity of the absolute value of the order parameter induces the following relation between a, b and α:

$$ab\sin\alpha = 2\pi/\varkappa H.$$

Each elementary unit cell of such a lattice of seeds contains one zero of the wave function (η_+ and η_- reduce to zero simultaneously) for $x = \frac{1}{2}a$ and $y = \frac{1}{2}b\sin\alpha$. Furthermore, if we take into account that in these dimensionless units we have $\phi_0 = 2\pi/\varkappa$, then the natural requirement that the lattice of seeds be simple leads to Abrikosov vortices, each carrying one flux quantum. Our overall program will therefore consist of the following: we substitute the explicit form of the solution we have found into (18) and find δ as a function of ρ and σ, which are related to the lattice parameters by the following expression:

$$\rho + i\sigma = \frac{b}{a}e^{i\alpha}.$$

This problem is made easier by the fact that $\varkappa$ in the compounds of interest is large (for UPt_3, $\varkappa \sim 20$), and so to accuracy $\sim 1/\varkappa^2$ we can neglect the field energy due to supercurrents, i.e., the term h_s^2 in the numerator of (18). As a result we obtain

$$\delta(\rho,\sigma) = \sum_{n,m}\sigma^{1/2}\exp[2\pi i\rho(n^2-m^2)]\exp[-2\pi\sigma(n^2+m^2)]\Big/ \times\left[1+\frac{10(1+C)+4C\omega-2D}{\omega^2(1+C+D)+4C\omega}\right]^2\left\{1+\frac{2(1+2\beta_2)}{\omega^2} \times\left[\frac{3}{4}-2\pi\sigma(n^2+m^2)+8\pi\sigma nm+16\pi^2\sigma^2m^2\right] +\frac{1}{\omega^4}\left[\frac{41}{16}-13\pi\sigma(n^2+m^2)+12\pi^2\sigma^2(n^4+m^4) +176\pi^2\sigma^2n^2m^2-192\pi^3\sigma^3n^2m^2(n^2+m^2)+256\pi^4\sigma^4n^4m^4\right]\right\} = A_0\delta_0(\rho,\sigma)+A_1\delta_1(\rho,\sigma)+A_2\delta_2(\rho,\sigma). \quad (19)$$

Here the summation is either over integer m and n or over half-integers;

$$\delta_0(\rho,\sigma) = \sigma^{1/2}\sum_{n,m}\exp[2\pi i\rho(n^2-m^2)]\exp[-2\pi\sigma(n^2+m^2)]$$

is a standard function of the lattice parameters for ordinary superconductors.[18] Its minimum is reached for $\rho = 1/2$, $\sigma = 3^{1/2}/2$, which corresponds to a hexagonal lattice.

In order to find the form of the lattice it is not necessary to minimize $\delta(\rho,\sigma)$ in the entire complex plane $\rho + i\sigma$, as symmetry considerations isolate the minimum in a certain region. First of all, both the vector pairs **b**, **a** and **b** + **a**, **a** determine the same lattice, which corresponds to the condition $\delta(\rho+1,\sigma) = \delta(\rho,\sigma)$. Secondly, in order to determine the form of the lattice we can choose the vectors **b** and $-$**a** such that $\delta(\rho,\sigma) = \delta(-\rho,\sigma)$.[18] These properties of the function $\delta(\rho,\sigma)$ allow us to seek its minimum only in the strip $0 \leqslant \rho \leqslant 1/2$. In addition, $\delta(1/2+\rho,\sigma) = \delta(1/2-\rho,\sigma)$, so that $\delta'_\rho(1/2,\sigma) = \delta'_\rho(0,\sigma) = 0$. Thirdly, the replacement $a \leftrightarrow b$ leads to the transformation $\rho \to \rho/(\rho^2+\sigma^2)$, $\sigma \to \sigma s/(\rho^2+\sigma^2)$ and describes a symmetry of $\delta(\rho,\sigma)$ relative to motions on the unit circle. This gives us the right to choose from the strip $0 \leqslant \rho \leqslant 1/2$ that part which lies in the unit circle. These properties do not depend on the specific form of $\delta(\rho,\sigma)$, and hold for the entire region $H_{c1} \leqslant H \leqslant H_{c2}$. For (19) the first two symmetry properties are obvious, while the third is proved by using the Poisson summation formula.

Let us determine σ by assuming that the minimum of $\delta(\rho,\sigma)$ corresponds to $\rho = 1/2$, i.e., to a position of the extreme which is stable by symmetry. The hexagonal lattice is not even an equilibrium configuration, since $\delta'_\sigma(1/2,3^{1/2}/2) = 0$. It is simplest to find the structure of the lattice in the limiting case of large ω ($\omega \sim 2(1+C)/C$ as $C \to 0$), when the solution (6) is almost single-component. Then $A_0 \sim 1$, $A_1 \sim 1/\omega^2$, $A_2 \sim 1/\omega^4$, and the original minimum $\delta_0(\rho,\sigma)$ is slightly shifted under the action of the perturbation $\delta_1(\rho,\sigma)$. To first order in $1/\omega^2$,

$$\sigma - \frac{3^{1/2}}{2} \sim \frac{30(1+2\beta_2)}{\omega^2}.$$

For arbitrary values of ω a deviation remains, as we have already noted; however, it is not possible to find it in explicit form. Nevertheless, because $\rho = 1/2$, the lattice of phase 6 will in any case consist of rhomboids, and will have, as we should also expect, the symmetry D_2. The magnetization curve $B(H)$ of the superconductor near H_{c2} has the form

$$B = \langle h\rangle = H + \langle h\rangle = H + \frac{2}{H_{c2}\delta}\frac{H-H_{c2}}{H_{c2}}.$$

The boundary conditions at the surface of a metal-insulator interface, which we neglected in (15) in our discussion of seeds in the bulk of the sample, must be taken into account in finding H_{c3}. The difference between this case and the usual (although $j_{sn} = 0$ as before) must lead to a change in the relation between H_{c3} and H_{c2}.

5. CONCLUSION

We see from the results of this paper that superconductivity will always be accompanied by unusual behavior of the anisotropy of the upper critical field when the order parameter generates other than one-dimensional group representations; this is true for crystals of any symmetry. Therefore the identification of such an anisotropy for the heavy-fermion superconductor UPt_3 (which is one of the most probable candidates for the realization of nontrivial superconductivi-

ty) in a plane passing through the hexagonal axis' could be a serious argument in favor of its superconductivity being nontrivial. Furthermore, for UPt_3 (see Ref. 6) there should exist an anisotropy in the neighborhood of the kink in the curve $H_{c2}(T)$ in the form of two overlapping ellipses; this is connected with the presence in a magnetic field of two unmixed phases near T_c, independent of their exact nature. Before a specific interpretation of superconductivity in $Cu_{1.8}Mo_6S_8$ can be given it is necessary to determine the low-temperature dependence of its heat capacity and other characteristics.

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APPENDIX

Here we present a proof of the fact that the solution corresponding to the minimum eigenvalue of the GL equations for a multicomponent order parameter is associated with $p_z = 0$. (In this Appendix we will assume $\hat{z}||\mathbf{H}$). The vector potential can be chosen in a gauge in which $\mathbf{A}\perp\mathbf{H}$ such that it depends only on x and y. Then the operator $\hat{p}_z$ commutes with the effective Hamiltonian from (2), and can be treated as a simple number. Let us discuss Eq. (22) with $p_z \neq 0$, and write it in the form

$$\lambda(p_z, H)\eta(p_z, H) = \hat{H}_{\text{eff}}\eta(p_z, H). \tag{A1}$$

As a result of applying the operator of spatial inversion, these equations are transformed to

$$\lambda(-p_z, H)\eta(-p_z, H) = \hat{H}_{\text{eff}}\eta(-p_z, H). \tag{A2}$$

The effective Hamiltonian is invariant with respect to the symmetry operations. Since the direction and magnitude of the magnetic field do not change, we have $\lambda(p_z,H) = \lambda(-p_z,H)$. From this it is clear that for $p_z \neq 0$, $\lambda(p_z,H)$ corresponds to two solutions $\eta(p_z,H)$ and $\eta(-p_z,H)$. They are independent since they involve different functions of the coordinate z. Consequently, if $p_z \neq 0$ for the minimum eigenvalue, then the multiplicity of the latter equals two.

Let us turn again to $\hat{H}_{\text{eff}}$. We will discuss it as a differential operator in all three spatial coordinates. Because of the positive definiteness of the gradient terms, this operator belongs to the class of elliptic operators. For elliptic operators with discrete spectra there exists a theorem which states that as we vary such an operator continuously the dimensionality of its kernel does not change. In fact,

$$\ker \hat{H} = \lim_{t\to\infty} \exp(-\hat{H}t). \tag{A3}$$

For a continuous variation of $\hat{H}$ the left and right hand sides must vary continuously; however, the left side is an integer. Consequently, it cannot change in general. As we are discussing superconducting samples with finite dimensions, we arrive at discrete spectra along z, so that this theorem applies to the operator $\hat{H}_{\text{eff}} - \lambda_{\min}$. This implies that the dimensionality of $\lambda_{\min}$ for any direction of the field $\hat{H}$ is the same as for a field oriented along the hexagonal axis. But, for this direction, according to (3), $p_z \neq 0$ explicitly increases λ, i.e., to this direction, as with all the others, there corresponds a dimensionality of $\lambda_{\min}$ equal to 1. Therefore we must choose $p_z = 0$. For the three-dimensional vector representations of the cubic group, the corresponding GL equations are such that the discussion presented above does not apply. However, in this case, once we have chosen initially different critical temperatures for the two-dimensional and one-dimensional representations, we can then allow them to approach one another continuously. Then the dimensionality of $\lambda_{\min}$ also will change continuously, and for a purely two-dimensional or one-dimensional representation it equals unity.

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