Contribution to the theory of critical fluctuations

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(Submitted 28 July 1989)
Zh. Eksp. Teor. Fiz. **97**, 892–900 (March 1990)

We propose a regularization of Landau's Hamiltonian, which enables us to transform the functional integral describing critical fluctuations into a path integral. The integral obtained resembles, but is not identical with, the integral occurring in (Feynman's) functional formulation of quantum mechanics. Its evaluation can be reduced to the solution of an infinite chain of coupled integral equations. The analysis of these equations enables us to find the critical exponents ν and η.

The Kadanov-Wilson-Fisher theory[1] is, evidently, the most consistent of the quantitative theory of critical phenomena which exist at the present time. However, the inadequate mathematical foundation of the initial premises of the renormalization group method[2] and the necessity to use a divergent asymptotic series in a parameter which, in fact, is not small, makes it impossible to consider the problem of critical fluctuations to be completely solved. We described in the present paper another approach to the evaluation of the starting functional integral.

We are dealing with the integral[1,3]

$$C(x_1-x_2)=\int \varphi(x_1)\varphi(x_2)\exp[-E(\varphi)]\mathcal{D}\varphi.$$

Here C is the correlation function of the scalar order parameter φ near the second-order phase-transition point; the function $E(\varphi)$ is the ratio of the increase in the thermodynamic potential induced by the fluctuations in $\varphi(x)$, to the temperature.

In connection with the unbounded increase of the correlation radius of the fluctuations in the order parameter near the transition point when we approach it, we can consider the fluctuation φ as a continuous field. One usually assumes that

$$E(\varphi)=\int H(\varphi)\,d^3x,$$

where $H(\varphi)$ is the Landau Hamiltonian:

$$H(\varphi)=\frac{1}{2m}(\nabla\varphi)^2+\frac{b}{2}\varphi^2+\frac{g}{2}\varphi^4, \qquad (1)$$

m, b, g being constants depending on the external conditions.

However, Eq. (1) which is local (depending on the value of the function and its derivatives only in the given point) can, in general, not be applied to short-wavelength fluctuations (starting with wavelengths of the order of molecular dimensions, if we are dealing with normal condensed media). According to the Appendix, when we have the regularization

$$E(\varphi)=\int\left[\frac{(\nabla\varphi)^2}{2m}+\frac{b}{2}\varphi^2\right]\times d^3x+\frac{1}{2}\int\int d^3x\,d^3y\,U(x-y)\varphi^2(x)\varphi^2(y)$$

we can write down the correlation function by means of a path integral rather than by means of three-dimensional scalar functions (as is the original expression):

$$C(x)=\frac{1}{2}\int_0^\infty dt\int\mathcal{D}q\,\delta[x-q(t)]\exp\left\{-\frac{m}{2}\int_0^t\left(\frac{dq}{d\tau}\right)^2 d\tau-\frac{1}{2}\int_0^t\int U[q(t_1)-q(t_2)]\,dt_1\,dt_2-\frac{bt}{2}\right\}. \qquad (2)$$

Here $q(\tau)$ is a curve (path) in three-dimensional space with the initial condition

$$q(0)=0.$$

The function $U(x)$ is assumed to be positive, rapidly decreasing as $x\to\infty$, and spherically symmetric.

If we take $U(x)=g\delta(x)$ we get the original Landau Hamiltonian, but then the integral

$$\int_0^t\int U[q(t_1)-q(t_2)]\,dt_1\,dt_2$$

becomes infinite. Hence it follows that the original local Hamiltonian is fundamentally inadmissible and its short-wavelength regularization method for the Landau Hamiltonian,[1,2] but the experimentally known universality of the main features of the critical phenomena[4] enables us to assume that the nature of the above mentioned regularization is unimportant (within well-defined limits) for the determination of the macroscopic characteristics of the medium.

The integral (2) resembles the integral arising in the functional formulation of quantum mechanics[6] (with imaginary time). However, the "action"

$$S(q,t)=\frac{m}{2}\int_0^t\left(\frac{dq}{d\tau}\right)^2 d\tau+\frac{1}{2}\int_0^t\int U[q(t_1)-q(t_2)]\,dt_1\,dt_2$$

is not instantaneous (it contains a double integral over the time). This fact prevents us from reducing the evaluation of the correlation function to a solution of a differential equation similar to the Schrödinger equation. One can, in principle, use near the critical point the functional analog of the Laplacian method (saddle-point method), but in the present paper we shall use another method.

We consider the function

$$P_n(x_1t_1|\ldots|x_nt_n)=\int\mathcal{D}q\exp[-S(q,t_n)]\prod_{j=1}^n\delta[x_j-q(t_j)]. \qquad (3)$$

 0038-5646/90/030500-05$03.00

We assume here that $0 \leqslant t_1 \leqslant \ldots \leqslant t_n$, and as before $q(0) = 0$. From the definition follows immediately the "argument canceling" rule:

$$P_n(x_1t_1|\ldots|x_nt_n)\,|_{t_j=t_{j+1}}$$
$$=\delta(x_j-x_{j+1})P_{n-1}(x_1t_1|\ldots|x_{j-1}t_{j-1}|x_{j+1}t_{j+1}|\ldots|x_nt_n). \quad (4)$$

In particular,

$$P_1(x, 0)=\delta(x). \quad (5)$$

We can obtain the following infinite set of coupled equations for the functions P_n:

$$\frac{\partial P_n(x_1t_1|\ldots|x_nt_n)}{\partial t_n}=\frac{1}{2m}\frac{\partial^2 P_n(x_1t_1|\ldots|x_nt_n)}{\partial x_n^2}$$
$$-\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} d\tau \int d^3y\, U(x_n-y)$$
$$\times P_{n+1}(x_1t_1|\ldots|x_{j-1}t_{j-1}|y\tau|x_jt_j|\ldots|x_nt_n) \quad (6)$$

(we assume t_0 to be equal to zero). The derivation of this set of equations is completely analogous to the derivation of the Schrödinger equation in the functional formulation of quantum mechanics[6] and therefore we do not give it. Equation (4) can be considered to be the initial condition for Eq. (6) [for $n = 1$ we must use (5)].

We transform the set of Eqs. (6) as follows. First of all we change variables,

$$P_n(x_1t_1|\ldots|x_nt_n)=f_n(\rho_1\tau_1|\ldots|\rho_n\tau_n),$$

where

$$\rho_j=x_j-x_{j-1}, \quad \tau_j=t_j-t_{j-1},$$
$$(x_0=\rho_0=0, \quad \tau_0=t_0=0),$$

and then we perform Fourier and Laplace transformations:

$$f_n(k_1p_1|\ldots|k_np_n)$$
$$=\int d^3\rho_1\ldots d^3\rho_n \int_0^\infty \ldots \int d\tau_1\ldots d\tau_n f_n(\rho_1\tau_1|\ldots|\rho_n\tau_n)$$
$$\times\exp\left[\sum_{j=1}^{n}(ik_j\rho_j-p_j\tau_j)\right].$$

We denote the results of the transformation with the same symbols as the original ones and distinguish them by their arguments. It is clear that four sets are possible: $\{\rho_j\}$ or $\{k_j\}$ and $\{\tau_j\}$ or $\{p_j\}$. In particular, it follows from (3) that

$$f_n(k_1\tau_1|\ldots|k_n\tau_n)$$
$$=\int \mathcal{D}q \exp\left\{-S(q, \tau_1+\ldots+\tau_n)+i\left[(k_1-k_2)q(\tau_1)\right.\right.$$
$$+(k_2-k_3)q(\tau_1+\tau_2)+\ldots+(k_{n-1}-k_n)$$
$$\left.\left.\times q\left(\sum_{j=1}^{n-1}\tau_j\right)+k_nq\left(\sum_{j=1}^{n}\tau_j\right)\right]\right\}. \quad (7)$$

We can write Eq. (7) in the form

$$f_n(k_1\tau_1|\ldots|k_n\tau_n)\,|_{\tau_j=0}$$
$$=f_{n-1}(k_1\tau_1|\ldots|k_{j-1}\tau_{j-1}|k_{j+1}\tau_{j+1}|\ldots|k_n\tau_n).$$

We note that the Fourier transform of the correlation function $C(k)$ can, according to (2), be written as

$$C(k)=\tfrac{1}{2}f_1(kp)\,|_{p=b/2} \quad (8)$$

(the Fourier transform is here also denoted by the same symbol as the original).

One checks easily that the set of Eqs. (6) in the new variables takes the form

$$\left(p_n+\frac{k_n^2}{2m}\right)f_n(k_1p_1|\ldots|k_np_n)=f_{n-1}(k_1p_1|\ldots|k_{n-1}p_{n-1})$$
$$-\sum_{j=1}^{n}\int d^3\varkappa\, \overline{U}(\varkappa)f_{n+1}(k_1p_1|\ldots|k_jp_j$$
$$\times|k_j-\varkappa, p_j|k_{j+1}-\varkappa, p_{j+1}|\ldots|k_n-\varkappa, p_n), \quad (9)$$

where

$$\overline{U}(\varkappa)=\frac{1}{(2\pi)^3}\int U(x)e^{i\varkappa x}d^3x,$$

and for $n = 1$ we must take $f_{n-1} = 1$.

In what follows we shall mainly consider the case

$$p_1=p_2=\ldots=p_n, \quad (10)$$

in which we shall simply write p for the common value of the p_j and we shall for the sake of simplicity drop this variable from the list of arguments of f_n. For instance, we shall write simply $f_2(k_1|k_2)$ for $f_2(k_1p|k_2p)$.

The solution of the set of Eqs. (9) is rather complicated. However, this is not necessary to determine the critical exponents. It will soon become clear that the critical exponents depend only on the nature of the singularity of the functions f_n for small $|k_j|$ near the critical point; but we do not actually need the detailed value of the solution. The remainder of this paper will be devoted to this simplified problem.

First of all we use the following approximation. We drop in the sum over j on the right-hand side of (9) all terms for which $j \leqslant n - n_0$, where n_0 is fixed. Of course, when $n \leqslant n_0$ the equations remain exact. One checks easily that when $n_0 = 1$ (one term) f_n reduces to the product of identical factors, each of which depends only on k_j and p. This approximation is unsatisfactory and will not be considered.

We turn at once to the case $n_0 = 2$. One checks easily that the general solution of the chain of equations in that case will be

$$f_n(k_1p_1|\ldots|k_np_n)=f_1(k_1p_1)\prod_{j=1}^{n-1}f(k_jp_j|k_{j+1}p_{j+1}),$$

where

$$[f_1(kp)]^{-1}=p+\frac{k^2}{2m}+\int d^3\varkappa\, \overline{U}(\varkappa-k)f(k|\varkappa), \quad (11)$$

while the function f satisfies the closed nonlinear integral equation

$$\left[p_2+\frac{k_2^2}{2m}+\int d^3\varkappa \overline{U}(\varkappa) f(k_2p_2|k_2-\varkappa, p_2)\right] f(k_1p_1|k_2p_2)$$
$$=1-\int d^3\varkappa\, \overline{U}(\varkappa) f(k_1p_1|k_1-\varkappa, p_1) f(k_1-\varkappa, p_1|k_2-\varkappa, p_2). \tag{12}$$

We take condition (10) and assume that for small $|k|$ the right-hand side of Eq. (11), for which we use the notation $Z(k)$ in what follows, can approximately be written as:

$$Z(k)=[f_1(kp)]^{-1}=\varepsilon+Ak^2, \tag{13}$$

where $\varepsilon=0$ in the transition point. It is implied here that $\varepsilon=\varepsilon(p)$, $A=A(p)$, but it will be more convenient to invert the first of these equations and assume the variable ε to be the independent variable: $p=p(\varepsilon)$, $A=A(\varepsilon)$. Moreover, it is convenient to use such units that, first, the length will be dimensionless (ε, k, and A will then be simultaneously dimensionless) and, secondly, the quantity A will be of the order of unity near the transition point. For the usual condensed media we are essentially dealing with a choice of the unit of length of the order of molecular sizes. In such units Eq. (13) must hold for $k^2\ll 1$.

We split the integral on the right-hand side of (11) into two terms, corresponding to the integration domains $|\varkappa|<Q$ and $|\varkappa|>Q$, where

$$\varepsilon^{1/2}\ll Q\ll 1. \tag{14}$$

In the first of the integrals thus obtained we change the integration variable to $\xi=\varkappa/\varepsilon^{1/2}$ and make the substitution

$$f(k_1|k_2)=(\varepsilon+Ak_2^2)^{-3/4} w(k_1|k_2/\varepsilon^{1/2}).$$

As a result we get

$$\varepsilon^{1/4}(1+A\xi_2^2)^{1/4} w(k_1|\xi_2)$$
$$=1-\int\limits_{|\varkappa|>Q} d^3\varkappa \bar{U}(\varkappa-k_1) f(k_1|\varkappa) f(\varkappa|\varkappa+\Delta k)$$
$$-\int\limits_{|\xi|<Q/\varepsilon^{1/2}} d^3\xi \bar{U}(\xi\varepsilon^{1/2}-k_1)\frac{w(k_1|\xi) w(\xi\varepsilon^{1/2}|\xi+\Delta\xi)}{(1+A\xi^2)^{3/4}[1+A(\xi+\Delta\xi)^2]^{3/4}}. \tag{15}$$

Here $\xi_2=k_2/\varepsilon^{1/2}$, $\Delta k=k_2-k_1$, $\Delta\xi=\Delta k/\varepsilon^{1/2}$. Thus there appears a second scale for measuring the lengths of the vectors k_j with the unit $\varepsilon^{1/2}$ (cf. the multiple scale method[6]).

Because of condition (14), the first integral on the right-hand side of (15) is practically independent of ε (as $\varepsilon\to 0$) and is thus not a function of ξ_2. For small k_1, k_2 it can be approximately taken to be constant. We can with the same accuracy replace $\bar{U}(\xi\varepsilon^{1/2}-k_1)$ by $\bar{U}(0)$. We can neglect the left-hand side of the equation for bounded k_1, ξ_2. We then get approximately

$$\int\limits_{|\xi|<Q/\varepsilon^{1/2}} d^3\xi \frac{w(k_1|\xi) w(\xi\varepsilon^{1/2}|\xi+\Delta\xi)}{(1+A\xi^2)^{3/4}[1+A(\xi+\Delta\xi)^2]^{3/4}}=\text{const}.$$

Hence it follows that w is bound as k_1, $k_2\to 0$ and when $|k_1|\lesssim\varepsilon^{1/2}$ is practically independent of k_1:

$$w(k_1|\xi_2)\approx\chi(\xi_2). \tag{16}$$

We now compare Eqs. (11) and (13). We again split the integral on the right-hand side of (11) into two terms corresponding this time to integration domains $|\varkappa|<N\varepsilon^{1/2}$ and $|\varkappa|>N\varepsilon^{1/2}$, respectively. We shall discuss the choice of the quantity N below.

The first of the two integrals considered can be written in the form

$$\varepsilon^{3/4}\int\limits_{|\xi|<N} d^3\xi \bar{U}(\xi\varepsilon^{1/2}-k)\frac{\chi(\xi)}{(1+A\xi^2)^{3/4}}.$$

In the second, for sufficiently large N, we can expand the integrand, considered as a function of $\varkappa$, k, and ε, in a series in ε. We thus get[1)] for $Z(k)$

$$Z(k)=p+\frac{k^2}{2m}+C_0\varepsilon^{3/4}+F(\varepsilon, k),$$

where C_0 is constant and F a regular function.

Putting

$$F(\varepsilon, k)=p_0+C_1\varepsilon+A'k^2+\ldots$$

(p_0, C_1, A' are constants), we get as $\varepsilon\to 0$

$$\varepsilon\approx\left(\frac{p'}{-C_0}\right)^{4/3}. \tag{17}$$

Here $p'=p+p_0$. The quantity p_0 determines the transition point "renormalized" due to fluctuations.

We shall assume that in the phase transition point $b=0$ and near it

$$b=\text{const}\cdot\Delta T,$$

where ΔT is the temperature measured from the transition point. It then follows from (17) that the correlation radius near the transition point increases proportional to $\Delta T^{-2/3}$ [see (8)]. Thus the critical exponent $\nu=\frac{2}{3}$. For the second critical exponent which characterizes the way the correlation function decreases at distances considerably less than the correlation radius we get, clearly, $\eta=0$.

The conclusions reached here are based upon a simplified variant of the set of Eqs. (9). However, at the basis of the explanation of the structure of the solution there were lying essentially only considerations about the dimensionality of the space which remain valid also when we go beyond the above mentioned simplification. We consider this problem in more detail.

We assume that near the transition point for small $|k|$ Eqs. (13) and (17) are correct as before. Slightly changing the notation, we write

$$f_1(k)=[B(p')^{4/3}+Ak^2]^{-1}. \tag{18}$$

As before, $p=-p_0$ corresponds to the transition point [see (8)] and $p'=p+p_0$. As usual we shall not explicitly indicate the p dependences of the quantities considered. Generally speaking, $A=A(p)$, $B=B(p)$, but as $p'\to 0$ we can consider them to be approximately constant.

When $k=0$ we have

$$f_1(0)=1/B(p')^{4/3}$$

and then, as $t\to\infty$

$$f_1(kt)|_{k=0}\approx\frac{1}{B\Gamma(4/3)} t^{1/3} e^{-p_0 t}.$$

We now conclude easily from Eq. (7) that as $p'\to 0$

$$f_n(k_1|\dots|k_n)|_{k_1=\dots=k_n=0}=\frac{1}{B\Gamma(4/3)}\int_0^\infty dt_1\dots\int_0^\infty dt_n\, s_n^{1/3}e^{-p's_n},$$

where

$$s_n=t_1+\dots+t_n.$$

We replace the integration variables $t_1,\dots,t_{n-1},t_n$ in the last integral by $t_1,\dots,t_{n-1},s_n$ and after that integrate by parts over t_j $(1\leqslant j\leqslant n-1)$. As a result we get

$$f_n(0|\dots|0)=\frac{1}{B\Gamma(4/3)}\int_0^\infty dt_1\dots\int_0^\infty dt_{n-1}t_j s_{n-1}^{1/3}e^{-p's_{n-1}}. \tag{19}$$

Adding Eq. (19) for different j we find

$$f_n(0|\dots|0)=\frac{1}{B\Gamma(4/3)\,(n-1)}\int_0^\infty dt_1\dots\int_0^\infty dt_{n-1}s_{n-1}^{4/3}e^{-p's_{n-1}}.$$

The last equation is similar to the original one, but with one less integration variable which enables us to apply again the transformation used. As a result we find easily:

$$f_n(0|\dots|0)=\frac{1}{B\Gamma(4/3)\,(n-1)!}\int_0^\infty t^{n-2/3}e^{-p't}\,dt$$

$$=\frac{\Gamma(n+1/3)}{B\Gamma(4/3)\Gamma(n)\,(p')^{n+1/3}}. \tag{20}$$

We consider the function

$$\varphi_n(k_1|\Delta k_2|\dots|\Delta k_{n-1}|k_n)$$

$$=f_n(k_1|\dots|k_n)Z(k_1)\left[Z(k_n)\prod_{j=2}^{n-1}Z(\Delta k_j)\right]^{1/4}, \tag{21}$$

where now

$$Z(k)=B(p')^{4/3}+Ak^2$$

and $\Delta k_j=k_{j+1}-k_j$. According to (20) these functions must be bounded as $p'\to 0$ and $k_j\to 0$.

Using (21) as a change of variables in Eq. (9), we get

$$\frac{\zeta_n(k_1|\Delta k_2|\dots|\Delta k_{n-1}|k_n)}{Z^{3/4}(k_n)}\varphi_n(k_1|\Delta k_2|\dots|\Delta k_{n-1}|k_n)$$

$$=\varphi_{n-1}(k_1|\Delta k_2|\dots|\Delta k_{n-2}|k_{n-1})\left[\frac{Z(\Delta k_{n-1})}{Z(k_{n-1})}\right]^{3/4}$$

$$-\sum_{j=1}^{n-1}\int d^3\varkappa\,\overline{U}(\varkappa)$$

$$\times\frac{\varphi_{n+1}(k_1|\Delta k_2|\dots|\Delta k_{j-1}|\varkappa|\Delta k_j|\dots|\Delta k_{n-1}|k_n+\varkappa)}{Z^{3/4}(\varkappa)Z^{3/4}(k_n+\varkappa)}. \tag{22}$$

We have taken here

$$\zeta_n(k_1|\Delta k_2|\dots|\Delta k_{n-1}|k_n)$$

$$=p+\frac{k_n^2}{2m}+\int d^3\varkappa\,\overline{U}(\varkappa)\frac{f_{n+1}(k_1|\dots|k_n|k_n+\varkappa)}{f_n(k_1|\dots|k_n)}. \tag{23}$$

We consider $\zeta_n|_{k_1=\dots=k_n=0}$. When $n=1$ it follows from (23), (9), and (18) that

$$\zeta_1(0)\approx B(p')^{4/3},\quad p'\to 0.$$

When $n>1$, $\zeta_n(0|\dots|0)$ is, by definition, the ratio of the quantities

$$pf_n(0|\dots|0)+\int d^3\varkappa\,\overline{U}(\varkappa)f_{n+1}(0|\dots|0|\varkappa) \tag{24}$$

and $f_n(0|\dots|0)$.

One can estimate (24) essentially in the same way as $\zeta_n(0|\dots|0)$, using the equation

$$f_{n+1}(k_1t_1|\dots|k_nt_n|\varkappa t_{n+1})|_{k_1=\dots=k_n=0}$$

$$=\int\mathcal{D}q\exp\{-S(q,s_n)+i\varkappa[q(s_n)-q(s_{n+1})]\}.$$

which follows from (7). We must then bear in mind that (24) becomes identically equal to 1 when $n=1$ [see (9)]. Without going into details we give the result:

$$\zeta_n(0|\dots|0)\approx\frac{3n-3}{3n-2}p',\quad p'\to 0,\quad n>1.$$

We can now easily check, using practically the same considerations which were used above when we considered the simplified variant of the set of Eqs. (9), that Eq. (22) does not have a singularity as $\varepsilon\to 0$, $k_j\to 0$ (this time $\varepsilon=B(p')^{4/3}$). The original assumption (18) is thus in accord with the conclusion that φ_n is bounded as $p'\to 0$, $k_j\to 0$.

Moreover, we can conclude from the form of the set of Eqs. (22) that for small k_j we cannot subject the variable k_1 to the scale change $k_1\to k_1/\varepsilon^{1/2}$ [cf. (16)], i.e., in the case considered φ_n depends, in fact, on k_1 and $k_j/\varepsilon^{1/2}$ $(j>1)$. To check this we must in Eq. (23) use (21) to change from f_n,f_{n+1} to φ_n,φ_{n+1}. In complete analogy with what was done earlier we now conclude that (18) follows from Eq. (9) with $n=1$ which finally confirms the "self-consistency" of this original assumption. At the same time it becomes clear that the above found values of the critical indices do not change when go over from the simplified (truncated) set of equations to the complete one.

In conclusion the author thanks A. F. Andreev, I. A. Batalin, V. P. Popov, and A. A. Sobyanin for discussing a number of problems in connection with this paper.

APPENDIX

We consider the following functional integral:

$$C(x_1,x_2)=\int\mathcal{D}\varphi\varphi(x_1)\varphi(x_2)\exp\left[-\int d^3x\frac{(\nabla\varphi)^2}{2m}\right]W(\varphi^2).$$

The scalar function φ in three-dimensional[2)] space serves as the integration variable: W is a functional that depends on φ^2. The homogeneity of W (invariance under space shifts) is not assumed in the general case.

We use a (functional) Fourier transformation.[3)] Let

$$W(\varphi^2)=\int\mathcal{D}f\exp\left[i\int d^3x\,\varphi^2(x)f(x)\right]\tilde{W}(f),$$

where f is an auxiliary functional variable. In that case

$$C(x_1,x_2)=\int\mathcal{D}f\,\mathcal{D}\varphi\,\varphi(x_1)\varphi(x_2)$$

$$\times\exp\left\{\int d^3x\left[-\frac{(\nabla\varphi)^2}{2m}+i\varphi^2f\right]\right\}\tilde{W}(f). \tag{A1}$$

The integral over φ is now Gaussian and therefore (see, e.g., Ref. 2)

$$C(x_1, x_2) = \int \mathcal{D}f G(x_1, x_2|f) W(f), \qquad \text{(A2)}$$

where G is the solution of the equation

$$[-\Delta_x/m - 2if(x)]G(x, y|f) = \delta(x-y),$$

and Δ is the Laplacian operator.

The solution of the last equation can be written down using a Feynman path integral. We consider the quantity

$$J(t, x, y|f) = \int \mathcal{D}q\delta[x-q(t)] \times \exp\left\{\int_0^t \left[-\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + if(q(\tau))\right]d\tau\right\},$$

where $q(\tau)$ is a path satisfying the initial condition $q(0) = y$. The integral J satisfies the equation (see, e.g., Ref. 5)

$$\frac{\partial J}{\partial t} = \left[\frac{\Delta_x}{2m} + if(x)\right]J$$

and the initial condition $J|_{t=0} = \delta(x-y)$. One can now easily check that

$$G(x, y|f) = \frac{1}{2}\int_0^\infty dt \int \mathcal{D}q\delta[x-q(t)] \times \exp\left\{\int_0^t \left[-\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + if(q(\tau))\right]d\tau\right\}. \qquad \text{(A3)}$$

Moreover,

$$\int_0^t d\tau f(q(\tau)) = \int_0^t d\tau \int d^3\xi\, \delta[\xi - q(\tau)]f(\xi).$$

We now conclude from (A2), (A3), and (A4) that [with $q(0) = y$]

$$C(x, y) = \frac{1}{2}\int_0^\infty dt \int \mathcal{D}q\delta[x - q(t)] \times \exp\left[-\frac{m}{2}\int_0^t \left(\frac{dq}{d\tau}\right)^2 d\tau\right] W(f)\Big|_{f(\xi) = \int_0^t \delta[\xi - q(\tau)]\,d\tau}$$

This is the required transformation of the original functional integral. Equation (2) is a particular case of it in which the correlator can be considered to be a function of the difference of its arguments because of the presence of spatial homogeneity.

[1)] The discussion given here is essentially based only upon considerations of dimensionality and is not completely correct: the N-dependence of the second integral was neglected. We can estimate this integral more accurately, but we have omitted the detailed calculations for simplicity.

[2)] In fact, the number of dimensions of space is unimportant for what follows.

[3)] One can obtain the same results by using instead of a Fourier transformation an expansion of W in the functional analog of a Taylor series, as was done, e.g., in Ref. 7.

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Translated by D. ter Haar