Effective refractive index of media experiencing a percolation-type phase transition

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The problem of light wave propagation in a solid in the heterophase state near the percolation threshold is examined. It is demonstrated that in the attainable vicinity of the threshold a description of multiple scattering requires analysis of the "strong coupling" state. The effective Hamiltonian of the problem is formulated and the renormalization group method is employed to calculate the asymptotic form of the average Green's function of a light wave in the medium.

1. INTRODUCTION

The problem of describing phase transitions in defect solids has received increasing attention in recent years. Materials with a so-called smeared phase transition containing large-scale defects represent a broad class of strongly-disordered solids. In spite of extensive research the issue of the nature of internal processes responsible for smeared phase transitions, specifically the existence or absence of a "true" phase transition in such materials, has remained open. A number o experimentally tested properties of materials containing smeared phase transitions can be described qualitatively based on concepts in which the disordered material is considered to be a set of regions experiencing a local phase transition; the characteristic parameters of these regions vary from one region to another. If we assume that the most significant fluctuations are those that occur in the local temperature field of the phase transition $T_c(\mathbf{x})$ whose distribution function $\mathscr{P}(T_c)$ has a single maximum at $T=\overline{T}_c$ and small dispersion $\langle \Delta T_c^2\rangle^{1/2} \ll \overline{T}_c$, while the correlator $\langle \Delta T_c(\mathbf{x})\Delta T_c(\mathbf{x}')\rangle$ has a single characteristic scale $R_0 \gg r_c$ (r_c is the radius of thermal fluctuations), such a model predicts the existence of a near-percolation-type phase transition in the system.

The model will be entirely equivalent to the continual percolation problem if the long-range fields induced by the new phase clusters are negligible. The low-level elastic interaction of the clusters corresponds to the weak striction case characteristic of a number of magnets, while dipole forces are not significant if the screening radius is sufficiently small $r_D < R_0$.

The most characteristic feature of the percolation-type phase transition is the growth in the average cluster dimension R of the new phase up through the dimensions of the specimen at the percolation threshold. In transparent disordered materials experiencing a percolation-type phase transition this will produce a peak in the small-angle light scattering intensity (Ref. 1).

Recent experiments on light scattering in disordered ferroelectric $PbSc_{1/2}Ta_{1/2}O_3$ crystals and in a transparent CTSL ferroelectric ceramic (8.5/65/35 composition) have revealed sharp small-angle light scattering intensity peaks[2,3] which indicate the presence of percolation-type processes.

Analysis suggests that under these experimental conditions light scattering is so strong that it is necessary to account for multiple scattering. The temperature and electrical field range near the percolation threshold then becomes quite accessible; for this range the value of the dimensionless parameter characterizing scattering multiplicity becomes greater than unity. In this strong fluctuation range of the medium both ordinary perturbation theory and the latter approximation are unsuitable.

The present study employs the renormalization-group (RG) method for summation of the diagrammatic series for the effective refractive index near the percolation threshold.

This method was first applied to describing critical light opalescence near the liquid-gas transition point in Ref. 4. However specific analyses have revealed that light scattering by thermal density fluctuations in the attainable temperature range $\tau = (T-T_c)/T_c > 10^{-6}$ near this point is comparatively weak and the contribution of multiple scattering grows under conditions where the strong coupling state is not attained. Light wave damping under actual experimental conditions can be calculated within the framework of infrared (IR) perturbation theory.[5] Unlike this case in solids it is necessary to investigate the strong coupling state in order to analyze experimental data on the correlation properties of the wave field (optical or acoustical) scattered by static heterophase fluctuations near the percolation threshold. Specifically this class of problems includes the problem of determining the effective elastic moduli for solid percolation-type composites (see, for example, Ref. 6).

From the theoretical viewpoint a description of the statistical structure of a scattered ("random") wave field reduces to a calculation of its correlators

$$G_n^{\alpha\beta\ldots\gamma}(\mathbf{x}_1,t_1,\ldots,\mathbf{x}_n,t_n)=\langle u_\alpha(\mathbf{x}_1,t_1)\ldots u_\gamma(\mathbf{x}_n,t_n)\rangle,$$

while a determination of the effective refractive index or the effective elasticity moduli is equivalent to a calculation of the field-averaged heterophase fluctuations of the retarded Green's function $G_2(\mathbf{x}_1,t_1,\mathbf{x}_2,t_2)$. In order to use the apparatus of the field theoretical RG to calculate G_2 it is first necessary to formulate the effective Hamiltonian for the wave propagating in the percolation medium.

2. DERIVATION OF THE EFFECTIVE HAMILTONIAN DESCRIBING THE SCATTERED LIGHT FIELD NEAR THE PERCOLATION THRESHOLD

We describe the propagation of a monochromatic wave of frequency ω in a heterophase material near the percolation threshold by the stochastic Helmholtz equation:

$$\Delta u+\tilde{\varepsilon}(\mathbf{x})k^2u=-f_0, \quad k=\omega/c. \tag{1}$$

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We assume that temporal variations in the states of large-scale clusters are sufficiently slow that the random permittivity field $\tilde{\varepsilon}(\mathbf{x},t)$ can be regarded as independent of time. Since we do not consider the polarization properties of the scattered light in the present study we will limit the analysis of the scalar equation (1).

The statistics of the field $\tilde{\varepsilon}(\mathbf{x})$ in the percolation medium are known from a solution of the continual problem from percolation theory.[7] When we refer to a percolation-type phase transition the random field $\tilde{\varepsilon}(\mathbf{x})$ is determined by the evolution of the inhomogeneous field of the order parameter $M(\mathbf{x})$ from varying external conditions (such as temperature). The local values of the random field depending on the symmetry of the system will be

$$\tilde{\varepsilon}(\mathbf{x})=\varepsilon_0+aM^n(\mathbf{x})$$

with $n=1$ or $n=2$, while the values of $M(\mathbf{x})$ are determined by the local temperature field of the phase transition $T_c(\mathbf{x})$ whose statistical properties are described in the Introduction. We note that since ordinarily phase transitions in solids are first-order transitions, the condition $r_c \ll R_0$ is easily achieved and the configurational fluctuations dominate over thermal fluctuations when averaging over the field $\varepsilon(\mathbf{x})\equiv\tilde{\varepsilon}(\mathbf{x})-\varepsilon_0$. Consequently there is no fundamental difference between the values $n=1$ and $n=2$ in the absence of long-range forces, since in both cases the spatial relation

$$\varepsilon(\mathbf{x})=a\langle M^n\rangle_T\theta(\mathbf{x})$$

is determined by the indicator function of the new phase: $\theta(\mathbf{x})=1$ or $\theta(\mathbf{x})=0$.

The simplest function describing the correlation properties of the scattered field $u(\mathbf{x})$ and which we will investigate here is the pair Green's function

$$G(\omega,\mathbf{x})=\langle L^{-1}\rangle_\varepsilon$$

averaged over fluctuations in $\varepsilon(\mathbf{x})$ with the operator

$$L=-\varepsilon_0k^2-\Delta-\varepsilon(\mathbf{x})k^2.$$

Using the standard representation in terms of Gaussian integrals we obtain the following expression for this function:

$$G(\mathbf{x}-\mathbf{x}')=\left\langle\frac{\int DuDu^+\{u(\mathbf{x})u^+(\mathbf{x}')\exp\left[-\int d\mathbf{x}u^+Lu\right]\}}{\int DuDu^+\exp\left[-\int d\mathbf{x}u^+Lu\right]}\right\rangle_\varepsilon. \tag{2}$$

We will represent the averaging operation $\langle\ldots\rangle_\varepsilon$ in (2) as a functional integral with respect to a certain field $\varphi(\mathbf{x})$ with the weight multiplier $\exp[-H(\varphi)]$. The functional $H(\varphi)$ will then, obviously, play the role of the effective Hamiltonian in the problem, thereby making it possible to develop the diagrammatic technique for the problem and, if the actual values of the physical parameters of the problem require, apply the theoretical-field RG technique. It will be convenient for our further analysis to rewrite (2) using the replica method as

$$G(\mathbf{x}-\mathbf{x}')=c\left\langle\int Du_kDu_k^+\left\{u_1(\mathbf{x})u_1^+(\mathbf{x}')\times\exp\left[-\int d\mathbf{x}\sum_{m=1}^{N}u_m^+Lu_m\right]\right\}\right\rangle_\varepsilon, \tag{3}$$

where $N\to 0$, with the constant c is defined by the expression:

$$c^{-1}=\left\langle\int Du_kDu_k^+\exp\left[-\int d\mathbf{x}\sum_{m=1}^{N}u_m^+Lu_m\right]\right\rangle_\varepsilon \tag{4}$$

The canonical representation of the percolation problem is given by the Ising lattice model with random bonds as $T\to 0$ (Ref. 8). The average values $\langle F\rangle$ in this model are calculated from:

$$\langle F\rangle=\left\langle\!\!\left\langle Z^{-1}\sum_{\{\mu\}}F(\{\mu\})\exp\left[J\sum_{\langle x,y\rangle}(\mu_x\mu_y-1)\sigma_{xy}\right]\right\rangle\!\!\right\rangle_\sigma. \tag{5}$$

Here $J\to+\infty$, $\mu_x=\pm 1$, σ_{xy} is a random function of the bond, with $\sigma_{xy}=0$ independently for each bond with a probability $1-p$; and summation over x, y is carried out over the nearest neighbors of the cubic lattice. Taking (5) into account we write the effective Hamiltonian for the lattice analog of our problem as

$$\begin{aligned}
&\mathcal{H}_{eff}=\mathcal{H}_0+\mathcal{H}_e+\mathcal{H}_{int},\\
&\mathcal{H}_0=-J\sum_{\langle x,y\rangle}(\mu_x\mu_y-1)\sigma_{xy},\\
&\mathcal{H}_{int}=\tilde{w}\sum_x\mu_x|u|^2,\\
&\mathcal{H}_e=\sum_x\sum_{m=1}^{N}u_m^+(x)[\tilde{L}_0u_m](x),\\
&|u|^2=\sum_{m=1}^{N}|u_m(x)|^2,
\end{aligned} \tag{6}$$

where $\tilde{L}_0$ is the lattice analog of the operator $L_0=-\Delta-\varepsilon_0k^2$.

We use the replica method to avoid the need to direct averaging over the field configurations σ_{xy}. The effective Hamiltonian can then be given as

$$\exp[-H_{eff}]=\exp[-H_e-H_{int}]\times\prod_{\langle x,y\rangle}\left\langle\!\!\left\langle\exp\left[J\sum_{\alpha=1}^{n}(\mu_x^\alpha\mu_y^\alpha-1)\sigma_{xy}\right]\right\rangle\!\!\right\rangle. \tag{7}$$

In writing (7) we used the independence of the values of σ_{xy} at different bonds; the angle brackets $\langle\langle\ldots\rangle\rangle$ denote averaging for the given bond. The limit $n\to 0$ is assumed in (7), and the quantities H_e and H_{int} are given by the relations

$$\begin{aligned}
&H_e=\sum_x\sum_{\alpha=1}^{n}\sum_{m=1}^{N}u_{m\alpha}^+(x)[\tilde{L}_0u_{m\alpha}](x),\\
&H_{int}=\tilde{w}\sum_x\sum_{\alpha=1}^{n}\mu_x^\alpha|u_\alpha|^2.
\end{aligned} \tag{8}$$

After averaging over the bonds in (7) we obtain:

$$\exp[-H_{eff}]=\exp[-H_e-H_{int}]\times\prod_{\langle x,y\rangle}\left\{1+\frac{p}{(1-p)}\exp J\sum_{\alpha=1}^{n}(\mu_x^\alpha\mu_y^\alpha-1)\right\}. \tag{9}$$

Passing to the limit $J \to +\infty$ we have:

$$\exp[-H_{eff}] = \exp[-H_e - H_{int}] \times \prod_{\langle x,y \rangle} \left\{ 1 + \frac{p}{(1-p)} \prod_{\alpha=1}^{n} \frac{1}{2} (\mu_x^\alpha \mu_y^\alpha + 1) \right\}, \tag{10}$$

from which we obtain the following expression for H_{eff}:

$$H_{eff} = H_0 + H_e + H_{int},$$
$$H_0 = -\sum_{x,y} I(x,y) \prod_{\alpha=1}^{n} (\mu_x^\alpha \mu_y^\alpha + 1), \tag{11}$$

with $I(x,y) = -2^{-(n+1)} \ln(1-p)$, if the nodes x, y are nearest neighbors, and $I(x,y) = 0$, otherwise.

The part of (11) that is quadratic in μ_x^α contains the sum of the squares of $\mu_x^\alpha \mu_x^\beta \ldots \mu_x^\gamma$, where $1 \leqslant \alpha < \beta < \ldots < \gamma \leqslant n$. We denote these as $\mu_x^{(\alpha)}$, where $(\alpha) = (\alpha,\beta,\ldots,\gamma)$ is the ordered multi-index. Introducing the set of fields $s_x^{(\alpha)}$ conjugate to $\mu_x^{(\alpha)}$ we write the statistical sum as

$$Z = c_1 \int Du_{k\alpha}\, Du_{k\alpha}^+\, Ds^{(\beta)} \sum_{\{\mu\}} \exp\left\{ H_e + H_{int} + \sum_x \sum_{(\alpha)} x_v^{(\alpha)} \mu_x^{(\alpha)} - \frac{1}{4} \sum_{x,y} \sum_{(\alpha)} s_x^{(\alpha)} [I^{-1}](x,y) s_y^{(\alpha)} \right\}. \tag{12}$$

Here I^{-1} is the inverse operator to I. We carry out summation over $\{\mu_x^{(\alpha)}\}$ and retain only the lowest terms in powers of $s^{(\alpha)}$ describing the interaction between the light wave and the percolation field. This routine is justified because the interaction term linear in $s^{(\alpha)}$ makes the most contribution to the infrared singularities of the perturbation theory. Then going over to the continuous fields $s^{(\alpha)}(\mathbf{x})$ and $u_{ma}(\mathbf{x})$ we obtain the effective Hamiltonian as

$$H_{eff} = H_p^* + H_e + H_{int}^*,$$
$$H_p^* = \int d\mathbf{x} \left\{ \frac{1}{2} \sum_{(\alpha)} [\tau_0 (s^{(\alpha)})^2 + \nabla s^{(\alpha)}\, \nabla s^{(\alpha)}] + g \sum_{(\alpha),(\beta),(\gamma)}{}' s^{(\alpha)} s^{(\beta)} s^{(\gamma)} + \ldots \right\}, \tag{13}$$
$$H_e = \int d\mathbf{x} \sum_{m=1}^{\mathcal{N}} u_m^+ L u_m,$$
$$H_{int}^* = w_0 \int d\mathbf{x}\, |u|^2 s^1.$$

Here $\Sigma'_{(\alpha),(\beta),(\gamma)}$ denotes summation over the ordered multi-indices (α), (β), (γ) such that in the set of indices $\{(\alpha), (\beta), (\gamma)\}$ each replica is encountered only twice (see Ref. 8). In the notation for H_{int}^*, H_e we have accounted for the fact that fields u_{ma} with different α do not mix and setting $\alpha = 1$ we have denoted u_{m1} by u_m. The Hamiltonian H_p^* in (13) which depends solely on the field $s^{(\alpha)}$ describes the 2^n-component Plotts model which corresponds to the continual percolation problem as $n \to 0$. It is convenient to go over to another representation of this problem:

$$H_{eff} = H_p + H_{int} + H_e, \tag{14}$$

in which H_p and H_{int} are determined by

$$H_p = \int dx \left\{ \frac{1}{2} [\tau_0 \varphi^2 + \nabla \varphi_i\, \nabla \varphi_i] + g_0 v_{ijk} \varphi_i \varphi_j \varphi_k + (\lambda_1^0 S_{ijkl} + \lambda_2^0 F_{ijkl}) \varphi_i \varphi_j \varphi_k \varphi_l \right\}, \tag{15}$$
$$H_{int} = w_0 \int dx \left(\frac{2}{S} \sum_{\alpha=1}^{s/2} (e_i^\alpha \varphi_i)\, |u|^2 \right),$$

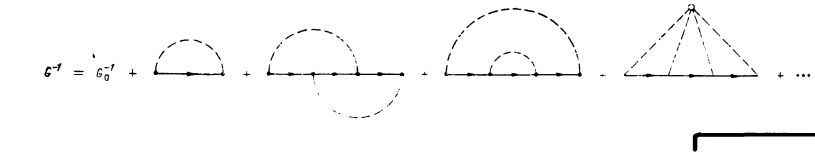
where

$$\varphi^2 = \varphi_i \varphi_i, \quad v_{ijk} = \sum_{\alpha=1}^{5} e_i^\alpha e_j^\alpha e_k^\alpha, \quad F_{ijkl} = \sum_{\alpha=1}^{s} e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha,$$
$$S_{ijkl} = {}^1/_3 (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \sum_{\alpha=1}^{s} e_i^\alpha = 0,$$
$$\sum_{\alpha=1}^{s} e_i^\alpha e_k^\alpha = s\delta_{ik}, \quad e_i^\alpha e_i^\beta = s\delta_{\alpha\beta} - 1;$$

summation over repeating characters i, j, k, l from 1 to $(s-1)$ is assumed. The limit $s \to 1$ in (15) corresponds to our percolation problem.

3. PERTURBATION THEORY AND THE EFFECTIVE EXPANSION PARAMETER

Using the effective Hamiltonian obtained in the preceding section we obtain a graphical representation of the perturbation series for the averaged retarded Green's function $G(\omega,\mathbf{x})$:



(16)

We are interested in the behavior of the series near the percolation threshold where the average cluster dimensions R grow without limit. The following analytic expression for the effective refractive index for $\varepsilon_0 kR \gg 1$ corresponds to the first four terms in (16):

$$n^2 = 1 + (i/8\pi)(\Delta\tilde{\varepsilon})^2 (k_0 R_0) \ln(-2ik_0 R) - [i/(128\pi^2)](\Delta\tilde{\varepsilon})^4 (k_0 R_0)^2 (k_0 R) \times \ln(-2ik_0 R) + \ldots, \tag{17}$$

where we have $k_0^2 = \varepsilon_0 (\omega/c)^2$, $\Delta\tilde{\varepsilon} = \Delta\varepsilon/\varepsilon_0$, and $\Delta\varepsilon$ is the difference in the phase permittivities. The effective refrac-

tive index in (17) is defined as $n = k_*/k_0$, where k_*, k_0 are the poles of the Fourier transforms of the dressed and bare Green's functions $G(\omega,\mathbf{q})$, $G_0(\omega,\mathbf{q})$.

We can easily see that the expansions (16), (17) coincide with the result of direct integration of the Helmholtz equation (1); here the relation between the correlators $\langle\varepsilon(\mathbf{x}_1)...\varepsilon(\mathbf{x}_n)\rangle_\varepsilon$ and the n-tails $K^n(\mathbf{x}_1,...,\mathbf{x}_n)$ of the continuous percolation problem is accounted for. For example,

$$\langle\varepsilon(\mathbf{x})\varepsilon(0)\rangle_\varepsilon=\langle\varepsilon^2(0)\rangle_T\langle\theta(\mathbf{x})\theta(0)\rangle_c$$
$$=\langle\varepsilon^2(0)\rangle_T K^{(2)}(\mathbf{x}),\ k_0R\gg 1,$$

where the $\langle...\rangle_T$, $\langle...\rangle_c$ designate averaging over the thermal and spatial fluctuations of the field $\varepsilon(\mathbf{x})$, respectively.[1]

It is clear from expression (17) that the effective expansion parameter for $k_0R\gg 1$ is the quantity f defined by the expression:

$$f\approx(\Delta\tilde{\varepsilon})^2(k_0R_0)(k_0R)/16\pi. \tag{18}$$

Unlike the case of light propagation near the liquid-gas transformation point when the effective scattering constant remains small near attainable values of $\tau\geqslant 10^{-6}$, near the percolation phase transition points in solids a transition to the strong coupling state ($f\gg 1$) is quite possible. The reason for this is obvious: The thermal fluctuation radius far from the critical point $r_c^{(0)}\approx 5\cdot 10^{-8}$ cm functions as the correlation scale for a liquid, while in solids there are no upper limits on the spatial scale R_0. For example an experimental estimate yields $R_0\sim 10^{-4}$ cm for $PbMg_{1/2}Nb_{2/3}O_3$ crystals experiencing a smeared phase transition.[9] The grain size, obviously, functions as the characteristic scale for the CTSL ferroelectric transparent ceramic; in this case the grain size also amounts to several microns. We employ the RG method to analyze light propagation in such materials near the percolation threshold when the parameter $f>1$. In order to apply this method to our problem it is necessary to first discuss the renormalizability of the effective Hamiltonian (14).

4. RENORMALIZATION OF THE EFFECTIVE HAMILTONIAN AND THE RENORMALIZATION GROUP EQUATIONS FOR INVARIANT CHARGES

An attempt to apply renormalization directly to Hamiltonian (14) encounters an obvious difficulty: The field-cubic vertex $\varphi_i(\mathbf{x})$ as well as the interaction vertex $\varphi|u|^2$ are dimensionless (logarithmic) with dimensionality $d=6$, while vertices of the form $\varphi\varphi\varphi\varphi$ are dimensionless for $d=4$. Such a situation has occurred previously in efforts to calculate the crossover conditions near the tricritical point[10,11] and in analyzing the critical behavior of a crystal containing extended defects.[12] In order to calculate the RG functions as expansions in powers of the small parameters Refs. 11 and 12 employed, in place of one of the interaction vertices, an operator (dependent on the additional parameter ξ) whose dimensions coincide with the initial vertex for $\xi=1$, while for $\xi=0$ it is logarithmic with the same dimensions as the remaining vertices. We proceed analogously and rather than directly renormalizing the Hamiltonian (14) we renormalize the following effective Hamiltonian:

$$\hat{H}_{eff}=\hat{H}_p+H_e+H_{int}; \tag{19}$$

H_e and H_{int} are determined in (13), (15), while the Hamiltonian $\hat{H}_p$ is expressed by the following formula:

$$\hat{H}_p=\int d\mathbf{x}\left\{\frac{1}{2}[\tau_0\varphi^2+\nabla\varphi_i\nabla\varphi_i]+\frac{1}{2}t_0(K\varphi_i)(K\varphi_i)\right.$$
$$\left.+g_0v_{ijk}(K\varphi_i)\varphi_j\varphi_k+[\lambda_1{}^0S_{ijkl}+\lambda_2{}^0F_{ijkl}]\varphi_i\varphi_j\varphi_k\varphi_l\right\}, \tag{20}$$

where the quantity $K\varphi_i$ in momentum space takes the form

$$(K\varphi_i)=(p^2)^{(1-\xi)/2}\varphi_i(\mathbf{p}). \tag{21}$$

For $\xi=0$ all interaction vertices in Hamiltonian (20) are logarithmic for $d=4$ and, consequently, it is possible to carry out of the renormalized quantities in powers of the small parameters $\varepsilon=(4-d)/2$ and ξ. We carry out renormalization of $\hat{H}_{\text{eff}}$ by means of the minimum subtraction scheme (see, for example, Ref. 13).

As is well known, in order to determine the form of the counterterms necessary for renormalization it is necessary to analyze the divergences of diagrams for the 1-irreducible Green's functions. Here we observe from the outset that since the theory contains no diagrams with closed lines of the wave field $u_m(\mathbf{x})$,renormalization of the part $\hat{H}_p(\varphi)$ of Hamiltonian (19) which is independent of the field $u_m(\mathbf{x})$ will be carried out independently. For $\hat{H}_p(\varphi)$ the surface divergences with integer-valued nonnegative indices δ have the following 1-irreducible diagrams: Self-energy diagrams with $\delta=2$, vertex diagrams with $\delta=0$ for the interactions $(K\varphi)\varphi\varphi$, $S\varphi^4$, $F\varphi^4$ and the composite operator diagrams $(K\varphi)(K\varphi)$ with $\delta=0$. Therefore we have determined that the Hamiltonian $H_p(\varphi)$ is multiplicatively renormalizable. We use $\{e_0\}$ to denote the sets of its bare parameters and we employ $\{e\}$ to represent its sets of renormalizable parameters:

$$\{e_0\}=(\tau_0,\ t_0,\ g_0,\ \lambda_1{}^0,\ \lambda_2{}^0),$$
$$\{e\}=(\tau,\ t,\ g,\ \lambda_1',\ \lambda_2',\ M),$$

where M is the renormalized mass introduced so that the dimensions satisfy $\dim t=\dim g=\dim\lambda_i'=0$. The general multiplicative renormalizability formula for $\hat{H}_p(\varphi)$ can then be given as

$$H_p{}^{(R)}(\varphi,\ \{e\})=\hat{H}_p(Z_\varphi\varphi,\ \{e_0\}), \tag{22}$$

where Z_φ is the renormalization constant of the field φ_i. Since we are interested in the behavior of the system in the critical range it is convenient to carry out calculations in the so-called massless scheme by setting $\tau=t=0$. Here it is not necessary to consider diagrams with the substitution lattices of the composite operator $(K\varphi)(K\varphi)$.

We obtain the relations between the bare and renormalized vertices in the standard fashion:

$$g_0=M^{(\varepsilon-\xi)}\tilde{g}Z_g,\quad \lambda_i{}^0=M^{2\varepsilon}Z_\varphi{}^{-4}(Z_{ik}\lambda_k'+Y_i), \tag{23}$$

where Y_i is independent of λ_i', and Z_A are the corresponding renormalization constants. The mass renormalization constant defined by the equality $\tau_0=Z_\tau\tau$ is equal to the renormalization constant of the composite operator φ^2: $Z_\tau=Z_{\varphi\varphi}$.

Reference 14 has demonstrated that for $\xi=1$ the $S\varphi^4$, $F\varphi^4$ interactions are not significant in describing the percolation properties in the critical range for $d=6-\tilde{\varepsilon}$, $\tilde{\varepsilon}\to 0$, and extrapolation of the corresponding operator dimensions directly to three-dimensional space does not affect the validity of this conclusion. We will see that the critical behavior will

be determined by $\tilde{g}(K\varphi)\varphi\varphi$ in our renormalization scheme in $(4-2\varepsilon)$-dimensional space. We carry out change of variables given by

$$Z_{ik}\lambda_k = Z_{ik}\lambda_k' + Y_i \tag{24}$$

and selected so that for $\lambda_i^0 = 0$ and we have $\lambda_i = 0$. This change of variables was used in Ref. 15. We show that the critical region is described by the infrared-stable fixed point with coordinates $\lambda_i^* = 0$. Since Gell-Mann–Low functions (GMLF) linearized in λ_i are sufficient for analyzing the stability of this point, it is sufficient to calculate $\tilde{Z}_{ik}$ in the zeroth order in λ_i. Here in the lowest order in $\tilde{g}$ the matrix

$$Z_{ik}(\lambda_i=0, \tilde{g}) = Z_{ik}(\lambda_i'=0, \tilde{g}),$$

while for the GMLF we obtain:

$$\beta_i = -2(\varepsilon-\eta)\lambda_i + \tilde{g}(\varepsilon+\xi)\lambda_k(\partial_g Z_{ik}), \tag{25}$$

where β_i is the GMLF for the charges λ_i, $\partial_A = \partial/\partial A$, $\eta = 2D_M \ln Z_\varphi$, $D_M = M\partial_M$.

We find the following values of the renormalization constants in the first order in the renormalized interaction constant g^2 for $s = 1$:

$$\begin{aligned} &Z_\varphi^2 = 1 + \frac{g^2}{32(\varepsilon+\xi)}, \quad Z_g Z_\varphi^3 = 1 + \frac{2g^2}{(\varepsilon+\xi)}, \\ &Z_\tau Z_\varphi^2 = 1 + \frac{g^2}{(\varepsilon+\xi)}, \quad Z_{11} = 1 - Z_{12}, \; Z_{22} = 1 - 2Z_{12}, \\ &Z_{12} = \frac{3}{2} Z_{21} = -\frac{6g^2}{(\varepsilon+\xi)}, \quad g = \tilde{g}/\pi. \end{aligned} \tag{26}$$

We then obtain in the standard fashion:

$$\begin{aligned} &\beta_g = g[-(\varepsilon+\xi-{}^3/_2\eta)+4g^2], \\ &\beta_1 = -2[(\varepsilon-\eta)\lambda_1 + 6g^2(\lambda_2-\lambda_1)], \\ &\beta_2 = -2[(\varepsilon-\eta)\lambda_2 + 4g^2(\lambda_1-3\lambda_2)], \end{aligned} \tag{27}$$

where β_g is the GMLF for vertex g.

The value of the index η^* corresponding to the IR-stable fixed point is numerically small (see below) and hence to first order in η^* we have for the coordinate g_* of the fixed point:

$$g_*^2 = (\varepsilon+\xi)/4. \tag{28}$$

For the critical indices we obtain

$$\begin{aligned} &\gamma_\tau = D_M \ln Z_\tau|_{g=g_*} = -2g_*^2 \approx -\frac{\varepsilon+\xi}{2}, \\ &\eta^* = 2D_M \ln Z_\varphi|_{g=g_*} = -\frac{g_*^2}{16} \approx -(\varepsilon+\xi)/64. \end{aligned} \tag{29}$$

Using (27), (28) we have the following for the stability indices for the fixed point with coordinates $g = g_*$, $\lambda_i = 0$:

$$\begin{aligned} &\omega_g = (\partial_g \beta_g)|_{g_*} = 8g_*^2, \\ &\omega_{1,2} = 2[\varepsilon - \eta^* + g_*^2(9 \pm 33^{1/2})]. \end{aligned} \tag{30}$$

Since ω_g and $\omega_{1,2} > 0$, this fixed point is IR-stable. We note that the coordinates of the fixed point obtained through expansion in the small parameters $\varepsilon = (4-d)/2$ and $\xi \ll 1$ and the corresponding critical indices as $\xi \to 1$ are in good agreement with the values obtained previously for the percolation problem within the framework of the $6-d = \tilde{\varepsilon}$-expansion for $\xi = 1$, $\tilde{\varepsilon} = 2 + 2\varepsilon$ (Ref. 16) and from a comparison to the results of a lower approximation of renormalization theory directly formulated in three-dimensional space when the value of the invariant charge at the fixed point is not parametrically small.[17]

We note in renormalizing the remaining parts of the Hamiltonian (19) dependent on the wave field $u_m(\mathbf{x})$ that the self-energy diagrams with two vertices w are the only ultraviolet-diverging vertices for $d = 4$ 1-irreducible n-tails of this field. The divergence index for these is equal to zero, and hence such diagrams give rise to additive renormalization Δm^2 of the squared "mass" of the wave field $m_0^2 = k_0^2$. Hence in our minimum subtraction scheme when the polar parts in ε, ξ are subtracted only from divergent diagrams (Ref. 13), $\Delta m^2 = w^2\Psi(g)$, where the function $\Psi(g)$ is a series in g^2, the renormalization constant of the wave field u_m is trivial: $Z_u = 1$, while renormalization of the vertex w is given by

$$w_0 = wM^\varepsilon Z_\varphi^{-1}, \tag{31}$$

where Z_φ is the previously determined renormalization constant of the percolation field φ_i.

Since any quantities $A(m_0, w_0, \{e_0\})$ dependent solely on the bare constants of the effective Hamiltonian (19) retain their values with changes in the renormalized mass M, the corresponding RG equation takes the following form for these quantities

$$\begin{aligned} \hat{R}A \equiv M\frac{\partial A}{\partial M}\Big|_{m_0, w_0\{e_0\}} &= \Big\{\beta_g\partial_g + \beta_w\partial_w + D_M + (D_M \ln \tau)D_\tau \\ &+ (D_M m^2)\frac{\partial}{\partial m^2}\Big\}A(m, w, \{e\}) = 0, \end{aligned} \tag{32}$$

where the operator is equal to

$$D_M = M\partial_M|_{m, w, \{e\}},$$

and the GMLF is equal to

$$\beta_w = -w[\varepsilon + [D_M \ln w]] = -w[\varepsilon - \eta(g)/2].$$

Using the explicit form of $\Delta m^2 = w^2\Psi(g)$ we write the RG operator in (32) in the following form:

$$\hat{R} = D_M + \beta_g\partial_g + \beta_w\partial_w - \gamma_\tau D_\tau - 2w^2\delta(g)\frac{\partial}{\partial m^2}, \tag{33}$$

where the function $\delta(g)$ is defined by the relation

$$\delta(g) = \Psi(g)[\varepsilon - {}^1/_2\eta(g)] - {}^1/_2\beta_g\partial_g\Psi(g). \tag{34}$$

Adzhemyan *et al.*[4] have noted that light scattering by thermal density fluctuations near the critical liquid-gas transformation point will produce a new invariant charge. Precisely the same situation also occurs in our case of light scattering by the percolating medium. The reason for the similarity is the formally identical nature of the ultraviolet and infrared asymptotic forms of the pair Green functions of the critical thermal fluctuation field and the percolation problem, as well as the linearity of the interaction vertices in the scattering field in both cases. Defining the second invariant charge as in Ref. 4 by the relation $w^2 = m^2 v$, we can represent (33) as an RG operator from doubly-charged theory:

$$\hat{R} = D_M + \beta_g\partial_g + \beta_v\partial_v - \gamma_\tau D_\tau - \gamma_m D_m, \tag{35}$$

where $\gamma_m = v\delta(g)$, while β_v—the GMLF for the new charge v—is equal to

$$\beta_v = v[\eta(g) - 2\varepsilon + 2v\delta(g)]. \tag{36}$$

The functions $\delta(g)$, $\eta(g)$ are series in g^2, so at the fixed point $g = g_*$ the value of the Fisher anomalous dimensions index of the percolation field is numerically small [see (29)]. The expansion of the function $\eta(g)$ begins with g^2, while the expansion $\delta(g)$ begins with the constant $\delta(g) = (16\pi^2)^{-1} + O(g^2)$. This fact together with the negative value of the Fisher index η^* for the percolation medium will produce a fixed point for an invariant charge v at $v_* > 0$. In a first approximation in the small parameters ε, ξ the value of the coordinate v_* is equal to

$$v_* = 16\pi^2(\varepsilon - \eta^*/2). \tag{37}$$

Unlike the case of thermal critical fluctuations the Fisher index must be accounted for in accordance with (29) as early as the lowest approximation. The GMLF β_g is independent of v and hence one of the stability indices of the fixed point with coordinates $g = g_*$, $v = v_*$ is identical to ω_g defined in (30), while the second is equal to $\omega_v \equiv \partial\beta_v/\partial v = 2^*_{\varepsilon-\eta}$. Since we have $\omega_g > 0$, $\omega_v > 0$, this fixed point is IR-stable, and since we have $v_* > 0$ it can also be attained from the range of bare values $v_0 = (w_0/m_0)^2$. The existence of such a fixed point makes it possible to investigate the behavior of the effective refractive index $n(k_0, R)$ near the percolation threshold and to calculate its scaling function as an expansion in powers of the small parameters ε and ξ.

5. GENERAL SOLUTION OF THE RG EQUATION AND THE ASYMPTOTIC FORM OF THE EFFECTIVE REFRACTIVE INDEX

As noted above in Sec. 3 the quantity $n(k_0, R)$ is determined by the position of the pole $k = m_*$ of the exact Green's function of the wave field, $n^2 = m_*^2/m^2$. Since the value of m_* is, obviously, determined by the bare constants of the Hamiltonian (19) it is a renormalization invariant quantity for which $\hat{R}m_* = 0$. Hence the RG equation for n^2 takes the following form:

$$(\beta_g\partial_g + \beta_v\partial_v - d_m D_y - d_\tau D_z - 2\gamma_m)n^2 = 0. \tag{38}$$

In (38) we arrived at dimensionless variables $y = m^2/M^2$ and $z = \tau/M^2$; the functions $d_m(v,g) = 2 + 2\gamma_m$, $d_\tau(g) = 2 + \gamma_\tau$ are the total dimensions of the quantities m^2 and τ.

The properties of the solutions of Eq. (38) can easily be analyzed using the method that was employed in Ref. 4 to formulate a general solution of the RG equation for the wave Green's function. For any solution of Eq. (38) we have

$$n^2(g, v, y, z) = \exp\left(2\int_t^1 \frac{\gamma_m}{t'}\,dt'\right) n^2(\tilde{g}, \tilde{v}, \tilde{y}, \tilde{z}), \tag{39}$$

where $\tilde{g}$, $\tilde{v}$, $\tilde{y}$, $\tilde{z}$ are normalized first integrals of (38) defined by the equations

$$\frac{dt}{t} = \frac{d\tilde{g}}{\beta_g(\tilde{g})} = \frac{d\tilde{v}}{\beta_v(\tilde{v}, \tilde{g})} = -\frac{d\tilde{y}}{\tilde{y}d_m(\tilde{g}, \tilde{v})} = -\frac{d\tilde{z}}{\tilde{z}d_\tau(\tilde{g})} \tag{40}$$

and the normalization conditions are $\tilde{b}_i(t = 1, \mathbf{b}) = b_i$, $\mathbf{b} = g, v, y, z)$. In the critical regime for $t \to 0$ we have

$$\tilde{g} \approx g_*,\ \tilde{v} \approx v_*,\ \tilde{z} \approx zt^{-d^*_\tau},\ \tilde{y} \approx yt^{-d^*_m}, \tag{41}$$

with $d^*_m = d_m(g_*, v_*)$, $d_\tau(g_*)v_p^{-1}$, where v_p is the correlation radius index of the percolation problem. Consequently n^2 is a scaling function of only the two variables y, z:

$$n^2(g, v, y, z) \approx t^{-\mu}n^2(g_*, v_*, yt^{-d_m^*}, zt^{-d_\tau^*}), \tag{42}$$

with $\mu = 2\gamma^*_m$.

In a natural way we have two ranges of variables in which the asymptotics forms of n are power functions:

1) $z \ll y \ll 1$, $n^2 = y^{-\alpha}P(zy^{-\beta})$, 2) $y \ll z \ll 1$, $n^2 = z^{-s}Q(yz^{-1/\beta})$.

The critical indices α, β, s are defined by the relations

$$\alpha = \mu/(2+\mu),\ \beta^{-1} = (2+\mu)\nu_p,\ s = \mu\nu_p, \tag{43}$$

while we use the following normalization conditions to find P, Q:

$$P\left(\frac{\tau}{m^2}\right) = m^{-2}m_*^2\left(g_*, v_*, 1, \frac{\tau}{m^2}\right)\Big|_{M=m},$$
$$Q\left(\frac{m^2}{\tau}\right) = m^{-2}m_*^2\left(g_*, v_*, \frac{m^2}{\tau}, 1\right)\Big|_{M^2=\tau}. \tag{44}$$

In order to calculate these functions accurate to $O(w_*^4, w_*^2, g_*^2)$ it is sufficient to determine the position of the pole $k = m_*$ of the renormalized Green's function while accounting for only the first diagram in (16). Using the value of the coordinate v_* to first order in ε, ξ we obtain

$$P\left(\frac{\tau}{m^2} \to 0\right) = \{1 + \alpha[2 - C + \ln(4\pi)] + \ldots\}e^{i\pi\varkappa\alpha},$$
$$Q\left(\frac{m^2}{\tau} \to 0\right) = 1 + \alpha[1 - C + \ln(4\pi)] + \ldots, \tag{45}$$
$$\varkappa = \operatorname{sign}(\omega),\ \alpha = (2\varepsilon - \eta^*)/(2 + 2\varepsilon - \eta^*),$$

where C is Euler's constant. finally the expression of the asymptotic forms of the effective refractive index in the "short-wave" and "long-wave" regions can be given as

$$n^2(k_0R \to \infty) = P_1 e^{i\pi\varkappa\alpha}(k_0R_0)^{-2\alpha},$$
$$n^2(k_0R \to 0) = Q_1(R/R_0)^\mu, \tag{46}$$

where P_1, Q_1 are real positive constants. By extrapolating $\varepsilon \to 1/2$, $\xi \to 1$ for the critical indices we have $\alpha \approx 1/3$, $\mu = 1 - \eta^* > 1$.

6. DISCUSSION

As noted in Sec. 3, n enters the asymptotic conditions (42), (46) in the experimentally accessible range near the percolation threshold for a number of disordered transparent crystals and ceramics. Of course there are other inhomogeneous media in which wave propagation can also be described based on the present conclusion that the wave velocity relation has a scaling character. Specifically, a number of interesting problems correspond to the case of wave propagation in a strongly-fluctuating Gaussian medium.[18] Since the Hamiltonian of the scattering field $H(\varphi)$ is renormalized independently, this situation is described by a fixed point with coordinates $g_* = 0$, $v_* = 16\pi^2\varepsilon$. The corresponding values of the critical indices in the asymptotic forms (46)

can be obtained by setting the Fisher index of the scattering field equal to zero.

It is also possible to generalize the RG scheme developed in the present study to a stochastic vector wave equation and obtain the scaling asymptotic forms for the effective elastic moduli of the percolation components. A solution to this problem, which has been the focus of significant attention in recent years,[19,22] will be provided in another paper.

In conclusion we wish to discuss a few general issues. On several occasions there have been comments regarding the promise of using field theory to investigate the correlation functions of a radiation field scattered by a strongly fluctuating medium. Moreover the RG method, to the best of our knowledge, was used in only the study[4] to find the average Green's function of a scalar wave scattered near the critical liquid-phase transformation point. The values of the physical parameters of this problem, of course, are such that the resulting critical conditions were wholly unattainable in any actual experiment.

This prompted the authors of Ref. 4 to conclude that the RG method was essentially useless and for this reason their study dropped the expansions of renormalized perturbation theory for the scaling asymptotic forms of the Green's function and the effective refractive index. However, as discussed above, the relative weakness of light scattering by thermal fluctuations in no way suggests a lack of physical media in which strong coupling can occur, and the results from the RG method are quite valuable. We believe it is important to emphasize the universality of the method. Specifically, this universality is manifested in a number of identified common traits of solutions of wave scattering problems by percolation clusters and thermal fluctuations. The formal relation between both problems lies in the fact that the first is obtained from the contribution of the s-component of the Potts model for $s \to 1$ contained in effective Hamiltonian (14) while the second is obtained from $s = 2$ when the s-model is equivalent to the Ising model[23] used in Ref. 4. The limit process in s occurs in the last calculation stage and hence the description of scattering by thermal clusters and percolation clusters is similar.

The universality of the RG method makes it possible to predict a certain similarity of results for a wide range of problems for which the effective Hamiltonian describing an inhomogeneous medium is renormalizable. In this case the values of the critical indices of the scaling functions are close to the Gaussian functions accurate to the Fisher index η^*.

The problem of calculating the average Green's function is the first and simplest in a number of problems relating to calculation of the correlators of the scattered radiation. There is reason to hope that the RG method will prove to be successful for calculating such physically important higher order correlators and the average intensity of a scattered wave and the average squared intensity fluctuation.[24]

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